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SOME FIXED POINT THEOREMS VIA GENERALIZED c-DISTANCE IN ORDERED CONE METRIC SPACES

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Abstract. In this paper, we prove some fixed point theorems by introducing the concept of generalized c-distance in partially ordered cone metric spaces.

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1. Introduction

The existence of fixed points for certain mappings in ordered metric spaces has been studied by Ran and Reurings [18]. In [13], Nieto and López extended the result of Ran and Reurings [18] for nondecreasing mappings and applied their results to obtain a unique solution for a first order differential equation. Afterwards, Huang and Zhang [8] introduced the concept of cone metric spaces by replacing the set of real numbers by an ordered real Banach space with a cone. So far, many researchers have established fixed point and common fixed point results for mappings under various contractive conditions in normal or non-normal cone metric spaces.Very

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recently, Cho *et al.* [5] introduced the concept of c-distance in cone metric spaces which is a cone version of w-distance of Kada *et al.* [11] and proved some fixed point theorems by using c-distance in partially ordered cone metric spaces. In this paper we introduce the concept of generalized c-distance in a partially ordered cone metric space and prove some fixed point theorems by using this new concept of generalized c-distance. Our results will improve and supplement some results in the existing literature.

2. Preliminaries

In this section we need to recall some basic notations, definitions, and necessary results from existing literature. Let E be a real Banach space and θ denote the zero element in E. A cone P is a subset of E such that

$$(i) \quad P\ is\ closed,\ nonempty\ and\ P \neq \{\theta\};$$

$$(ii) \quad a,b \in \mathbb{R},\ a,b \geq 0,\ x,y \in P \ \Rightarrow\ ax+by \in P;$$

$$(iii) \quad P \cap (-P) = \{\theta\}.$$

For any cone $P \subseteq E$, we can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ (equivalently, $y \succ x$) if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P. The cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$,

$$\theta \preceq x \preceq y\ implies\ \|x\| \leq k\,\|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P. Rezapour and Hamlbarani [16] proved that there are no normal cones with normal constant $k < 1$.

Definition 2.1. [8] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

$$(i)\ \theta \preceq d(x,y)\ for\ all\ x,y \in X\ and\ d(x,y) = \theta\ if\ and\ only\ if\ x = y;$$

$$(ii)\ d(x,y) = d(y,x)\ for\ all\ x,y \in X;$$

$$(iii)\ d(x,y) \preceq d(x,z) + d(z,y)\ for\ all\ x,y,z \in X.$$

Then d is called a cone metric on X, and (X,d) is called a cone metric space.

Definition 2.2. [8] Let (X,d) be a cone metric space. Let (x_n) be a sequence in X and $x \in X$. If for every $c \in E$ with $\theta \ll c$ there is a natural number n_0 such that for all $n > n_0$, $d(x_n,x) \ll c$, then (x_n) is said to be convergent and (x_n) converges to x, and x is the limit of (x_n). We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ $(n \to \infty)$.

Definition 2.3. [8] Let (X,d) be a cone metric space, (x_n) be a sequence in X. If for any $c \in E$ with $\theta \ll c$, there is a natural number n_0 such that for all $n,m > n_0$, $d(x_n,x_m) \ll c$, then (x_n) is called a Cauchy sequence in X.

Definition 2.4. [8] Let (X,d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 2.5. [19] *Every cone metric space (X,d) is a topological space. For $c \gg \theta$, $c \in E$, $x \in X$ let $B(x,c) = \{y \in X : d(y,x) \ll c\}$ and $\beta = \{B(x,c) : x \in X, c \gg \theta\}$. Then $\tau_c = \{U \subseteq X : \forall x \in U, \exists B \in \beta, x \in B \subseteq U\}$ is a topology on X.*

Definition 2.6. [19] Let (X,d) be a cone metric space. A map $T : (X,d) \to (X,d)$ is called sequentially continuous if $x_n \in X, x_n \to x$ implies $Tx_n \to Tx$.

Lemma 2.7. [19] *Let (X,d) be a cone metric space, and $T : (X,d) \to (X,d)$ be any map. Then, T is continuous if and only if T is sequentially continuous.*

Lemma 2.8. [17] *Let E be a real Banach space with a cone P. Then*

(i) *If $a \ll b$ and $b \ll c$, then $a \ll c$.*

(ii) *If $a \preceq b$ and $b \ll c$, then $a \ll c$.*

Lemma 2.9. [8] *Let E be a real Banach space with cone P. Then one has the following.*

(i) *If $\theta \ll c$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.*

(ii) *If a_n, b_n are sequences in E such that $a_n \to a$, $b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 1$, then $a \preceq b$.*

Proposition 2.10. [10] If E is a real Banach space with cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$ then $a = \theta$.

