Available online at http://scik.org
Adv. Fixed Point Theory, 4 (2014), No. 4, 491-508
ISSN: 1927-6303

FIXED POINT THEOREMS FOR SET-VALUED QUASI-CONTRACTION MAPS IN A G-METRIC SPACE

MATTHEW BRIJESH SOOKOO, SREEDHARA RAO GUNAKALA*

Department of Mathematics and Statistics, University of the West Indies, St. Augustine, Trinidad and Tobago

Abstract. In this paper, we achieve a fixed point theorem for G-metric set-valued quasi-contraction maps in a G-metric space. The result was obtained using a similar approach to that used by Amini-Harandi [1] and it extends the set-valued fixed point theory from metric spaces to G-metric spaces.

Keywords: fixed point theorem; quasi-contraction maps; set-valued maps; G-metric set-valued quasi-contractions.

2010 AMS Subject Classification: 47H10.

1. Introduction

A set-valued mapping T from a set X to another set Y is a rule that associates one or more elements of Y with every element of X. If T is a function and D_T is the domain of T then a fixed point or an invariant point of the function T is an element $x \in D_T$ that is mapped to itself. That is $T(x) = x$. A fixed point theorem is a result giving the conditions for which the function T will have at least one fixed point.

*Corresponding author

E-mail address: sreedhara.rao@sta.uwi.edu

Received October 11, 2013

The D-metric space was introduced in 1992 by Dhage [2] as an attempt to generalize the existing metric space results. In 2003 Mustafa and Sims [3] exposed some imperfections in the topological properties of the D-metric space, annulling the validity of the majority of results that were obtained in those spaces. In 2006, Mustafa and Sims attempted to address the D-metric space deficiencies by introducing a new structure of generalized metric spaces called G-metric spaces [4].

2. Preliminaries

The main aim of this section is to state some basic definitions and results that are essential for general knowledge and serves as a convenient means of reference material for subsequent use.

Definition 2.1. [2] Let X be a non-empty set and let $D : X \times X \times X \to [0,\infty)$ be a function satisfying the following conditions, for all $a,x,y,z \in X$

(i) $D(x,y,z) \geq 0$;

(ii) $D(x,y,z) = 0$ if and only if $x = y = z$;

(iii) $D(x,y,z) = D(x,z,y) = D(y,x,z) = D(y,z,x) = D(z,x,y) = D(z,y,x)$;

(iv) $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$.

Then D is called a D-metric on X and the pair (X,D) is called a D-metric space.

Definition 2.2. [4] Let X be a non-empty set, and let $G : X \times X \times X \to [0,\infty)$ be a function satisfying the following axioms,

(i) $G(x,y,z) = 0$ if and only if $x = y = z$;

(ii) $G(x,x,y) > 0$ for all $x,y \in X$ with $x \neq y$;

(iii) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;

(iv) $G(x,y,z) = G(x,z,y) = G(y,x,z) = G(y,z,x) = G(z,x,y) = G(z,y,x)$;

(v) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X,G) is called a generalized metric space or a G-metric space.

Definition 2.3. [4] Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. Then $\{x_n\}$ is G-convergent to x if,

$$\lim_{m,n\to\infty} G(x,x_n,x_m) = 0.$$

That is, for any $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$, for all $n,m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or,

$$\lim_{n\to\infty} x_n = x.$$

The following lemma follows directly from Definition 2.3.

Lemma 2.1. [4] *Let (X,G) be a G-metric space. Then the following are equivalent*

(i) $\{x_n\}$ *is G-convergent to x;*

(ii) $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$*;*

(iii) $G(x,x,x_n) \to 0$ *as* $n \to \infty$*;*

(iv) $G(x_m,x_n,x) \to 0$ *as* $m,n \to \infty$*.*

We now proceed to define a Cauchy sequence in a G-metric space.

Definition 2.4. [4] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if for each $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $G(x_m,x_n,x_p) < \varepsilon$, for all $m,n,p \geq n_0$. That is $G(x_m,x_n,x_p) \to 0$ as $m,n,p \to \infty$.

The following Lemma is a consequence of Definition 2.4.

Lemma 2.2. [4] *Let (X,G) be a G-metric space. Then $\{x_n\}$ is called G-Cauchy if and only if for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $n,m \geq N$.*

Definition 2.5. Let (X,d) be a metric space. The family of all non-empty closed and bounded subsets of X is denoted by $CB(X)$.

Definition 2.6 [1] Let (X,d) be a metric space. The set-valued map $T : X \to CB(X)$ is said to be a q-set-valued quasi-contraction if,

$$d_H(Tx,Ty) \leq q.\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$$

for any $x,y \in X$ where $0 \le q < 1$ and d_H denotes the Hausdorff metric on $CB(X)$ induced by d. That is for all $A,B \in CB(X)$,

$$d_H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}.$$

The following is a fixed point theorem for the set-valued quasi-contraction maps in metric spaces.

