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APPROXIMATE BEST PROXIMITY PAIRS IN METRIC SPACE FOR CONTRACTION MAPS

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Abstract. The purpose of this paper is prove theorems for various types of well known generalized contractions on metric spaces with use of two general lemmas are given regarding approximate best proximity pair.

Keywords: approximate best proximity pair; approximate best proximity pair property; diameter approximate best proximity pair; generalized contractions.

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1. Introduction

Nowadays, fixed point and operator theory play an important role in different areas of mathematics, and its applications, particularly in mathematics, physics, differential equation, game theory and dynamic programming. Also, There are plenty of problems in applied mathematics which can be solved by means of best proximity pair theory. Still, practice proves that in many real situations an approximate solution is more than sufficient, so the existence of best proximity pair is not strictly required, but that of nearly best proximity pair. Another type of practical situations that lead to this approximation is when the conditions that have to be imposed in order

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to guarantee the existence of best proximity pairs are far too strong for the real problem one has to solve.

In 2003, Kirk *et al.* [1], obtained some result on best proximity pairs. Also, In 2006, Eldred and Veeramani obtained some result on its [2].

Now, as in [3] (see also [4]-[7]), we can find the best proximity points of the sets A and B, by considering a map $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Best proximity pair also evolves as a generalization of the concept of fixed point of mappings. Because if $A \cap B \neq \emptyset$ every best proximity point is a fixed point of T.

In 2011, Mohsenalhosseini *et al.* [8], introduced the approximate best proximity pairs and proved the approximate best proximity pairs property for it. Now, we study some well known types of operators on metric spaces, and we give some qualitative and quantitative results regarding approximate best proximity pairs of such operators.

Let (X,d) be a metric space.

Definition 1.1. [8] *Let $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$, $x \in A \cup B$. Then $x \in A \cup B$ is an approximate best proximity point of the pair (A,B), if $d(x,Tx) \leq d(A,B) + \varepsilon$, for some $\varepsilon > 0$.*

Remark 1.1. *In this paper we will denote the set of all approximate best proximity pairs of pair (A,B), for a given ε, by :*

$$P_T^{\varepsilon}(A,B) = \{x \in A \cup B : \ d(x,Tx) \leq d(A,B) + \varepsilon \ for \ some \ \varepsilon > 0\}.$$

We say that the pair (A,B) is an approximate best proximity pair property if $P_T^{\varepsilon}(A,B) \neq \emptyset$.

Example 1.1. [8] Suppose Let $X = \mathbf{R}^2$ and $A = \{(x,y) \in X : (x-y)^2 + y^2 \leq 1\}$ and $B = \{(x,y) \in X : (x+y)^2 + y^2 \leq 1\}$ with $T(x,y) = (-x,y)$ for $(x,y) \in X$. Then $d((x,y),T(x,y)) \leq d(A,B) + \varepsilon$ for some $\varepsilon > 0$. Hence $P_T^{\varepsilon}(A,B) \neq \emptyset$.

Theorem 1.1. [8] *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and*

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = d(A,B) \ for \ some \ x \in A \cup B.$$

Then the pair (A,B) is an approximate best proximity pair.

Definition 1.2. [8] Let $T: A\cup B \to A\cup B$, be continues map such that $T(A)\subseteq B$, $T(B)\subseteq A$ and $\varepsilon > 0$. We define diameter $P_T^a(A,B)$ by

$$diam(P_T^{\varepsilon}(A,B)) = \sup\{d(x,y):\ x,y \in P_T^{\varepsilon}(A,B)\}.$$

2. Approximate best proximity pair for various types of operators

In this section we will formulate and prove, using Theorem 1.1, qualitative results for various types of operators on a metric space, results that establish the conditions under which the mappings considered have the approximate best proximity pair property.