Definition 2.11. [5] Let (X,d) be a cone metric space. Then a function $q : X \times X \to E$ is called a c-distance on X if the following are satisfied :

(i): $\theta \preceq q(x,y)$ for all $x,\, y \in X$;

(ii): $q(x,z) \preceq q(x,y) + q(y,z)$ for all $x,y,z \in X$;

(iii): for each $x \in X$ and $n \geq 1$, if $q(x,y_n) \preceq u$ for some $u = u_x \in P$, then $q(x,y) \preceq u$ whenever (y_n) is a sequence in X converging to a point $y \in X$;

(iv): for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$.

Example 2.12. [5] Let (X,d) be a cone metric space and P be a normal cone. Put $q(x,y) = d(x,y)$ for all $x,y \in X$. Then q is a c-distance.

Example 2.13. [5] Let (X,d) be a cone metric space and P be a normal cone. Put $q(x,y) = d(u,y)$ for all $x,y \in X$, where $u \in X$ is a fixed point. Then q is a c-distance.

Example 2.14. [5] Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0,\infty)$ and define a mapping $d : X \times X \to E$ by $d(x,y) = \mid x-y \mid$ for all $x,y \in X$. Then (X,d) is a cone metric space. Define a mapping $q : X \times X \to E$ by $q(x,y) = y$ for all $x,y \in X$. Then q is a c-distance.

Remark 2.15. [5] (1) $q(x,y) = q(y,x)$ does not necessarily hold for all $x,y \in X$.
(2) $q(x,y) = \theta$ is not necessarily equivalent to $x = y$ for all $x,y \in X$.

3. Main results

In this section we always suppose that E is a real Banach space, P is a cone in E with $int(P) \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P.

We begin with a definition.

Definition 3.1. Let (X,d) be a cone metric space and $j \in \mathbb{N}$. A function $q : X \times X \to E$ is called a generalized c-distance of order j on X if the following conditions are satisfied:

(q1): $\theta \preceq q(x,y)$, for all $x,y \in X$;

(q2): $q(x,z) \preceq \sum_{i=0}^{j} q(x_i,x_{i+1})$, for all $x,z \in X$ and for all distinct points $x_i \in X$, $i \in \{1,2,3,\cdots,j\}$ each of them different from $x(=x_0)$ and $z(=x_{j+1})$;

(q3): for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \preceq u$ for some $u = u_x \in P$, then $q(x, y) \preceq u$ whenever (y_n) is a sequence in X converging to a point $y \in X$;

(q4): for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

It is to be noted that every c-distance is a generalized c-distance of order 1. In fact, every c-distance may also be considered as a generalized c-distance of any order $j \in \mathbb{N}$. However the converse is not true, in general. In this connection we consider the following examples.

Example 3.2. Let $E = \mathbb{R}^2$, the Euclidean plane and $P = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ a cone in E. Let $X = \{\alpha, \beta, \gamma, \delta\} \subseteq \mathbb{R}$ and define $d : X \times X \to E$ by

$$d(x, y) = (a \mid x - y \mid, b \mid x - y \mid)$$

for all $x, y \in X$, where a, b are positive constants. Then (X, d) is a cone metric space. Let $q : X \times X \to E$ be defined by

$$q(\alpha, \beta) = q(\beta, \alpha) = (9, 9), q(\alpha, \gamma) = q(\gamma, \alpha) = q(\beta, \gamma) = q(\gamma, \beta) = (3, 3),$$

$$q(\alpha, \delta) = q(\delta, \alpha) = q(\beta, \delta) = q(\delta, \beta) = q(\gamma, \delta) = q(\delta, \gamma) = (5, 5)$$

$$and\ q(x, x) = (0.6, 0.6)\ for\ every\ x \in X.$$

Then q satisfies condition $(q2)$ of Definition 3.1 for $j = 2$. The conditions $(q1)$ and $(q3)$ are immediate. To show $(q4)$, for any $c \in E$ with $\theta \ll c$, put $e = (\frac{1}{2}, \frac{1}{2})$. Then

$$q(z, x) \ll e\ and\ q(z, y) \ll e\ imply\ d(x, y) \ll c.$$

Thus q is a generalized c-distance of order 2 on X but it is not a c-distance on X since it lacks the triangular property:

$$q(\alpha, \beta) = (9, 9) \npreceq q(\alpha, \gamma) + q(\gamma, \beta) = (3, 3) + (3, 3) = (6, 6).$$