Theorem 2.1 [1] *Let (X,d) be a complete metric space. Let $T : X \to CB(X)$ be a q-set-valued quasi-contraction with $q < \frac{1}{2}$. Then T has a fixed point.*

3. Main results

In this section we introduce the concept of G-metric set-valued quasi-contractions in G-metric spaces and present our main result which extends Theorem 2.1. to G-metric spaces.

Definition 3.1. Let (X,G) be a G-metric space. The family of all non-empty closed and bounded subsets of X is denoted by $CB_G(X)$.

Definition 3.2. Let (X,d) be a metric space, (X,G) be a G-metric space and $CB_G(X)$ be the family of all non-empty closed and bounded subsets of X in a G-metric space.

(a) The distance between any point $x \in X$ and any two non-empty subsets $A,B \in CB_G(X)$ is denoted by $G(x,A,B)$ and is defined by,

$$G(x,A,B) = d(x,A) + d(x,B) + d(A,B),$$

where, $d(x,A) = inf\{d(x,y) : y \in A\}$, $d(x,B) = inf\{d(x,y) : y \in B\}$ and $d(A,B) = inf\{d(a,b) : a \in A, b \in B\}$.

(b) Let $A,B,C \in CB_G(X)$. The Hausdorff G-metric or Hausdorff G-metric distance is denoted by $G_H(A,B,C)$ and is defined by,

$$G_H(A,B,C) = \max\left\{\sup_{x\in A} G(x,B,C), \sup_{x\in B} G(x,C,A), \sup_{x\in C} G(x,A,B)\right\}.$$

Definition 3.3. Let (X,G) be a G-metric space, the set-valued map $T : X \to CB_G(X)$ is said to be a G-metric q-set-valued quasi-contraction if for any $x,y,z \in X$,

$$G_H(Tx,Ty,Tz) \leq q.\max\left\{\begin{array}{ccc} G(x,y,z), & G(x,Tx,Ty), & G(x,Ty,Tz), \\ G(x,Tx,Tz), & G(y,Tx,Ty), & G(y,Ty,Tz), \\ G(y,Tx,Tz), & G(z,Tx,Ty), & G(z,Ty,Tz), \\ & G(z,Tx,Tz) & \end{array}\right\},$$

where $0 \leq q < 1$ and G_H denotes the Hausdorff metric on $CB_G(X)$ induced by G. That is, for all $A,B,C \in CB_G(X)$,

$$G_H(A,B,C) = \max\left\{\sup_{x\in A} G(x,B,C), \sup_{x\in B} G(x,C,A), \sup_{x\in C} G(x,A,B)\right\}.$$

The following is our main result in G-metric spaces, it is a fixed point theorem for G-metric set-valued quasi-contraction mappings.

Theorem 3.1. *Let (X,G) be a complete G-metric space. Suppose that $T : X \to CB_G(X)$ is a G-metric q-set-valued quasi-contraction with $q < \frac{1}{2}$. Then T has a fixed point. That is there exist $u \in X$ such that $u = Tu (u \in Tu)$.*

Proof. We first observe that for each $A,B,C \in CB_G(X)$, $a \in A$ and $\alpha > 0$ with $G_H(A,B,C) < \alpha$, there exist $b \in B$ and $c \in C$ such that $G(a,b,c) < \alpha$. Now let $r > 0$ be such that $q < r < \frac{1}{2}$. Then by Definition 3.3, we find that

$$G_H(Tx,Ty,Tz) < r.\max\left\{\begin{array}{ccc} G(x,y,z), & G(x,Tx,Ty), & G(x,Ty,Tz), \\ G(x,Tx,Tz), & G(y,Tx,Ty), & G(y,Ty,Tz), \\ G(y,Tx,Tz), & G(z,Tx,Ty), & G(z,Ty,Tz), \\ & G(z,Tx,Tz) & \end{array}\right\}.$$

If we replace x,y and z by x_0,x_1 and x_2 respectively, then we get

$$G_H(Tx_0,Tx_1,Tx_2) < r.\max\left\{\begin{array}{ccc} G(x_0,x_1,x_2), & G(x_0,Tx_0,Tx_1), & G(x_0,Tx_1,Tx_2), \\ G(x_0,Tx_0,Tx_2), & G(x_1,Tx_0,Tx_1), & G(x_1,Tx_1,Tx_2), \\ G(x_1,Tx_0,Tx_2), & G(x_2,Tx_0,Tx_1), & G(x_2,Tx_1,Tx_2), \\ & G(x_2,Tx_0,Tx_2) & \end{array}\right\}.$$

But by observation $G_H(A,B,C) < \alpha$, $a \in A$ and $\alpha > 0$. This implies that there exist $b \in B$ and $c \in C$ such that $G(a,b,c) < \alpha$. Setting $x_1 \in Tx_0$, $x_2 \in Tx_1$ and $x_3 \in Tx_2$, we get

$$G(x_1,x_2,x_3) < r.\max \left\{ \begin{array}{ccc} G(x_0,x_1,x_2), & G(x_0,Tx_0,Tx_1), & G(x_0,Tx_1,Tx_2), \\ G(x_0,Tx_0,Tx_2), & G(x_1,Tx_0,Tx_1), & G(x_1,Tx_1,Tx_2), \\ G(x_1,Tx_0,Tx_2), & G(x_2,Tx_0,Tx_1), & G(x_2,Tx_1,Tx_2), \\ & G(x_2,Tx_0,Tx_2) & \end{array} \right\}.$$