Definition 2.1. [9] A mapping $T: X \to X$ is a α-contraction if

$$\exists \alpha \in (0,1) \textit{ such that } d(Tx,Ty) \leq \alpha d(x,y),\ \forall x,y \in X.$$

Definition 2.2. A mapping $T: A\cup B \to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a P-α contraction if

$$\exists \alpha \in (0,1) \textit{ such that } d(Tx,Ty) \leq \alpha d(x,y),\ \forall x,y \in A\cup B.$$

Theorem 2.1. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T: A\cup B \to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a P-α-contraction.*
Then:

$$\forall \varepsilon > 0,\ P_T^{\varepsilon}(A,B) \neq \emptyset.$$

Proof. Let $\varepsilon > 0$ and $x \in A\cup B$.

$$\begin{aligned} d(T^n x, T^{n+1}x) &= d(T(T^{n-1}x), T(T^n x) \\ &\leq ad(T^{n-1}x), T^n x) \\ &\leq \\ &\ \ \vdots \\ &\leq a^n d(x,Tx). \end{aligned}$$

Since $a \in (0,1)$, we find that

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0, \ \forall x \in A \cup B.$$

Now by Theorem 1.1 it follows that $P_T^{\varepsilon}(A,B) \neq \emptyset, \forall \varepsilon > 0$. ■

Theorem 2.2. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-α-contraction and $\varepsilon > 0$. We assume that:*

(i) $P_T^{\varepsilon}(A,B) \neq \emptyset$;

(ii) $\forall \theta > 0, \ \exists \phi(\theta) > 0$ *such that*;

$$d(x,y) - d(Tx,Ty) \ \leq \ \theta \Rightarrow d(x,y) \leq \phi(\theta), \ \forall x,y \in P_T^{\varepsilon}(A,B) \neq \emptyset.$$

Then:

$$diam(P_T^{\varepsilon}(A,B)) \leq \phi(2d(A,B) + \varepsilon).$$

Proof. Let $\varepsilon_1, \varepsilon_2 > 0$ and $x, y \in P_T^{\varepsilon}(A,B)$. It follows that

$$d(x,Tx) \leq d(A,B) + \varepsilon_1, d(y,Ty) \leq d(A,B) + \varepsilon_2.$$

Note that

$$\begin{aligned} d(x,y) &\leq d(x,Tx) + d(Tx,Ty) + d(y,Ty) \\ &\leq d(Tx,Ty) + 2d(A,B) + \varepsilon_1 + \varepsilon_2. \end{aligned}$$

Put $\varepsilon = Max\{\varepsilon_1, \varepsilon_2\}$. It follows that

$$d(x,y) - d(Tx,Ty) \leq 2d(A,B) + \varepsilon.$$

Now by (ii) it follows that $d(x,y) \leq \phi(2d(A,B) + \varepsilon)$, so

$$diam(P_T^{\varepsilon}(A,B)) \leq \phi(2d(A,B) + \varepsilon). \ ■$$

In 1968, Kannan (see [10] [11]) proved a fixed point theorem for operators which need not be continuous. We it apply on metric space for the approximate best proximity pair.

Definition 2.3. A mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-Kannan operator if there exists $\alpha \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, T(x)) + d(y, T(y))],\ \forall x, y \in A \cup B.$$

Theorem 2.3. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-Kannan operator. Then:*

$$\forall \varepsilon > 0,\ P_T^{\varepsilon}(A, B) \neq \emptyset.$$

Proof. Let $\varepsilon > 0$ and $x \in A \cup B$.

$$\begin{aligned} d(T^n x, T^{n+1} x) &= d(T(T^{n-1} x), T(T^n x) \\ &\leq \alpha[d(T^{n-1} x, T^n x) + d(T^n x, T^{n+1} x)] \\ &= \alpha d(T^{n-1} x, T^n x) + \alpha d(T^n x, T^{n+1} x)]. \end{aligned}$$

