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AN INFINITE FAMILY NONEXPANSIVE MAPPINGS AND A RELAXED COCOERCIVE MAPPING

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Abstract. In this article, we investigate the problem of finding a common element of the set of solutions of a variational problem and the set of common fixed points of an infinite family nonexpansive mappings based on a viscosity approximation algorithm in a Hilbert space.

Keywords: variational inequality; nonexpansive mapping; fixed point; Hilbert space.

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1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a closed convex subset of H and let $A : C \to H$ be a nonlinear map. Let P_C be the projection of H onto the convex subset C. The classical variational inequality which denoted by $VI(C,A)$ is to find $u \in C$ such that

$$\langle Au, v-u \rangle \geq 0, \quad \forall v \in C, \tag{1.1}$$

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It is know that P_C satisfies

$$\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2 \tag{1.2}$$

for $x,y\in H$. Moreover, P_Cx is characterized by the properties: $P_Cx\in C$ and $\langle x-P_Cx, P_Cx-y\rangle \geq 0$ for all $y\in C$. One can see that the variational inequality (1.1) is equivalent to a fixed point problem. The function $u\in C$ is a solution of the variational inequality (1.1) if and only if $u\in C$ satisfies the relation $u=P_C(u-\lambda Au)$, where $\lambda>0$ is a constant.

In the real world, many important problems have reformulations which require finding solutions of the variational inequality, for instance, evolution equations, complementarity problems, mini-max problems, and optimization problems; see [1-23] and the references therein.

Recall that the following definitions. A mapping A is said to be γ-cocoercive, if for each $x,y\in C$, we have

$$\langle Ax-Ay, x-y\rangle \geq \gamma\|Ax-Ay\|^2, \quad \text{for a constant } \mu>0.$$

Clearly, every μ-cocoercive map A is $1/\mu$-Lipschitz continuous. A is said to be relaxed γ-cocoerceive, if there exists a constant $u>0$ such that

$$\langle Ax-Ay, x-y\rangle \geq (-\gamma)\|Ax-Ay\|^2, \quad \forall x,y\in C.$$

A is said to be relaxed (γ,r)-cocoercive, if there exist two constants $u,v>0$ such that

$$\langle Ax-Ay, x-y\rangle \geq (-\gamma)\|Ax-Ay\|^2 + r\|x-y\|^2, \quad \forall x,y\in C.$$

A mapping $S:C\to C$ is said to be nonexpansive if $\|Sx-Sy\|\leq\|x-y\|$ for all $x,y\in C$. Next, we denote by $F(S)$ the set of fixed points of S. A mapping $f:C\to C$ is said to be a contraction if there exists a coefficient α $(0<\alpha<1)$ such that

$$\|f(x)-f(y)\|\leq\alpha\|x-y\|,$$

for $\forall x,y\in C$. A linear bounded operator B is strongly positive if there exists a constant $\bar{\gamma}>0$ with the property

$$\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x\in H.$$

A set-valued mapping $T:H\to 2^H$ is called monotone if for all $x,y\in H$, $f\in Tx$ and $g\in Ty$ imply $\langle x-y, f-g\rangle\geq 0$. A monotone mapping $T:H\to 2^H$ is maximal if the graph of $G(T)$

of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0,\ \forall u \in C\}$ and define

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [1].

2. Preliminaries

Recently iterative methods for nonexpansive mappings have been applied to solve convex minimization problems. A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle, \tag{2.1}$$

where B is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S and b is a given point in H. In [11], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n b, \quad n \geq 0,$$

converges strongly to the unique solution of the minimization problem (2.1) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. More recently, Marino and Xu [12] introduced a new iterative scheme by the viscosity approximation method:

$$x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0.$$

They proved the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^*\rangle \geq 0, \quad x \in C,$$

which is the optimality condition for the minimization problem $\min_{x\in C}\frac{1}{2}\langle Bx,x\rangle - h(x)$, where C is the fixed point set of a nonexpansive mapping S, h is a potential function for δf (i.e., $h'(x) = \delta f(x)$ for $x \in H$.)

Concerning a family of nonexpansive mappings has been considered by many authors. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. The problem of finding an optimal point that minimizes a given cost function over common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance. A simple algorithmic solution to the problem of minimizing a quadratic function over common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation.