Example 3.3. Let $E = \mathbb{R}$ and $P = \{x \in E : x \geq 0\}$ a cone in E. Let $X = \mathbb{N}$ and define a mapping $d : X \times X \to E$ by

$$d(x, y) = \mid x - y \mid$$

for all $x,y \in X$. Then (X,d) is a cone metric space. Let $q : X \times X \to E$ be defined by

$$q(x,y) = \begin{cases} 0 \;\; if \;\; x = y, \\ \\ 3 \;\; if \;\; x,y \in \{1,2\} \; and \; x \neq y, \\ \\ 1 \; if \; (x \in \{1,2\}^c \; or \; y \in \{1,2\}^c) \; and \; x \neq y. \end{cases}$$

Then q satisfies condition $(q2)$ of Definition 3.1 for $j \geq 2$. The conditions $(q1)$ and $(q3)$ are immediate. To show $(q4)$, for any $c \in E$ with $0 \ll c$, put $e = \frac{1}{2}$. Then

$$q(z,x) \ll e \; and \; q(z,y) \ll e \; imply \; d(x,y) \ll c.$$

Thus q is a generalized c-distance of order j on X but it is not a c-distance on X since it lacks the triangular property:

$$q(1,2) = 3 > q(1,3) + q(3,2) = 1 + 1 = 2.$$

Remark 3.4. Generalized c-distances form a bigger category than that of c-distances.

Lemma 3.5. *Let (X,d) be a cone metric space and q be a generalized c-distance of order j on X. Let (x_n) and (y_n) be sequences in X. Suppose that (α_n) and (β_n) are sequences in P converging to θ, and let $x,y,z \in X$. Then the following hold :*

(i) If $q(x_n,y_n) \preceq \alpha_n$ and $q(x_n,z) \preceq \beta_n$ for any $n \in \mathbb{N}$, then (y_n) converges to z;

(ii) If $q(x_n,y) \preceq \alpha_n$ and $q(x_n,z) \preceq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $q(x,y) = \theta$ and $q(x,z) = \theta$, then $y = z$;

(iii) If $q(x_n,x_m) \preceq \alpha_n$ for any $n,m \in \mathbb{N}$ with $m > n$, then (x_n) is a Cauchy sequence.

Proof. (i) Let $c \in E$ with $\theta \ll c$. Then there exists $\delta > 0$ such that $\| x \| < \delta$ implies $c - x \in int\,(P)$. Since (α_n) and (β_n) are converging to θ, there exists $n_0 \in \mathbb{N}$ such that $\| \alpha_n \| < \delta$ and $\| \beta_n \| < \delta$ for all $n > n_0$. Thus $c - \alpha_n \in int\,(P)$ and $c - \beta_n \in int\,(P)$ for all $n > n_0$ and so $\alpha_n \ll c$ and $\beta_n \ll c$ for all $n > n_0$. By hypothesis, $q(x_n,y_n) \preceq \alpha_n \ll c$ and $q(x_n,z) \preceq \beta_n \ll c$ for all $n > n_0$. Now from $(q4)$ with $e = c$ it follows that $d(y_n,z) \ll c$ for all $n > n_0$. Therefore (y_n) converges to z.

Clearly, (ii) is immediate from (i).

(iii) Let $c \in E$ with $\theta \ll c$. Then by the arguments similar to that used above, there exists a positive integer n_0 such that $q(x_n, x_m) \preceq \alpha_n \ll c$ for all $m > n$ with $n > n_0$. This implies that $q(x_n, x_{n+1}) \preceq \alpha_n \ll c$ and $q(x_n, x_{m+1}) \preceq \alpha_n \ll c$ for all $m > n$ with $n > n_0$. From $(q4)$ with $e = c$ it follows that $d(x_{n+1}, x_{m+1}) \ll c$ for all $m > n$ with $n > n_0$. This shows that (x_n) is a Cauchy sequence in X.