Therefore $(1 - \alpha) d(T^n x, T^{n+1} x) \leq \alpha d(T^{n-1} x, T^n x)$. Then

$$\begin{aligned} d(T^n x, T^{n+1} x) &\leq \frac{\alpha}{1 - \alpha} d(T^{n-1} x, T^n x) \\ &\ \ \vdots \\ &\leq (\frac{\alpha}{1 - \alpha})^n d(x, Tx). \end{aligned}$$

But $\alpha \in (0, \frac{1}{2})$ hence $\frac{\alpha}{1 - \alpha} \in (0, 1)$. Therfore

$$\lim_{n \to \infty} d(T^n x, T^{n+1} x) = 0,\ \forall x \in A \cup B.$$

Now by Theorem 1.1 it follows that $P_T^{\varepsilon}(A, B) \neq \emptyset, \forall \varepsilon > 0$. ■

In 1972, Chatterjea (see [12]) considered another which again does not impose the continuity of the operator. We it apply on metric space for the approximate best proximity pair.

Definition 2.4. A mapping $T : A\cup B \to A\cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-Chatterjea operator if there exists $\alpha \in (0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq \alpha[d(x,T(y)) + d(y,T(x))],\ \forall x,y \in A\cup B.$$

Theorem 2.4. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A\cup B \to A\cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-Chatterjea operator. Then:*

$$\forall \varepsilon > 0,\ P_T^{\varepsilon}(A,B) \neq \emptyset.$$

Proof. Let $\varepsilon > 0$ and $x \in A\cup B$.

$$\begin{aligned} d(T^n x, T^{n+1}x) &= d(T(T^{n-1}x), T(T^n x)) \\ &\leq \alpha[d(T^{n-1}x, T(T^n x)) + d(T^n x, T(T^{n-1}x))] \\ &= \alpha[d(T^{n-1}x, T^{n+1}x) + d(T^n x, T^n x)] = \alpha d(T^{n-1}x, T^{n+1}x). \end{aligned}$$

On the other hand, we have

$$d(T^{n-1}x, T^{n+1}x) \leq d(T^{n-1}x, T^n x) + d(T^n x, T^{n+1}x).$$

Then

$$(1-\alpha)d(T^n x, T^{n+1}x) \leq \alpha d(T^{n-1}x, T^n x).$$

It follows that

$$\begin{aligned} d(T^n x, T^{n+1}x) &\leq \frac{\alpha}{1-\alpha} d(T^{n-1}x, T^n x) \\ &\vdots \\ &\leq (\frac{\alpha}{1-\alpha})^n d(x, Tx). \end{aligned}$$

Since $\alpha \in (0, \frac{1}{2})$, we find $\frac{\alpha}{1-\alpha} \in (0,1)$. It follows that

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0,\ \forall x \in A\cup B.$$

Using Theorem 1.1, we see that $P_T^{\varepsilon}(A,B) \neq \emptyset, \forall \varepsilon > 0$. ■

We, by combining the three independent contraction conditions above obtain another approximate best proximity pair result for operators which satisfy the following.

Definition 2.5. A mapping $T : A\cup B \to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a P-Zamfirescu operator if there exists $\alpha_1,\beta,\gamma\in R$, $\alpha_1\in[0,1[,\beta\in[0,\frac{1}{2}[,\gamma\in[0,\frac{1}{2}[$ such that for all $x,y\in A\cup B$ at least one of the following is true:

(i) $d(Tx,Ty)\leq\alpha_1 d(x,y)$;

(ii) $d(Tx,Ty)\leq\beta[d(x,T(x))+d(y,T(y))]$;

(iii) $d(Tx,Ty)\leq\gamma[d(x,T(y))+d(y,T(x))]$.