Recently Yao et al. [13] considered a general iterative algorithm for an infinite family of non-expansive mapping in the framework of Hilbert spaces. To be more precisely, they introduced the following general iterative algorithm.

$$x_{n+1} = \lambda_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \lambda_n A)W_n x_,$$

where f is a contraction on H, A is a stronglyy positive bounded linear operator, W_n are nonexpansive mappings which are generated by an finite family of nonexpansive mapping $T_1, T_2, \ldots$. To be more precisely,

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
&\vdots\\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I,\\
&u_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 T_2 U_{u,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned}\tag{2.2}$$

where $\{\gamma_1\},\{\gamma_2\},\ldots$ are real numbers such that $0 \leq \gamma \leq 1$, $T_1,T_2,\ldots$ be an infinite family of mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Concerning W_n we have the following lemmas which are important to prove our main results.

Lemma 2.1 [14] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1,T_2,\ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty}F(T_n)$ is nonempty, and let $\gamma_1,\gamma_2,\ldots$ be real numbers such that $0<\gamma_n \leq \eta < 1$ for any $n \geq 1$. Then, for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.1, one can define the mapping W of C into itself as follows. $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, for every $x \in C$. Such a W is called the W-mapping generated by $T_1,T_2,\ldots$ and $\gamma_1,\gamma_2,\ldots$. Throughout this paper, we will assume that $0<\gamma_n \leq \eta < 1$ for all $n \geq 1$.

Lemma 2.2 [14] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1,T_2,\ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty}F(T_n)$ is nonempty, and let $\gamma_1,\gamma_2,\ldots$ be real numbers such that $0<\gamma_n \leq \eta < 1$ for any $n \geq 1$. Then, $F(W)=\cap_{n=1}^{\infty}F(T_n)$.*

In this paper, we introduce a composite iterative process as following:

$$\begin{cases} x_1 \in C \\ y_n = P_C\big(\beta_n\gamma f(x_n)+(I-\beta_n B)W_nP_C(I-r_nA)x_n\big), \\ x_{n+1}=\alpha_n x_n+(1-\alpha_n)y_n, \quad n \geq 1, \end{cases} \tag{2.3}$$

where A is a relaxed cocoercive mapping, B is a strongly positive linear bounded operator, f is a contraction on C and W_n is a mapping generated by (2.2). We prove the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to a common element of the set of common fixed points of an infinite nonexpansive mappings and the set of solutions of the variational inequalities for relaxed (γ,r)-cocoercive maps, which solves another variation inequality $\langle \gamma f(q)-Bq, q-p\rangle \leq 0, \quad p \in \cap_{i=1}^{\infty}F(T_i)\cap VI(C,A)$ and is also the optimality condition for the minimization problem $\min_{x\in C}\frac{1}{2}\langle Bx,x\rangle - h(x)$, where C is the intersection of the common fixed points set of a nonexpansive mappings and the set of solutions of the variational inequalities for relaxed (γ,r)-cocoercive maps, h is a potential function for δf (i.e., $h'(x)=\delta f(x)$ for $x \in H$.)

The results are obtained in this paper improve and extend the recent ones announced by many authors; see the literatures.

In order to prove our main results, we need the following lemmas.

Lemma 2.3 [12] *Assume B is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.4 [11] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where γ_n is a sequence in (0,1) and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.5. *In a real Hilbert space H, there holds the the following inequality*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,$$

for all $x, y \in H$.

Lemma 2.6 [15] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let β_n be a sequence in [0,1] with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

3. Main results

Theorem 3.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $A : C \to H$ be relaxed (γ, r)-cocoercive and μ-Lipschitz continuous. Let $f : C \to C$ be a contraction with the coefficient α $(0 < \alpha < 1)$ and $\{T_i\}_{i=1}^{\infty}$ be an infinite nonexpansive mappings* *from C into itself generated by (2.2) such that $F = \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$. Let B be a strongly positive linear bounded self-adjoint operator of C into itself with coefficient $\bar{\gamma} > 0$ such that $\|B\| \le 1$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Assume that $x_1 \in C$ and $\{x_n\}$ is generated by*

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n) P_C\big(\beta_n \gamma f(x_n) + (I - \beta_n B) W_n P_C (I - r_n A) x_n\big), \quad n \ge 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ are chosen such that

(i) $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(iii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(iv) $\{r_n\} \subset [a,b]$ *for some* a,b *with* $0 < a < b < \frac{2(r-\gamma\mu^2)}{\mu^2}$, $r > \gamma\mu^2$.