Theorem 3.6. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X, d) is a complete cone metric space. Let q be a generalized c-distance of order j on X and $f : X \to X$ be a continuous and nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following conditions hold:*
(i) there exist $a_1,\ a_2,\ a_3 \geq 0$ with $a_1 + a_2 + a_3 < 1$ such that

$$q(fx, fy) \preceq a_1\, q(x, y) + a_2\, q(x, fx) + a_3\, q(y, fy) \tag{3.1}$$

for all $x, y \in X$ with $x \sqsubseteq y$;
(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$.
Then f has a fixed point in X. Moreover, if $u = fu$, then $q(u, u) = \theta$.

Proof. Since $x_0 \sqsubseteq fx_0$ and f is nondecreasing with respect to $\sqsubseteq$, we have

$$x_0 \sqsubseteq fx_0 \sqsubseteq f^2 x_0 \sqsubseteq \cdots \sqsubseteq f^n x_0 \sqsubseteq f^{n+1} x_0 \sqsubseteq \cdots. \tag{3.2}$$

Let $x_n = fx_{n-1} = f^n x_0$ for $n = 1, 2, 3, \cdots$. Then (x_n) is a nondecreasing sequence in X with respect to $\sqsubseteq$. We can suppose that $x_n \neq x_m$ for all distinct $n, m \in \{0, 1, 2, \cdots\}$. In fact, if $x_n = x_m$ for some $n, m \in \{0, 1, 2, \cdots\}$, $m \neq n$ then assuming $m > n$, it follows from (3.2) that

$$x_n = x_{n+1} = \cdots = x_m.$$

Now $x_n = x_{n+1}$ implies that $x_n = fx_n$. So, x_n is a fixed point of f. Thus in the sequel of the proof we can assume that $x_n \neq x_m$ for all distinct $n, m \in \{0, 1, 2, \cdots\}$.
For any natural number n, we have by using condition (3.1) that

$$\begin{aligned} q(x_n, x_{n+1}) &= q(fx_{n-1}, fx_n) \\ &\preceq a_1\, q(x_{n-1}, x_n) + a_2\, q(x_{n-1}, fx_{n-1}) + a_3\, q(x_n, fx_n) \\ &= a_1\, q(x_{n-1}, x_n) + a_2\, q(x_{n-1}, x_n) + a_3\, q(x_n, x_{n+1}). \end{aligned}$$

So, it must be the case that

$$q(x_n, x_{n+1}) \preceq r\, q(x_{n-1}, x_n) \tag{3.3}$$

where $r = \frac{a_1+a_2}{1-a_3} \in [0,1)$.

By repeated application of (3.3), we obtain

$$q(x_n, x_{n+1}) \preceq r^n\, q(x_0, x_1). \tag{3.4}$$

Let $m, n \in \mathbb{N}$ with $m > n$. Taking $m = n + p$ where $p = 1, 2, 3, \cdots$ and using (3.1) and (3.4), we have

$$\begin{aligned} q(x_n, x_m) &= q(fx_{n-1}, fx_{m-1}) \\ &\preceq a_1 q(x_{n-1}, x_{m-1}) + a_2 q(x_{n-1}, fx_{n-1}) + a_3 q(x_{m-1}, fx_{m-1}) \\ &= a_1 q(x_{n-1}, x_{m-1}) + a_2 q(x_{n-1}, x_n) + a_3 q(x_{m-1}, x_m) \\ &\preceq a_1 q(x_{n-1}, x_{m-1}) + a_2 r^{n-1} q(x_0, x_1) + a_3 r^{m-1} q(x_0, x_1) \\ &\preceq a_1 q(x_{n-1}, x_{m-1}) + (a_2 + a_3) r^{n-1} q(x_0, x_1), \end{aligned}$$

since $r^{m-1} \leq r^{n-1}$.

Continuing in this way, we obtain at the n-th step that

$$\begin{aligned} q(x_n, x_m) &\preceq a_1^n q(x_0, x_p) + (a_2 + a_3)[r^{n-1} + a_1 r^{n-2} + \cdots + a_1^{n-1}] q(x_0, x_1) \\ &= a_1^n q(x_0, x_p) + \beta_n q(x_0, x_1), \end{aligned} \tag{3.5}$$

where $\beta_n = (a_2 + a_3)[r^{n-1} + a_1 r^{n-2} + \cdots + a_1^{n-1}]$.