Theorem 2.5. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A\cup B \to A\cup B$ satisfying $T(A)\subseteq B$, $T(B)\subseteq A$ is a P-Zamfirescu operator. Then:*

$$\forall\varepsilon>0,\ P_T^{\varepsilon}(A,B)\neq\emptyset.$$

Proof: Let $x,y\in A\cup B$. Supposing ii holds, we have that

$$\begin{aligned} d(Tx,Ty) &\leq \beta[d(x,T(x))+d(y,T(y))] \\ &\leq \beta d(x,Tx)+\beta[d(y,x)+d(x,Tx)+d(Tx,Ty)] \\ &= 2\beta d(x,Tx)+\beta d(x,y)+\beta d(Tx,Ty). \end{aligned}$$

Thus

$$d(Tx,Ty)\leq\frac{2\beta}{1-\beta}d(x,Tx)+\frac{\beta}{1-\beta}d(x,y). \tag{2.1}$$

Supposing iii holds, we have that

$$\begin{aligned} d(Tx,Ty) &\leq \gamma[d(x,Ty)+d(y,Tx)] \\ &\leq \gamma[d(x,y)+d(y,Ty)]+\gamma[d(y,Ty)+d(Ty,Tx)] \\ &= \gamma d(Tx,Ty)+2\gamma d(y,Ty)+\gamma d(x,y). \end{aligned}$$

Thus

$$d(Tx,Ty)\leq\frac{2\gamma}{1-\gamma}d(y,Ty)+\frac{\gamma}{1-\gamma}d(x,y). \tag{2.2}$$

Similarly, we have

$$\begin{aligned} d(Tx,Ty) &\leq \gamma[d(x,Ty)+d(y,Tx)] \\ &\leq \gamma[d(x,Tx)+d(Tx,Ty)]+\gamma[d(y,x)+d(x,Tx)] \\ &= \gamma d(Tx,Ty)+2\gamma d(x,Tx)+\gamma d(x,y). \end{aligned}$$

Then

$$d(Tx,Ty) \leq \frac{2\gamma}{1-\gamma}d(x,Tx)+\frac{\gamma}{1-\gamma}d(x,y). \tag{2.3}$$

In view of i), (2.1), (2.2), (2.3), we have

$$\eta = max\{\alpha_1, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\},$$

and it is easy to see that $\eta \in [0,1[$. For T satisfying at least one of the conditions i), ii), iii) we have that

$$d(Tx,Ty) \leq 2\eta d(x,Tx)+\eta d(x,y) \tag{2.4}$$

and

$$d(Tx,Ty) \leq 2\eta d(y,Ty)+\eta d(x,y) \tag{2.5}$$

hold. Using these conditions implied by i) - iii) and taking $x \in X$, we have

$$\begin{aligned} d(T^n x, T^{n+1}x) &= d(T(T^{n-1}x), T(T^n x)) \\ &\leq^{(2.4)} 2\eta d(T^{n-1}x, T(T^{n-1}x)+\eta d(T^{n-1}x, T^n x) \\ &= 3\eta d(T^{n-1}x, T^n x). \end{aligned}$$

Then

$$d(T^n x, T^{n+1}x) \leq \cdots \leq (3\eta)^n d(x,Tx).$$

Therefore, we have

$$Lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0,\ \forall x \in A \cup B.$$

Using Theorem 1.1, we find that $P_T^{\varepsilon}(A,B) \neq \emptyset, \forall \varepsilon > 0$. ■

Now, we consider the contraction condition given in 2004 by V. Berinde, who also formulated a corresponding fixed point theorem, see, for example, [10].

Definition 2.6. A mapping $T : A\cup B \to A\cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-weak contraction operator if there exists $\alpha_1 \in]0,1[$ and $L \geq 0$ such that

$$d(Tx,Ty) \leq \alpha_1 d(x,y) + Ld(y,T(x))], \ \forall x,y \in A\cup B.$$

Theorem 2.6. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A\cup B \to A\cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a p-weak contraction. Then:*

$$\forall \varepsilon > 0, \ P_T^{\varepsilon}(A,B) \neq \emptyset.$$

Proof. Let $x \in A\cup B$.