Then $\{x_n\}$ converges strongly to $q \in F$, where $q = P_F(\gamma f + (I - B))(q)$, which solves the variation inequality $\langle \gamma f(q) - Bq, p - q \rangle \le 0, \quad \forall p \in F$.

Proof. First, we show the mapping $I - r_n A$ is nonexpansive. Indeed, from the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on A and the condition (iv), we have

$$\begin{aligned}
&\|(I - r_n A)x - (I - r_n A)y\|^2 \\
&= \|(x-y) - r_n(Ax - Ay)\|^2 \\
&\le \|x-y\|^2 - 2r_n[-\gamma\|Ax - Ay\|^2 + r\|x-y\|^2] + r_n^2\|Ax - Ay\|^2 \\
&\le \|x-y\|^2 + 2r_n\mu^2\gamma\|x-y\|^2 - 2r_n r\|x-y\|^2 + \mu^2 r_n^2\|x-y\|^2 \\
&= (1 + 2r_n\mu^2\gamma - 2r_n r + \mu^2 r_n^2)\|x-y\|^2 \\
&\le \|x-y\|^2,
\end{aligned}$$

which implies the mapping $I - r_n A$ is nonexpansive. Since the condition (i), we may assume, with no loss of generality, that $\beta_n < \|B\|^{-1}$ for all n. From Lemma 2.3, we know that if $0 < \rho \le \|B\|^{-1}$, then $\|I - \rho B\| \le 1 - \rho\bar{\gamma}$. Letting $p \in F$ and putting $y_n = P_C\big(\beta_n \gamma f(x_n) + (I - \beta_n B) W_n P_C (I - r_n A) x_n\big)$, we have

$$\begin{aligned}
\|y_n - p\| &\le \beta_n\|\gamma f(x_n) - Bp\| + (1 - \beta_n\bar{\gamma})\|W_n P_C (I - r_n A)x_n - p\| \\
&\le \beta_n\gamma\|f(x_n) - f(p)\| + \beta_n\|\gamma f(p) - Bp\| + (1 - \beta_n\bar{\gamma})\|x_n - p\| \\
&= [1 - \beta_n(\bar{\gamma} - \gamma\alpha)]\|x_n - p\| + \beta_n\|\gamma f(p) - Bp\|.
\end{aligned}$$

On the other hand, we have

$$\|x_{n+1}-p\| \le \alpha_n\|x_n-p\|+(1-\alpha_n)\|y_n-p\|$$
$$\le \alpha_n\|x_n-p\|+(1-\alpha_n)[(1-\beta_n(\bar{\gamma}-\gamma\alpha))\|x_n-p\|+\beta_n\|\gamma f(p)-Bp\|]$$

By simple inductions, we have $\|x_n-p\| \le \max\{\|x_0-p\|, \frac{\|Bp-\gamma f(p)\|}{\bar{\gamma}-\gamma\alpha}\}$, which gives that the sequence $\{x_n\}$ is bounded. Set $\rho_n = P_C(I-r_nA)x_n$. Notice that

$$\begin{aligned}\|\rho_n-\rho_{n+1}\| \le&\|(I-r_nA)x_n-(I-r_{n+1}A)x_{n+1}\|\\ =&\|(x_n-r_nAx_n)-(x_{n+1}-r_nAx_{n+1})+(r_{n+1}-r_n)Ax_{n+1}\| \\ \le&\|x_n-x_{n+1}\|+|r_{n+1}-r_n|M_1,\end{aligned} \tag{3.1}$$

where M_1 is an appropriate constant such that $M_1 \ge \sup_{n\ge 1}\{\|Ax_n\|\}$. It follows that

$$\begin{aligned}\|y_n-y_{n+1}\| \le (1-\beta_{n+1}\bar{\gamma})(\|\rho_{n+1}-\rho_n\|+\|W_{n+1}\rho_n-W_n\rho_n\|)\\ +|\beta_{n+1}-\beta_n|M_2+\gamma\beta_{n+1}\alpha\|x_{n+1}-x_n\|,\end{aligned} \tag{3.2}$$