We now show that

$$q(x_0, x_p) \preceq \frac{1}{1 - a_1^j} \left(\frac{1}{1-r} + \beta_j \right) M,$$

where $M = q(x_0, x_1) + q(x_0, x_2) + \cdots + q(x_0, x_j) \in P$.

If $p \leq j$, then

$$\begin{aligned} q(x_0,x_p) &\preceq (1+\beta_j)\,q(x_0,x_p) \\ &\preceq [(1+r+r^2+\cdots)+\beta_j]\,q(x_0,x_p) \\ &= \left(\frac{1}{1-r}+\beta_j\right) q(x_0,x_p) \\ &\preceq \left(1+a_1^j+(a_1^j)^2+\cdots\right)\left(\frac{1}{1-r}+\beta_j\right) q(x_0,x_p) \\ &\preceq \frac{1}{1-a_1^j}\left(\frac{1}{1-r}+\beta_j\right) M. \end{aligned}$$

If $p > j$, then there exists $s \in \mathbb{N}$ such that $p = sj+t$, where $0 \leq t < j$.

If $t = 0$, then by using conditions (3.4) and (3.5)

$$\begin{aligned} q(x_0,x_p) &\preceq q(x_0,x_1)+q(x_1,x_2)+\cdots+q(x_{j-1},x_j)+q(x_j,x_p) \\ &\preceq q(x_0,x_1)+rq(x_0,x_1)+\cdots+r^{j-1}q(x_0,x_1) \\ &\quad +a_1^j q(x_0,x_{p-j})+\beta_j q(x_0,x_1) \\ &= \left(\sum_{\nu=0}^{j-1} r^\nu+\beta_j\right) q(x_0,x_1)+a_1^j q(x_0,x_{p-j}). \end{aligned} \tag{3.6}$$

By repeated application of (3.6), we obtain at $(s-1)$-th step that

$$\begin{aligned} q(x_0,x_p) &\preceq \left[1+a_1^j+(a_1^j)^2+\cdots+(a_1^j)^{s-2}\right]\left(\sum_{\nu=0}^{j-1} r^\nu+\beta_j\right) q(x_0,x_1) \\ &\quad +(a_1^j)^{(s-1)} q(x_0,x_j) \\ &\preceq \left[1+a_1^j+(a_1^j)^2+\cdots+(a_1^j)^{s-2}\right]\left(\sum_{\nu=0}^{j-1} r^\nu+\beta_j\right) q(x_0,x_1) \\ &\quad +(a_1^j)^{(s-1)}\left(\sum_{\nu=0}^{j-1} r^\nu+\beta_j\right) q(x_0,x_j) \\ &\preceq \left[1+a_1^j+(a_1^j)^2+\cdots+(a_1^j)^{s-1}\right]\left(\sum_{\nu=0}^{j-1} r^\nu+\beta_j\right) M \\ &\preceq \frac{1}{1-a_1^j}\left(\frac{1}{1-r}+\beta_j\right) M. \end{aligned}$$

If $t \neq 0$, then

$$\begin{aligned} q(x_0,x_p) &\preceq q(x_0,x_1)+q(x_1,x_2)+\cdots+q(x_{j-1},x_j)+q(x_j,x_p) \\ &\preceq \left(\sum_{\nu=0}^{j-1} r^\nu+\beta_j\right) q(x_0,x_1)+a_1^j q(x_0,x_{p-j}). \end{aligned} \tag{3.7}$$

By repeated application of (3.7), we obtain at s-th step that

$$\begin{aligned} q(x_0,x_p) &\preceq \left[1+a_1^j+(a_1^j)^2+\cdots+(a_1^j)^{s-1}\right]\left(\sum_{\nu=0}^{j-1} r^\nu+\beta_j\right) q(x_0,x_1) \\ &\quad +(a_1^j)^s q(x_0,x_t) \\ &\preceq \left[1+a_1^j+(a_1^j)^2+\cdots+(a_1^j)^{s-1}\right]\left(\sum_{\nu=0}^{j-1} r^\nu+\beta_j\right) q(x_0,x_1) \\ &\quad +(a_1^j)^s\left(\sum_{\nu=0}^{j-1} r^\nu+\beta_j\right) q(x_0,x_t) \\ &\preceq \left[1+a_1^j+(a_1^j)^2+\cdots+(a_1^j)^{s}\right]\left(\sum_{\nu=0}^{j-1} r^\nu+\beta_j\right) M \\ &\preceq \frac{1}{1-a_1^j}\left(\frac{1}{1-r}+\beta_j\right) M. \end{aligned}$$