Continuing in this manner, by induction, we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, which implies,

$$G(x_n,x_{n+1},x_{n+2}) < r.\max \left\{ \begin{array}{cc} G(x_{n-1},x_n,x_{n+1}), & G(x_{n-1},Tx_{n-1},Tx_n), \\ G(x_{n-1},Tx_n,Tx_{n+1}), & G(x_{n-1},Tx_{n-1},Tx_{n+1}), \\ G(x_n,Tx_{n-1},Tx_n), & G(x_n,Tx_n,Tx_{n+1}), \\ G(x_n,Tx_{n-1},Tx_{n+1}), & G(x_{n+1},Tx_{n-1},Tx_n), \\ G(x_{n+1},Tx_n,Tx_{n+1}), & G(x_{n+1},Tx_{n-1},Tx_{n+1}) \end{array} \right\}.$$

We have several cases.

Case I: If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$ then $x_n = x_{n+1} \in Tx_n$. That is x_n is a fixed point of T and the proof is completed.

Case II: If $x_{n+1} = x_{n+2}$ for some $n \in \mathbb{N}$, then $x_{n+1} = x_{n+2} \in Tx_{n+1}$. That is x_{n+1} is a fixed point of T and the proof is completed.

Case III: $x_n \neq x_{n+1} \neq x_{n+2}$ for each $n \in \mathbb{N}$. Now $x_n \in Tx_{n-1}$, $x_{n+1} \in Tx_n$ and $x_{n+2} \in Tx_{n+1}$.

Therefore,

$$G(x_n,x_{n+1},x_{n+2}) < r.\max\left\{\begin{array}{cc} G(x_{n-1},x_n,x_{n+1}), & G(x_{n-1},Tx_{n-1},Tx_n), \\ G(x_{n-1},Tx_n,Tx_{n+1}), & G(x_{n-1},Tx_{n-1},Tx_{n+1}), \\ G(x_n,Tx_{n-1},Tx_n), & G(x_n,Tx_n,Tx_{n+1}), \\ G(x_n,Tx_{n-1},Tx_{n+1}), & G(x_{n+1},Tx_{n-1},Tx_n), \\ G(x_{n+1},Tx_n,Tx_{n+1}), & G(x_{n+1},Tx_{n-1},Tx_{n+1}) \end{array}\right\}$$

$$\leq r.\max\left\{\begin{array}{cc} G(x_{n-1},x_n,x_{n+1}), & G(x_{n-1},x_n,x_{n+1}), \\ G(x_{n-1},x_{n+1},x_{n+2}), & G(x_{n-1},x_n,x_{n+2}), \\ G(x_n,x_n,x_{n+1}), & G(x_n,x_{n+1},x_{n+2}), \\ G(x_n,x_n,x_{n+2}), & G(x_{n+1},x_n,x_{n+1}), \\ G(x_{n+1},x_{n+1},x_{n+2}), & G(x_{n+1},x_n,x_{n+2}) \end{array}\right\}$$

$$= r.\max\left\{\begin{array}{c} G(x_{n-1},x_n,x_{n+1}), \\ G(x_{n-1},x_n,x_{n+2}), \\ G(x_{n-1},x_{n+1},x_{n+2}) \end{array}\right\}.$$

Thus, we have

$$G(x_n,x_{n+1},x_{n+2}) < r.\max\left\{\begin{array}{c} G(x_{n-1},x_n,x_{n+1}), \\ G(x_{n-1},x_n,x_{n+2}), \\ G(x_{n-1},x_{n+1},x_{n+2}) \end{array}\right\}. \tag{3.1}$$

Note that we can modify Equation (3.1) by replacing n by $n-2$ to get that

$$G(x_n,x_{n-1},x_{n-2}) < r.\max\left\{\begin{array}{c} G(x_{n-3},x_{n-2},x_{n-1}), \\ G(x_{n-3},x_{n-2},x_n), \\ G(x_{n-3},x_{n-1},x_n) \end{array}\right\}. \tag{3.2}$$

From Equation (3.2), we have three choices

$$G(x_n,x_{n-1},x_{n-2}) \leq rG(x_{n-3},x_{n-2},x_{n-1}),$$

$$G(x_n,x_{n-1},x_{n-2}) \leq rG(x_{n-3},x_{n-2},x_n),$$

$$G(x_n,x_{n-1},x_{n-2}) \leq rG(x_{n-3},x_{n-1},x_n).$$

Now we show by induction that for each $n \geq 3$ there exist $1 \leq i < j \leq n$, where $j \in \{i+1, i+2\}$ such that, $\left(\text{note that } r \leq \frac{r}{1-r}\right)$

$$G(x_n, x_{n-1}, x_{n-2}) \leq \left(\frac{r}{1-r}\right)^{n-2} G(x_0, x_i, x_j). \tag{3.3}$$