where M_2 is an appropriate constant such that $M_2 \ge \max\{\sup_{n\ge 1}\{\|BW_n\rho_n\|\}, \gamma\sup_{n\ge 1}\{\|f(x_n)\|\}\}$. Since T_i and $U_{n,i}$ are nonexpansive, we find from (2.2)

$$\begin{aligned}\|W_{n+1}\rho_n-W_n\rho_n\| &= \|\gamma_1T_1U_{n+1,2}\rho_n-\gamma_1T_1U_{n,2}\rho_n\|\\ &\le \gamma_1\|U_{n+1,2}\rho-U_{n,2}\rho_n\|\\ &= \gamma_1\|\gamma_2T_2U_{u+1,3}\rho_n-\gamma_2T_2U_{n,3}\rho_n\|\\ &\le \gamma_1\gamma_2\|U_{u+1,3}\rho_n-U_{n,3}\rho_n\|\\ &\le \cdots\\ &\le \gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}\rho_n-U_{n,n+1}\rho_n\|\\ &\le M_3\prod_{i=1}^{n}\gamma_i,\end{aligned} \tag{3.3}$$

where $M_3 \ge 0$ is an appropriate constant such that $\|U_{n+1,n+1}\rho_n-U_{n,n+1}\rho_n\| \le M_3$, for all $n \ge 0$. Substitute (3.1) and (3.3) into (3.2) yields that

$$\begin{aligned}\|y_n-y_{n+1}\| \le [1-\beta_{n+1}(\bar{\gamma}-\alpha\gamma)]\|x_{n+1}-x_n\|\\ +M_4(|r_{n+1}-r_n|+|\beta_{n+1}-\beta_n|+\prod_{i=1}^{n}\gamma_i),\end{aligned}$$

where M_4 is an appropriate appropriate constant such that $M_4 \geq \max\{M_1, M_2, M_3\}$. From the conditions (i) and (iii), we have

$$\limsup_{n\to\infty}\{\|y_{n+1}-y_n\|-|x_{n+1}-x_n\|\}\leq 0. \tag{3.4}$$

By virtue of Lemma 2.6, we obtain that

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.5}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.6}$$

It follows that

$$\lim_{n\to\infty}\|W_n\rho_n-y_n\|=0. \tag{3.7}$$

For $p \in F$, we have

$$\begin{aligned}
&\|\rho_n-p\|^2\\
&=\|P_C(I-r_nA)x_n-P_C(I-r_nA)p\|^2\\
&\leq\|(x_n-p)-r_n(Ax_n-Ap)\|^2\\
&=\|x_n-p\|^2-2r_n\langle x_n-p,Ax_n-Ap\rangle+r_n^2\|Ax_n-Ap\|^2\\
&\leq\|x_n-p\|^2-2r_n[-\gamma\|Ax_n-Ap\|^2+r\|x_n-p\|^2]+r_n^2\|Ax_n-Ap\|^2\\
&\leq\|x_n-p\|^2+2r_n\gamma\|Ax_n-Ap\|^2-2r_nr\|x_n-p\|^2+r_n^2\|Ax_n-Ap\|^2\\
&\leq\|x_n-p\|^2+(2r_n\gamma+r_n^2-\frac{2r_nr}{\mu^2})\|Ax_n-Ap\|^2.
\end{aligned} \tag{3.8}$$

Observe that

$$\begin{aligned}
\|y_n-p\|^2&=\|\beta_n(\gamma f(x_n)-Bp)+(I-\beta_nB)(W_n\rho_n-p)\|^2\\
&\leq(\beta_n\|\gamma f(x_n)-Bp\|+(1-\beta_n\bar{\gamma})\|\rho_n-p\|)^2\\
&\leq\beta_n\|\gamma f(x_n)-Bp\|^2+\|\rho_n-p\|^2+2\beta_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.
\end{aligned} \tag{3.9}$$

We find that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)(y_n-p)\|^2\\ &\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\ &\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)[\beta_n\|\gamma f(x_n)-Bp\|^2+\|\rho_n-p\|^2\\ &\quad +2\beta_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|].\end{aligned} \tag{3.10}$$

Substituting (3.8) into (3.10), we arrive at

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\leq \|x_n-p\|^2+\beta_n\|\gamma f(x_n)-Bp\|^2+(2r_n\gamma+r_n^2-\frac{2r_nr}{\mu^2})\|Ax_n-Ap\|^2\\ &\quad +2\beta_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.\end{aligned} \tag{3.11}$$