$$\begin{aligned} d(T^n x, T^{n+1}x) &= d(T(T^{n-1}x), T(T^n x)\\ &\leq \alpha_1 d(T^{n-1}x, T^n x) + Ld(T^n x, T^n x)\\ &= \alpha_1 d(T^{n-1}x, T^n x) \leq \cdots \leq \alpha_1^n d(x,Tx). \end{aligned}$$

In view of $\alpha_1 \in]0,1[$, we find that

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0, \ \forall x \in A\cup B.$$

Using Theorem 1.1, we find that $P_T^{\varepsilon}(A,B) \neq \emptyset, \forall \varepsilon > 0$. ■

In 1974, Ciric [13] obtained a contraction condition for which the map satisfying it is still a Picard operator. We it apply on metric space for the approximate best proximity pair.

Definition 2.7. A mapping $T : A\cup B \to A\cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-quasi contraction if there exists $h \in]0,1[$ such that

$$d(Tx,Ty) \leq h \cdot max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}, \ \forall x,y \in A\cup B.$$

Corollary 2.1. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A\cup B \to A\cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-quasi contraction with $0 < h < \frac{1}{2}$. Then:*

$$\forall \varepsilon > 0, \ P_T^{\varepsilon}(A,B) \neq \emptyset.$$

Proof. By Proposition 3 of [14] any P-quasi contraction with $0 < h < \frac{1}{2}$ is a weak contraction. Therefore by Theorem 2.6, $P_T^{\varepsilon}(A,B) \neq \emptyset, \forall \varepsilon > 0$.

3. Diameter approximate best proximity pair for various types of operators

For the same operators we have studied in the previous section, we will formulate and prove useing Theorem 2.2, in order to obtain results for diameter approximate best proximity pair.

Theorem 3.1. [8] *Let $T : A \cup B \to A \cup B$, such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $\varepsilon > 0$. If there exists a $\alpha \in [0,1]$ such that for all $(x,y) \in A \times B$*

$$d(Tx,Ty) \leq \alpha d(x,y).$$

Then

$$diam(P_T^{\varepsilon}(A,B)) \leq \frac{2\varepsilon}{1-\alpha} + \frac{2d(A,B)}{1-\alpha}.$$

Theorem 3.2. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-Kannan operator and $\varepsilon > 0$. Then*

$$diam(P_T^{\varepsilon}(A,B)) \leq 2\varepsilon(1+\alpha) + 2\alpha d(A,B).$$

Proof. Let $\varepsilon > 0$. Condition 1) in Theorem 2.2, is satisfied, as one can see in the proof of Theorem 2.3, we only verify that condition 2) in Theorem 2.2, holds. Let $\theta > 0$ and $x,y \in P_T^{\varepsilon}(A,B))$ and assume that $d(x,y) - d(Tx,Ty) \leq \theta$. Then

$$d(x,y) \quad \leq \quad \alpha[d(x,Tx) + d(y,Ty)] + \theta.$$

As $x,y \in P_T^{\varepsilon}(A,B))$, we know that

$$d(x,Tx) \leq d(A,B) + \varepsilon_1, d(y,Ty) \leq d(A,B) + \varepsilon_2.$$

Putting $\varepsilon = \{\varepsilon_1, \varepsilon_2\}$, we see that $d(x,y) \leq 2\alpha d(A,B) + 2\alpha\varepsilon + \theta$. So for every $\theta > 0$ there exists $\phi(\theta) = \theta + 2\alpha(d(A,B) + \varepsilon) > 0$ such that

$$d(x,y) - d(Tx,Ty) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta).$$

Using Theorem 2.2, we obtain that

$$diam(P_T^{\varepsilon}(A,B)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$diam(P_T^{\varepsilon}(A,B)) \leq 2\varepsilon(1+\alpha) + 2\alpha d(A,B), \forall \varepsilon > 0.$$

Theorem 3.3. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-Chatterjea operator and $\varepsilon > 0$. Then:*

$$diam(P_T^{\varepsilon}(A,B)) \leq \frac{2\varepsilon(1+\alpha) + 2\alpha d(A,B)}{1-2\alpha}.$$