For $n = 3$, the three choices become

$$G(x_1, x_2, x_3) \leq rG(x_0, x_1, x_2),$$

$$G(x_1, x_2, x_3) \leq rG(x_0, x_1, x_3),$$

$$G(x_1, x_2, x_3) \leq rG(x_0, x_2, x_3).$$

That is, $G(x_3, x_2, x_1) \leq \frac{r}{1-r} G(x_0, x_i, x_j) = \left(\frac{r}{1-r}\right)^{3-2} G(x_0, x_i, x_j)$ for some $1 \leq i < j \leq 3$, $j \in \{i+1, i+2\}$. Thus Equation (3.3) holds for $n = 3$. $\big($Note that for $n = 4$ we can modify Equation (3.1) by replacing n by $n-3$ and for $n = 5$ we can modify Equation (3.1) by replacing n by $n-4$ and so on$\big)$.

Next, we assume that Equation (3.3) holds for all values less than n, we will show that it holds for n. Now Equation (3.3) trivially holds if $G(x_n, x_{n-1}, x_{n-2}) \leq rG(x_{n-3}, x_{n-2}, x_{n-1})$. Therefore we consider the choices when

$$G(x_n, x_{n-1}, x_{n-2}) \leq rG(x_{n-3}, x_{n-2}, x_n) \text{ and } G(x_n, x_{n-1}, x_{n-2}) \leq rG(x_{n-3}, x_{n-1}, x_n).$$

First, we suppose that

$$G(x_n, x_{n-1}, x_{n-2}) \leq rG(x_{n-3}, x_{n-2}, x_n). \tag{3.4}$$

Then by Definition 2.2 and our assumption, we have

$$\begin{aligned}
G(x_{n-3}, x_{n-2}, x_n) &= G(x_n, x_{n-2}, x_{n-3}) \\
&\leq G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-2}, x_{n-3}) \\
&\leq G(x_n, x_{n-1}, x_{n-2}) + G(x_{n-1}, x_{n-2}, x_{n-3}) \\
&\leq rG(x_{n-3}, x_{n-2}, x_n) + \left(\frac{r}{1-r}\right)^{n-3} G(x_0, x_i, x_j).
\end{aligned}$$

Therefore, we have

$$(1-r)G(x_{n-3}, x_{n-2}, x_n) \leq \left(\frac{r}{1-r}\right)^{n-3} G(x_0, x_i, x_j)$$

This implies that

$$G(x_{n-3},x_{n-2},x_n) \le \frac{1}{1-r}\left(\frac{r}{1-r}\right)^{n-3} G(x_0,x_i,x_j)$$

Substituting in Equation (3.4), we find that

$$G(x_n,x_{n-1},x_{n-2}) \le \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j).$$

This implies that Equation (3.3) holds. Second, we suppose

$$G(x_n,x_{n-1},x_{n-2}) \le rG(x_{n-3},x_{n-1},x_n).$$

Then by Definition 2.2 and our assumption, we have

$$\begin{aligned} G(x_{n-3},x_{n-1},x_n) &= G(x_n,x_{n-1},x_{n-3}) \\ &\le G(x_n,x_{n-2},x_{n-2}) + G(x_{n-2},x_{n-1},x_{n-3}) \\ &\le G(x_n,x_{n-1},x_{n-2}) + G(x_{n-1},x_{n-2},x_{n-3}) \\ &\le rG(x_{n-3},x_{n-1},x_n) + \left(\frac{r}{1-r}\right)^{n-3} G(x_0,x_i,x_j). \end{aligned}$$

Therefore, we have

$$(1-r)G(x_{n-3},x_{n-1},x_n) \le \left(\frac{r}{1-r}\right)^{n-3} G(x_0,x_i,x_j).$$

This implies that $G(x_{n-3},x_{n-1},x_n) \le \frac{1}{1-r}\left(\frac{r}{1-r}\right)^{n-3} G(x_0,x_i,x_j)$. Hence, we have

$$G(x_n,x_{n-1},x_{n-2}) \le \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j).$$

This implies that Equation (3.3) holds. We proceed to show that T has a fixed point. Firstly we show that the sequence $\{x_n\}$ is bounded, then we show $\{x_n\}$ is Cauchy. To show that $\{x_n\}$ is bounded, put $\delta_1 = G(x_0,x_1,Tx_1)$. Now from Equation (3.1), we have three choices. Either

$$G(x_1,x_2,x_3) \le rG(x_0,x_1,x_2),$$

$$G(x_1,x_2,x_3) \le rG(x_0,x_1,x_3) \text{ or,}$$

$$G(x_1,x_2,x_3) \le rG(x_0,x_2,x_3).$$

Suppose that $G(x_1,x_2,x_3) \leq rG(x_0,x_1,x_2)$. Note that

$$\begin{aligned} G(x_0,x_2,x_3) &= G(x_3,x_2,x_0), \\ &\leq G(x_3,x_0,x_0) + G(x_0,x_2,x_0), \\ &\leq G(x_3,x_1,x_2) + G(x_0,x_1,x_2), \\ &\leq rG(x_0,x_1,x_2) + G(x_0,x_1,x_2). \end{aligned}$$