It follows from the condition (iv) that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{3.12}$$

Observe that

$$\begin{aligned}\|\rho_n-p\|^2 =&\|P_C(I-r_nA)x_n-P_C(I-r_nA)p\|^2\\ \leq&\langle(I-r_nA)x_n-(I-r_nA)p,\rho_n-p\rangle\\ =&\frac{1}{2}\{\|(I-r_nA)x_n-(I-r_nA)p\|^2+\|\rho_n-p\|^2\\ &-\|(I-r_nA)x_n-(I-r_nA)p-(\rho_n-p)\|^2\}\\ \leq&\frac{1}{2}\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|(x_n-\rho_n)-r_n(Ax_n-Ap)\|^2\}\\ =&\frac{1}{2}\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|x_n-\rho_n\|^2-r_n^2\|Ax_n-Ap\|^2\\ &+2r_n\langle x_n-\rho_n,Ax_n-Ap\rangle\},\end{aligned}$$

which yields that

$$\|\rho_n-p\|^2 \leq\|x_n-p\|^2-\|\rho_n-x_n\|^2+2r_n\|\rho_n-x_n\|\|Ax_n-Ap\|. \tag{3.13}$$

Substituting (3.13) into (3.10), we have

$$
\begin{aligned}
&(1-\alpha_n)\|\rho_n - x_n\|^2\\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n\|\gamma f(x_n) - Bp\|^2 + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\|\\
&\quad + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|\\
&\le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \beta_n\|\gamma f(x_n) - Bp\|^2\\
&\quad + 2r_n\|\rho_n - x_n\|\|Ax_n - Ap\| + 2\beta_n\|\gamma f(x_n) - Bp\|\|\rho_n - p\|.
\end{aligned}
$$

From the conditions (i), (ii), (3.6) and (3.12), we have

$$\lim_{n\to\infty}\|\rho_n - x_n\| = 0. \tag{3.14}$$

On the other hand, we have $\|\rho_n - W_n\rho_n\| \le \|x_n - \rho_n\| + \|x_n - y_n\| + \|y_n - W_n\rho_n\|$. It follows from (3.5), (3.7) and (3.14) that $\lim_{n\to\infty}\|W_n\rho_n - \rho_n\| = 0$. From Remark 3.1 of [23], we have for any $\varepsilon > 0$, there is N such that $\|W\rho - W_n\rho\| \le \varepsilon$ for all $\rho \in \{\rho_n\}$ and for all $n \ge N$. Therefore, we have $\|W\rho_n - W_n\rho_n\| \to 0$ as $n \to \infty$. Notice that

$$\|W\rho_n - \rho_n\| \le \|W_n\rho_n - \rho_n\| + \|W_n\rho_n - W\rho_n\|,$$

from which it follows that

$$\lim_{n\to\infty}\|W\rho_n - \rho_n\| = 0. \tag{3.15}$$

Since $P_F(\gamma f + (I-B))$ is a contraction, we find that $P_F(\gamma f + (I-B))$ has a unique fixed point, say $q \in H$. That is, $q = P_F(\gamma f + (I-B))(q)$. To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle = \lim_{i\to\infty}\langle \gamma f(q) - Bq, x_{n_i} - q\rangle.$$

As $\{x_{n_i}\}$ is bounded, we have that there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to p. We may assume that without loss of generality that $x_{n_i} \rightharpoonup p$. Hence we have $p \in F$. Indeed, let us first show that $p \in VI(C,A)$. Put

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1, & w_1 \in C\\ \emptyset, & w_1 \notin C. \end{cases}$$