Proof. Let $\varepsilon > 0$. We will only verify that condition 2) in Theorem 2.2 holds. Let $\theta > 0$ and $x,y \in P_T^{\varepsilon}(A,B))$ and assume that $d(x,y) - d(Tx,Ty) \leq \theta$. Then:

$$\begin{aligned} d(x,y) &\leq \alpha[d(x,Ty) + d(y,Tx)] + \theta \\ &\leq \alpha d(x,Ty) + \alpha d(y,Tx) + \theta \\ &\leq \alpha[d(x,y) + d(y,Ty)] + \alpha[d(y,x) + d(x,Tx)] + \theta. \end{aligned}$$

As $x,y \in P_T^{\varepsilon}(A,B))$, we know that

$$d(x,Tx) \leq d(A,B) + \varepsilon_1, d(y,Ty) \leq d(A,B) + \varepsilon_2.$$

Put $\varepsilon = \{\varepsilon_1, \varepsilon_2\}$. It follows that $d(x,y) \leq 2\alpha d(x,y) + 2\alpha d(A,B) + 2\alpha\varepsilon + \theta$. Then

$$(1-2\alpha)d(x,y) \leq 2\alpha d(A,B) + 2\alpha\varepsilon + \theta.$$

Therefore, we have

$$d(x,y) \leq \frac{2\alpha(d(A,B) + \varepsilon) + \theta}{(1-2\alpha)}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\alpha(d(A,B)+\varepsilon)+\theta}{(1-2\alpha)} > 0$ such that

$$d(x,y) - d(Tx,Ty) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta).$$

Using Theorem 2.2, we see that

$$diam(P_T^{\varepsilon}(A,B)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$diam(P_T^{\varepsilon}(A,B)) \leq \frac{2\varepsilon(1+\alpha)+2\alpha d(A,B)}{1-2\alpha}, \forall \varepsilon > 0.$$

Theorem 3.4. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-Zamfirescu operator and $\varepsilon > 0$. Then:*

$$diam(P_T^{\varepsilon}(A,B)) \leq \frac{2\varepsilon(1+\eta)+2\eta d(A,B)}{1-\eta}.$$

where $\eta = max\{\alpha_1, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, and α_1, β, γ as in Definition 2.5.

Proof. In the proof of Theorem2.5, we have already shown that if T satisfies at least one of the conditions F_{Z1}),F_{Z2}), F_{Z3}). From Definition 2.5, we see that

$$d(Tx,Ty) \leq 2\eta d(x,Tx) + \eta d(x,y)$$

and

$$d(Tx,Ty) \leq 2\eta d(y,Ty) + \eta d(x,y)$$

hold. Let $\varepsilon > 0$. We will only verify that condition 2) in Theorem 2.2 is satisfied, as 1) holds, see the Proof of Theorem 2.5. Let $\theta > 0$ and $x, y \in P_T^{\varepsilon}(A,B))$ and assume that $d(x,y) - d(Tx,Ty) \leq \theta$. Then

$$d(x,y) - d(Tx,Ty) \quad \leq \quad 2\eta d(x,Tx) + \eta d(x,y) + \theta \Rightarrow$$

$$(1-\eta)d(x,y) \leq 2\eta d(A,B) + 2\eta\varepsilon + \theta$$

$$d(x,y) \leq \frac{2\eta(d(A,B)+\varepsilon)+\theta}{1-\eta}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\eta(d(A,B)+\varepsilon)+\theta}{1-\eta} > 0$ such that

$$d(x,y) - d(Tx,Ty) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta).$$

Using Theorem 2.2, we see that

$$diam(P_T^{\varepsilon}(A,B)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which implies that

$$diam(P_T^{\varepsilon}(A,B)) \leq \frac{2\varepsilon(1+\eta)+2\eta d(A,B)}{1-\eta}, \forall \varepsilon > 0.$$