That is,

$$G(x_0,x_2,x_3) \leq (1+r)G(x_0,x_1,x_2) \leq \left(1+\frac{r}{1-r}\right)G(x_0,x_1,x_2).$$

Considering the second choice $G(x_1,x_2,x_3) \leq rG(x_0,x_1,x_3)$ Note that

$$\begin{aligned} G(x_0,x_2,x_3) &= G(x_3,x_2,x_0), \\ &\leq G(x_3,x_0,x_0) + G(x_0,x_2,x_0), \\ &\leq G(x_3,x_1,x_2) + G(x_0,x_1,x_2), \\ &\leq rG(x_0,x_1,x_3) + G(x_0,x_1,x_2), \\ &\leq rG(x_0,x_3,x_3) + rG(x_3,x_1,x_3) + G(x_0,x_1,x_2), \\ &\leq rG(x_0,x_1,x_2) + rG(x_0,x_1,x_2) + G(x_0,x_1,x_2). \end{aligned}$$

That is,

$$G(x_0,x_2,x_3) \leq (1+2r)G(x_0,x_1,x_2) \leq \left(1+2\left(\frac{r}{1-r}\right)\right)G(x_0,x_1,x_2).$$

Finally the third choice $G(x_1,x_2,x_3) \leq rG(x_0,x_2,x_3)$ Now,

$$\begin{aligned} G(x_0,x_2,x_3) &= G(x_3,x_2,x_0) \\ &\leq G(x_3,x_0,x_0) + G(x_0,x_2,x_0) \\ &\leq G(x_3,x_1,x_2) + G(x_0,x_1,x_2) \\ &\leq rG(x_0,x_2,x_3) + G(x_0,x_1,x_2) \\ &\leq \left(\frac{r}{1-r}\right)G(x_0,x_2,x_3) + G(x_0,x_1,x_2) \end{aligned}$$

This implies that $\left(1-\frac{r}{1-r}\right)G(x_0,x_2,x_3) \le G(x_0,x_1,x_2)$. Therefore, we have

$$G(x_0,x_2,x_3) \le \left(\frac{1}{1-\frac{r}{1-r}}\right)G(x_0,x_1,x_2).$$

Put

$$\delta_2 = \left(\frac{1+2\left(\frac{r}{1-r}\right)}{1-\frac{r}{1-r}}\right)G(x_0,x_1,x_2).$$

Note that for the three cases $\delta_1 \le \delta_2$ and $G(x_0,x_2,x_3) \le \delta_2$. Let us define the nondecreasing sequence $\{\delta_n\}$ such that

$$\max\left\{G(x_0,x_i,x_j)\right\} \le \delta_n$$

for $1 \le i < j \le n$, where $j \in \{i+1,i+2\}$. Now

$$\begin{aligned}
G(x_0,x_{n-1},x_n) &= G(x_n,x_{n-1},x_0),\\
&\le G(x_n,x_{n-2},x_{n-2}) + G(x_{n-2},x_{n-1},x_0),\\
&\le G(x_n,x_{n-1},x_{n-2}) + G(x_0,x_{n-2},x_{n-1}),\\
&\le \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j) + G(x_0,x_{n-2},x_{n-1}).
\end{aligned}$$

That is,

$$G(x_0,x_{n-1},x_n) \le \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j) + \delta_{n-1}. \tag{3.5}$$

Also since $j \in \{i+1,i+2\}$, we have $G(x_0,x_i,x_j) \le G(x_0,x_i,x_{i+1}) + G(x_0,x_i,x_{i+2})$, for $i < j < n$,

$$\begin{aligned}
\left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j) &\le \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_{i+1}) + \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_{i+2})\\
&\le 2\left(\frac{r}{1-r}\right)^{n-2}\delta_{n-1}.
\end{aligned}$$

Therefore Equation (3.5) becomes

$$G(x_0,x_{n-1},x_n) \le \left[1+2\left(\frac{r}{1-r}\right)^{n-2}\right]\delta_{n-1}.$$

For $i < j = n$, we have

$$\left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j) \le \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_{n-1},x_n) + \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_{n-2},x_n).$$

Therefore Equation (3.5) becomes

$$\left[1-\left(\frac{r}{1-r}\right)^{n-2}\right]G(x_0,x_{n-1},x_n) \leq \left[1+\left(\frac{r}{1-r}\right)^{n-2}\right]\delta_{n-1}$$
$$\leq \left[1+2\left(\frac{r}{1-r}\right)^{n-2}\right]\delta_{n-1}.$$

That is,

$$G(x_0,x_{n-1},x_n) \leq \left[\frac{1+2\left(\frac{r}{1-r}\right)^{n-2}}{1-\left(\frac{r}{1-r}\right)^{n-2}}\right]\delta_{n-1}$$