Since A is relaxed (γ, r)-cocoercve and the condition (iii), we have

$$\langle Ax - Ay, x - y\rangle \geq (-\gamma)\|Ax - Ay\|^2 + r\|x - y\|^2 \geq (r - \gamma\mu^2)\|x - y\|^2 \geq 0,$$

which yields that A is monotone. Thus T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Aw_1 \in N_C w_1$ and $\rho_n \in C$, we have

$$\langle w_1 - \rho_n, w_2 - Aw_1\rangle \geq 0.$$

On the other hand, from $\rho_n = P_C(I - r_n A)x_n$, we have $\langle w_1 - \rho_n, \rho_n - (I - r_n A)x_n\rangle \geq 0$ and hence $\langle w_1 - \rho_{n_i}, w_2\rangle \geq \langle w_1 - \rho_{n_i}, A\rho_{n_i} - Ax_{n_i}\rangle - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - x_{n_i}}{r_{n_i}}\rangle$, which implies that $\langle w_1 - p, w_2\rangle \geq 0$. We have $p \in T^{-1}0$ and hence $p \in VI(C, A)$. Next, let us show $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Since Hilbert spaces are Opial's spaces, from (3.15), we have

$$\begin{aligned}
\liminf_{i\to\infty} \|\rho_{n_i} - p\| <& \liminf_{i\to\infty} \|\rho_{n_i} - Wp\| \\
=& \liminf_{i\to\infty} \|\rho_{n_i} - W\rho_{n_i} + W_n\rho_{n_i} - Wp\| \\
\leq& \liminf_{i\to\infty} \|W\rho_{n_i} - Wp\| \\
\leq& \liminf_{i\to\infty} \|\rho_{n_i} - p\|,
\end{aligned}$$

which derives a contradiction. Thus, we have $p \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. On the other hand, we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle \gamma f(q) - Bq, x_n - q\rangle =& \lim_{n\to\infty} \langle \gamma f(q) - Bq, x_{n_i} - q\rangle \\
=& \langle \gamma f(q) - Bq, p - q\rangle \leq 0.
\end{aligned} \tag{3.16}$$

It follows from Lemma 2.5 that

$$\begin{aligned}
\|y_n - q\|^2 \leq& \frac{(1 - \beta_n\bar{\gamma})^2 + \beta_n\gamma\alpha}{1 - \beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{2\beta_n}{1 - \alpha_n\gamma\alpha}\langle \gamma f(q) - Bq, y_n - q\rangle \\
=& \frac{(1 - 2\beta_n\bar{\gamma} + \beta_n\alpha\gamma)}{1 - \beta_n\gamma\alpha}\|x_n - q\|^2 + \frac{\beta_n^2\bar{\gamma}^2}{1 - \beta_n\gamma\alpha}\|x_n - q\|^2 \\
&+ \frac{2\beta_n}{1 - \beta_n\gamma\alpha}\langle \gamma f(q) - Bq, y_n - q\rangle \\
\leq& [1 - \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n\gamma\alpha}]\|x_n - q\|^2 \\
&+ \frac{2\beta_n(\bar{\gamma} - \alpha\gamma)}{1 - \beta_n\gamma\alpha}[\frac{1}{\bar{\gamma} - \alpha\gamma}\langle \gamma f(q) - Bq, y_n - q\rangle + \frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}M_5],
\end{aligned} \tag{3.17}$$

where M_5 is an appropriate constant. On the other hand, we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)(y_n-p)\|^2 \\ &\leq \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2 \end{aligned} \tag{3.18}$$

Substitute (3.17) into (3.18) yields that

$$\begin{aligned} \|x_{n+1}-p\|^2 \leq{}& [1-(1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n-q\|^2 \\ &+(1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(q)-Aq,y_n-q\rangle+\frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_5]. \end{aligned} \tag{3.19}$$

Put $l_n=(1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha_n\gamma)}{1-\beta_n\alpha\gamma}$ and $t_n=\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(q)-Aq,y_n-q\rangle+\frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M_5$. That is,

$$\|x_{n+1}-q\|^2 \leq (1-l_n)\|x_n-q\|^2+l_nt_n. \tag{3.20}$$

Notice that

$$\begin{aligned} \langle\gamma f(q)-Aq,y_n-q\rangle &= \langle\gamma f(q)-Aq,y_n-x_n\rangle+\langle\gamma f(q)-Aq,x_n-q\rangle \\ &\leq \|\gamma f(q)-Aq\|\|y_n-x_n\|+\langle\gamma f(q)-Aq,x_n-q\rangle. \end{aligned}$$

From (3.5) and (3.16) that

$$\limsup_{n\to\infty}\langle\gamma f(q)-Aq,y_n-q\rangle \leq 0.$$

Apply Lemma 2.4 to (3.20) to conclude $x_n \to q$ as $n\to\infty$.

Conflict of Interests

The author declares that there is no conflict of interests.

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