Thus, for the case $p > j$, we have

$$q(x_0,x_p) \preceq \frac{1}{1-a_1^j}\left(\frac{1}{1-r}+\beta_j\right) M.$$

It now follows from (3.5) that for all $m,n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} q(x_n,x_m) &\preceq \frac{a_1^n}{1-a_1^j}\left(\frac{1}{1-r}+\beta_j\right) M+\beta_n q(x_0,x_1) \\ &\preceq b_n M, \end{aligned}$$

where $b_n = \frac{a_1^n}{1-a_1^j}\left(\frac{1}{1-r}+\beta_j\right)+\beta_n \to 0$ as $n \to \infty$. By using Lemma 3.5(iii), we conclude that (x_n) is a Cauchy sequence in X. Since X is complete, there exists an element $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Since f is continuous,

$$fu = f(\lim_n x_n) = \lim_n fx_n = \lim_n x_{n+1} = u.$$

Thus u is a fixed point of f.

Again,

$$\begin{aligned} q(u,u) = q(fu,fu) &\preceq a_1 q(u,u) + a_2 q(u,fu) + a_3 q(u,fu) \\ &= (a_1+a_2+a_3)\, q(u,u). \end{aligned}$$

Since $a_1+a_2+a_3<1$, by Proposition 2.10, it follows that $q(u,u)=\theta$.

Theorem 3.7. *Let $(X,\sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space. Let q be a generalized c-distance of order j on X and $f: X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following conditions hold:*
(i) there exist $a_1,\ a_2,\ a_3 \geq 0$ with $a_1+a_2+a_3<1$ such that

$$q(fx,fy) \preceq a_1 q(x,y) + a_2 q(x,fx) + a_3 q(y,fy) \tag{3.8}$$

for all $x,y\in X$ with $x \sqsubseteq y$;
(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;
(iii) $inf\,\{q(x,y)+q(fx,y)+q(x,fx): x\in X\} \succ \theta$ for all $y\in X$ with $y\neq fy$. Then f has a fixed point in X. Moreover, if $u=fu$, then $q(u,u)=\theta$.

Proof. If we take $x_n = f^n x_0 = f x_{n-1}$, then as in the proof of Theorem 3.6 we have

$$x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq x_{n+1} \sqsubseteq \cdots.$$

Moreover,

$$q(x_n,x_{n+1}) \preceq r^n q(x_0,x_1) \tag{3.9}$$

where $r = \frac{a_1+a_2}{1-a_3} \in [0,1)$.

By an argument similar to that used in Theorem 3.6, for $m,n\in\mathbb{N}$ with $m>n$ we have

$$q(x_n,x_m) \preceq b_n M \tag{3.10}$$

where $b_n = \frac{a_1^n}{1-a_1^j}\left(\frac{1}{1-r}+\beta_j\right)+\beta_n$, $\beta_n = (a_2+a_3)[r^{n-1}+a_1 r^{n-2}+\cdots+a_1^{n-1}]$ and $M = q(x_0,x_1)+q(x_0,x_2)+\cdots+q(x_0,x_j)\in P$.

By using Lemma 3.5(iii), we conclude that (x_n) is a Cauchy sequence in X. Since X is complete,

there exists an element $u \in X$ such that $x_n \to u$ as $n \to \infty$.
By (3.10) and $(q3)$, we have

$$q(x_n, u) \preceq b_n M, \ for \ all \ n. \tag{3.11}$$

If $u \neq fu$, then by hypothesis (iii), (3.9) and (3.11), we have

$$\begin{aligned}
\theta &\prec inf\{q(x,u) + q(fx,u) + q(x,fx) : x \in X\} \\
&\preceq inf\{q(x_n,u) + q(fx_n,u) + q(x_n,fx_n) : n \in \mathbb{N}\} \\
&= inf\{q(x_n,u) + q(x_{n+1},u) + q(x_n,x_{n+1}) : n \in \mathbb{N}\} \\
&\preceq inf\{b_n M + b_{n+1} M + r^n q(x_0,x_1) : n \in \mathbb{N}\} \\
&= \theta.
\end{aligned}$$

This is a contradiction. Therefore, u is a fixed point of f. We can prove $q(u,u) = \theta$ by the final part of the proof of Theorem 3.6.