Let

$$\delta_n = \left[\frac{1+2\left(\frac{r}{1-r}\right)^{n-2}}{1-\left(\frac{r}{1-r}\right)^{n-2}}\right]\delta_{n-1}.$$

Note that $\delta_{n-1} \leq \delta_n$ and $G(x_0,x_{n-1},x_n) \leq \delta_n$. The sequence $\{x_n\}$ is bounded if and only if

$$\delta = \lim_{n\to\infty}\delta_n = \frac{\prod_{n=1}^{\infty}\left[1+2\left(\frac{r}{1-r}\right)^{n-2}\right]}{\prod_{n=1}^{\infty}\left[1-\left(\frac{r}{1-r}\right)^{n-2}\right]} < \infty.$$

Now the series,

$$\sum_{n=1}^{\infty}\left[1+2\left(\frac{r}{1-r}\right)^{n-2}\right] \text{ and } \sum_{n=1}^{\infty}\left[1-\left(\frac{r}{1-r}\right)^{n-2}\right]$$

are convergent since $\frac{r}{1-r} < 1$. Therefore,

$$\prod_{n=1}^{\infty}\left[1+2\left(\frac{r}{1-r}\right)^{n-2}\right] < \infty, \prod_{n=1}^{\infty}\left[1-\left(\frac{r}{1-r}\right)^{n-2}\right] < \infty \text{ and } \prod_{n=1}^{\infty}\left[1-\left(\frac{r}{1-r}\right)^{n-2}\right] > 0.$$

Hence, $\delta < \infty$.

Case IV: If $x_n = x_{n+2}$ for each $n \in \mathbb{N}$, we proceed as we did in case III.

$$G(x_n,x_{n+1},x_{n+2}) < r.\max\left\{\begin{matrix} G(x_{n-1},x_n,x_{n+1}), & G(x_{n-1},Tx_{n-1},Tx_n), \\ G(x_{n-1},Tx_n,Tx_{n+1}), & G(x_{n-1},Tx_{n-1},Tx_{n+1}), \\ G(x_n,Tx_{n-1},Tx_n), & G(x_n,Tx_n,Tx_{n+1}), \\ G(x_n,Tx_{n-1},Tx_{n+1}), & G(x_{n+1},Tx_{n-1},Tx_n), \\ G(x_{n+1},Tx_n,Tx_{n+1}), & G(x_{n+1},Tx_{n-1},Tx_{n+1}) \end{matrix}\right\}$$

$$\leq r.\max\left\{\begin{matrix} G(x_{n-1},x_n,x_{n+1}), & G(x_{n-1},x_n,x_{n+1}), \\ G(x_{n-1},x_{n+1},x_{n+2}), & G(x_{n-1},x_n,x_{n+2}), \\ G(x_n,x_n,x_{n+1}), & G(x_n,x_{n+1},x_{n+2}), \\ G(x_n,x_n,x_{n+2}), & G(x_{n+1},x_n,x_{n+1}), \\ G(x_{n+1},x_{n+1},x_{n+2}), & G(x_{n+1},x_n,x_{n+2}) \end{matrix}\right\}$$

$$= r.\max\left\{\begin{matrix} G(x_{n-1},x_{n+2},x_{n+1}), & G(x_{n-1},x_{n+2},x_{n+1}), \\ G(x_{n-1},x_{n+1},x_{n+2}), & G(x_{n-1},x_{n+2},x_{n+2}), \\ G(x_{n+2},x_{n+2},x_{n+1}), & G(x_{n+2},x_{n+1},x_{n+2}), \\ G(x_{n+2},x_{n+2},x_{n+2}), & G(x_{n+1},x_{n+2},x_{n+1}), \\ G(x_{n+1},x_{n+1},x_{n+2}), & G(x_{n+1},x_{n+2},x_{n+2}) \end{matrix}\right\}$$

$$= r.G(x_{n-1},x_{n+2},x_{n+1}).$$

Thus, we have

$$G(x_n,x_{n+1},x_{n+2}) < rG(x_{n-1},x_{n+2},x_{n+1}). \tag{3.6}$$

Note that we can modify Equation (3.6) by replacing n by $n-2$ to get

$$G(x_n,x_{n-1},x_{n-2}) < rG(x_{n-3},x_n,x_{n-1}) \tag{3.7}$$

Now we show by induction that for each $n \geq 3$ there exist $1 \leq i < j \leq n$, where $j \in \{i+1,i+2\}$ such that, (note that $r \leq \frac{r}{1-r}$)

$$G(x_n,x_{n-1},x_{n-2}) < \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j). \tag{3.8}$$

For $n=3$, we have

$$G(x_1,x_2,x_3) < rG(x_0,x_3,x_2) \leq \frac{r}{1-r}G(x_0,x_2,x_3) = \left(\frac{r}{1-r}\right)^{3-2} G(x_0,x_i,x_j).$$

for $i=2, j=3$. Therefore Equation (3.8) holds for $n=3$. Next, we assume that Equation (3.8) holds for all values less than n. We show that it holds for n. By our assumption

$$\begin{aligned} G(x_{n-3},x_n,x_{n-1}) &= G(x_n,x_{n-1},x_{n-3}) \\ &\leq G(x_n,x_{n-2},x_{n-2}) + G(x_{n-2},x_{n-1},x_{n-3}) \\ &\leq G(x_n,x_{n-1},x_{n-2}) + G(x_{n-1},x_{n-2},x_{n-3}) \\ &< rG(x_{n-3},x_n,x_{n-1}) + \left(\frac{r}{1-r}\right)^{n-3} G(x_0,x_i,x_j). \end{aligned}$$