Theorem 3.5. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a P-weak contraction with $\alpha_1 + L < 1$, and $\varepsilon > 0$. Then:*

$$diam(P_T^{\varepsilon}(A,B)) \leq \frac{(L+2)\varepsilon + Ld(A,B)}{1-\alpha_1 - L}.$$

Proof. Let $\varepsilon > 0$. We will only verify that condition 2) in Theorem 2.2 holds. Let $\theta > 0$ and $x, y \in P_T^{\varepsilon}(A,B))$ and assume that $d(x,y) - d(Tx,Ty) \leq \theta$. Then

$$\begin{aligned} d(x,y) &\leq d(Tx,Ty) + \theta \\ &\leq \alpha_1 d(x,y) + Ld(y,Tx) + \theta \\ &\leq \alpha_1 d(x,y) + Ld(x,y) + Ld(x,Tx) + \theta \\ &\leq (\alpha_1 + L)d(x,y) + L(d(A,B) + \varepsilon) + \theta. \end{aligned}$$

Therefore, we have

$$d(x,y) \leq \frac{L(d(A,B)+\varepsilon)+\theta}{1-\alpha_1 - L}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{L(d(A,B)+\varepsilon)+\theta}{1-\alpha_1-L} > 0$ such that

$$d(x,y) - d(Tx,Ty) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta).$$

Using Theorem 2.2, we find that

$$diam(P_T^{\varepsilon}(A,B)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$diam(P_T^{\varepsilon}(A,B)) \leq \frac{(L+2)\varepsilon + Ld(A,B)}{1-\alpha_1 - L}, \forall \varepsilon > 0.$$

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] W.A. Kirk, S. Reich, P. Veeramani, Proximinal retracts and best proximity pair theorems, Numer. Funct. Anal. Optim. 24 (2003), 851-862.

[2] A.A. Eldred, P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323 (2006), 1001-1006.

[3] I. Singer, Best approximation in normed linear spaces by elements of linear subspaces. Translated from the Romanian by Radu Georgescu. Die Grundlehren der mathematischen Wissenschaften, Band 171 Publishing House of the Academy of the Socialist Republic of Romania, Bucharest; Springer-Verlag, New York-Berlin (1970).

[4] V. Vetrivel, P. Veeramani, P. Bhattacharyya, Some extensions of Fan's best approxomation theorem, Numer. Funct. Anal. Optim. 13 (1992), 397-402.

[5] K. Włodarczyk, R. Plebaniak, A. Banach, Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces, Nonlinear Anal. 70 (2009), 3332-3341.

[6] K. Włodarczyk, R. Plebaniak, A. Banach, Erratum to: "Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces" [Nonlinear Anal. 70 (2009), 3332-3341], Nonlinear Anal. 71 (2009), 3583-3586.

[7] K. Włodarczyk, R. Plebaniak, C. Obczyǹski, Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces, Nonlinear Anal. 72 (2010), 794-805.

[8] S. A. M. Mohsenalhosseini, H. Mazaheri, and M.A. Dehghan, Approximate best proximity pairs in metric space, Abst. Appl. Anal. 2011 (2011), Article ID 596971.

[9] I.A. Rus, Generalized Contractions and Applications, Cluj University Press, Cluj- Napoca, 2001.

[10] V. Berinde, On the Approximation of Fixed Points of Weak Contractive Mappings, Carpathian J. Math. 19 (2003), 7-22.

[11] R. Kannan, Some Results on Fixed Points, Bull. Calcutta Math. Soc. 10 (1968), 71-76.

[12] S.K. Chatterjea, Fixed-point Theorems, C.R. Acad. Bulgare Sci. 25 (1972), 727-730.

[13] L.B. Ciric, A generalization of Banachs contraction principle, Proc. Amer. Math. Soc. 45 (1974), 267-273.

[14] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum 2004.