In the following theorem we omit the continuity assumption of f.

Theorem 3.8. *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that (X,d) is a complete cone metric space. Let q be a generalized c-distance of order j on X and $f : X \to X$ be a nondecreasing mapping with respect to $\sqsubseteq$. Suppose that the following conditions hold:*
(i) there exist $a_1,\ a_2 \geq 0$ with $a_1 + a_2 < 1$ such that

$$q(fx, fy) \preceq a_1 q(x,y) + a_2 q(x, fx) \tag{3.12}$$

for all $x, y \in X$ with $x \sqsubseteq y$;
(ii) there exists $x_0 \in X$ such that $x_0 \sqsubseteq fx_0$;
(iii) if (x_n) is a nondecreasing sequence in X such that $x_n \to x$, then $x_n \sqsubseteq x$ for all n.
Then f has a fixed point in X. Moreover, if $u = fu$, then $q(u,u) = \theta$.

Proof. As in the proof of Theorem 3.6 we construct a nondecreasing sequence (x_n) where $x_n = f^n x_0 = f x_{n-1}$.
Moreover,

$$q(x_n, x_{n+1}) \preceq r^n q(x_0, x_1) \tag{3.13}$$

where $r = a_1 + a_2 \in [0,1)$.

By an argument similar to that used in Theorem 3.6, for $m,n \in \mathbb{N}$ with $m > n$ we have

$$q(x_n, x_m) \preceq b_n M \tag{3.14}$$

where $b_n = \frac{a_1^n}{1-a_1^j}\left(\frac{1}{1-r} + \beta_j\right) + \beta_n$, $\beta_n = a_2[r^{n-1} + a_1 r^{n-2} + \cdots + a_1^{n-1}]$ and $M = q(x_0,x_1) + q(x_0,x_2) + \cdots + q(x_0,x_j) \in P$.

By Lemma 3.5(iii), (x_n) is a Cauchy sequence in X. Since X is complete, there exists an element $u \in X$ such that $x_n \to u$ as $n \to \infty$.

By (3.14) and $(q3)$, we have

$$q(x_n, u) \preceq b_n M, \; for\ all\ n. \tag{3.15}$$

Since (x_n) is nondecreasing and converges to u, by the given condition (iii), we have $x_n \sqsubseteq u$ for all n.

Thus for all $n \in \mathbb{N}$, we have by using (3.13) and (3.15)

$$\begin{aligned} q(x_n, fu) = q(fx_{n-1}, fu) &\preceq a_1 q(x_{n-1}, u) + a_2 q(x_{n-1}, fx_{n-1}) \\ &= a_1 q(x_{n-1}, u) + a_2 q(x_{n-1}, x_n) \\ &\preceq a_1 b_{n-1} M + a_2 r^{n-1} q(x_0, x_1) \\ &\preceq \alpha_n M, \end{aligned} \tag{3.16}$$

where $\alpha_n = a_1 b_{n-1} + a_2 r^{n-1} \to 0$ as $n \to \infty$. By using Lemma 3.5(ii), it follows from (3.15) and (3.16) that $fu = u$. Hence u is a fixed point of f. We can prove $q(u,u) = \theta$ by the argument similar to that used in Theorem 3.6.

Theorem 3.9. *In addition to hypothesis of Theorem 3.6 or Theorem 3.7 or Theorem 3.8, suppose that any two elements of X are comparable. Then there exists a unique fixed point of f.*

Proof. We first note that the set of fixed points of f is nonempty. We will show that if u and v are fixed points of f, then $u = v$. Since the elements of X are comparable, we may assume that

$u \sqsubseteq v$. In case of either Theorem 3.6 or Theorem 3.7, we have

$$\begin{aligned} q(u,v) = q(fu,fv) &\preceq a_1 q(u,v) + a_2 q(u,fu) + a_3 q(v,fv) \\ &= a_1 q(u,v) + a_2 q(u,u) + a_3 q(v,v) \\ &= a_1 q(u,v), \end{aligned}$$

since $q(u,u) = \theta$ and $q(v,v) = \theta$.

This gives that, $q(u,v) = \theta$. By Lemma 3.5(ii), $q(u,u) = \theta$ and $q(u,v) = \theta$ imply that $u = v$.

In case of Theorem 3.8, we can obtain the same conclusion by taking $a_3 = 0$ in above.

Conflict of Interests

The author declares that there is no conflict of interests.

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