Therefore, we have

$$(1-r)G(x_{n-3},x_n,x_{n-1}) < \left(\frac{r}{1-r}\right)^{n-3} G(x_0,x_i,x_j).$$

This implies that $G(x_{n-3},x_n,x_{n-1}) < \frac{1}{1-r}\left(\frac{r}{1-r}\right)^{n-3} G(x_0,x_i,x_j)$. Substitute in Equation (3.7) to get,

$$G(x_n,x_{n-1},x_{n-2}) < \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j).$$

This implies that Equation (3.8) holds. To show that $\{x_n\}$ is bounded, put $\delta_1 = G(x_0,x_1,Tx_1)$. From Equation (3.6), we have

$$G(x_1,x_2,x_3) < rG(x_0,x_2,x_3).$$

Now

$$\begin{aligned} G(x_0,x_2,x_3) &= G(x_3,x_2,x_0), \\ &\leq G(x_3,x_0,x_0) + G(x_0,x_2,x_0), \\ &\leq G(x_3,x_1,x_2) + G(x_0,x_1,x_2), \\ &< rG(x_0,x_2,x_3) + G(x_0,x_1,x_2), \\ &< \left(\frac{r}{1-r}\right) G(x_0,x_2,x_3) + G(x_0,x_1,x_2). \end{aligned}$$

This implies that $\left(1-\frac{r}{1-r}\right)G(x_0,x_2,x_3) < G(x_0,x_1,x_2)$. Therefore, we have $G(x_0,x_2,x_3) < \left(\frac{1}{1-\frac{r}{1-r}}\right)G(x_0,x_1,x_2)$. Put

$$\delta_2 = \left(\frac{1+2\left(\frac{r}{1-r}\right)}{1-\frac{r}{1-r}}\right)G(x_0,x_1,x_2).$$

Note that $\delta_1 < \delta_2$ and $G(x_0,x_2,x_3) \leq \delta_2$. Let us define the nondecreasing sequence $\{\delta_n\}$ such that $\max\left\{G(x_0,x_i,x_j)\right\} \leq \delta_n$ for $1 \leq i < j \leq n$, where $j \in \{i+1, i+2\}$. Now

$$\begin{aligned} G(x_0,x_{n-1},x_n) &= G(x_n,x_{n-1},x_0), \\ &\leq G(x_n,x_{n-2},x_{n-2}) + G(x_{n-2},x_{n-1},x_0), \\ &\leq G(x_n,x_{n-1},x_{n-2}) + G(x_0,x_{n-2},x_{n-1}), \\ &< \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j) + G(x_0,x_{n-2},x_{n-1}). \end{aligned}$$

That is,

$$G(x_0,x_{n-1},x_n) < \delta_{n-1} + \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j). \tag{3.9}$$

Also since $j \in \{i+1, i+2\}$, we have $G(x_0,x_i,x_j) \leq G(x_0,x_i,x_{i+1}) + G(x_0,x_i,x_{i+2})$, for $i < j < n$, we have

$$\begin{aligned} \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j) &\leq \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_{i+1}) + \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_{i+2}) \\ &\leq 2\left(\frac{r}{1-r}\right)^{n-2} \delta_{n-1}. \end{aligned}$$

Therefore Equation (3.9) becomes

$$G(x_0,x_{n-1},x_n) < \left[1+2\left(\frac{r}{1-r}\right)^{n-2}\right]\delta_{n-1}.$$

For $i < j = n$, we have

$$\left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j) \leq \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_{n-1},x_n) + \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_{n-2},x_n).$$

Therefore Equation (3.9) becomes

$$\left[1-\left(\frac{r}{1-r}\right)^{n-2}\right]G(x_0,x_{n-1},x_n) < \left[1+\left(\frac{r}{1-r}\right)^{n-2}\right]\delta_{n-1}$$
$$\leq \left[1+2\left(\frac{r}{1-r}\right)^{n-2}\right]\delta_{n-1}.$$

That is,

$$G(x_0,x_{n-1},x_n) < \left[\frac{1+2\left(\frac{r}{1-r}\right)^{n-2}}{1-\left(\frac{r}{1-r}\right)^{n-2}}\right]\delta_{n-1}.$$

Let $\delta_n = \left[\frac{1+2\left(\frac{r}{1-r}\right)^{n-2}}{1-\left(\frac{r}{1-r}\right)^{n-2}}\right]\delta_{n-1}$. Note that $\delta_{n-1} < \delta_n$ and $G(x_0,x_{n-1},x_n) < \delta_n$. The sequence $\{x_n\}$ is bounded if and only if

$$\delta = \lim_{n\to\infty}\delta_n = \frac{\prod_{n=1}^{\infty}\left[1+2\left(\frac{r}{1-r}\right)^{n-2}\right]}{\prod_{n=1}^{\infty}\left[1-\left(\frac{r}{1-r}\right)^{n-2}\right]} < \infty.$$

Now the series

$$\sum_{n=1}^{\infty}\left[1+2\left(\frac{r}{1-r}\right)^{n-2}\right] \text{ and } \sum_{n=1}^{\infty}\left[1-\left(\frac{r}{1-r}\right)^{n-2}\right]$$

are convergent since $\frac{r}{1-r} < 1$. Therefore,

$$\prod_{n=1}^{\infty}\left[1+2\left(\frac{r}{1-r}\right)^{n-2}\right] < \infty, \prod_{n=1}^{\infty}\left[1-\left(\frac{r}{1-r}\right)^{n-2}\right] < \infty \text{ and } \prod_{n=1}^{\infty}\left[1-\left(\frac{r}{1-r}\right)^{n-2}\right] > 0.$$

Hence, $\delta < \infty$.

We now show that $\{x_n\}$ is a Cauchy sequence for both Case III and Case IV.

For Case III. Suppose $M = sup\left\{G(x_m,x_n,x_p) : m,n,p \in \mathbb{N}\right\}$. From Equation (3.3), we have

$$G(x_n,x_{n-1},x_{n-2}) \leq \left(\frac{r}{1-r}\right)^{n-2} G(x_0,x_i,x_j) \leq \left(\frac{r}{1-r}\right)^{n-2} M.$$

Now for m,n sufficiently large with $m<n$, we have

$$
\begin{aligned}
G(x_n,x_n,x_m) &= G(x_m,x_n,x_n)\\
&\le G(x_m,x_{m+1},x_{m+1})+G(x_{m+1},x_n,x_n)\\
&\le G(x_m,x_{m+1},x_{m+2})+G(x_{m+1},x_{m+2},x_{m+3})\\
&< \sum_{k=m}^{n-2} G(x_k,x_{k+1},x_{k+2})\\
&\le \sum_{k=m}^{n-2}\left(\frac{r}{1-r}\right)^k M\\
&< \varepsilon.
\end{aligned}
$$

In a similar manner, we obtain $G(x_n,x_n,x_m)<\varepsilon$ for Case IV. This implies that for both Case III and Case IV, $\{x_n\}$ is a Cauchy sequence. Now since $\{x_n\}$ is a Cauchy sequence and (X,G) is complete, for both Case III and Case IV there exist $u\in X$ such that $\lim_{n\to\infty} x_n = u$. Now

$$
\begin{aligned}
G(u,u,Tu) &= \lim_{n\to\infty}(x_{n+1},x_{n+1},Tu)\\
&\le \lim_{n\to\infty} G_H(Tx_n,Tx_n,Tu)\\
&\le \lim_{n\to\infty} q.max\left\{\begin{matrix} G(x_n,x_n,u), & G(x_n,Tx_n,Tx_n), & G(x_n,Tx_n,Tu),\\ G(x_n,Tx_n,Tu), & G(x_n,Tx_n,Tx_n), & G(x_n,Tx_n,Tu),\\ G(x_n,Tx_n,Tu), & G(u,Tx_n,Tx_n), & G(u,Tx_n,Tu),\\ & G(u,Tx_n,Tu) & \end{matrix}\right\}\\
&\le \lim_{n\to\infty} q.max\left\{\begin{matrix} G(x_n,x_n,u), & G(x_n,x_{n+1},x_{n+1}), & G(x_n,x_{n+1},Tu),\\ G(x_n,x_{n+1},Tu), & G(x_n,x_{n+1},x_{n+1}), & G(x_n,x_{n+1},Tu),\\ G(x_n,x_{n+1},Tu), & G(u,x_{n+1},x_{n+1}), & G(u,x_{n+1},Tu),\\ & G(u,x_{n+1},Tu) & \end{matrix}\right\}\\
&= q.G(u,u,Tu).
\end{aligned}
$$

Therefore, we have

$$
G(u,u,Tu)\le q.G(u,u,Tu). \tag{3.10}
$$

Since $q < \frac{1}{2}$ the only way Equation (3.10) will hold is if $G(u,u,Tu) = 0$, which implies $u = Tu$ (that is $u \in Tu$). Hence u is a fixed point of T in X.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Amini-Harandi, Fixed point theory for set-valued quasi-contraction maps in metric spaces, Appl. Math. Lett. 24 (2011), 1791-1794.

[2] B. C. Dhage, Generalised metric spaces and mappings with fixed point, Bull. Calcutta Math. Soc. 84 (1992), 329-336.

[3] Z. Mustafa and B. Sims, Some remarks concerning *D*-metric spaces, School of Mathematical and Physical Sciences, Proceedings of the Internatinal Conferences on Fixed Point Theorey and Applications, Valencia, Spain (2003) 189-198.

[4] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289-297.