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A NEW THREE-STEP ITERATIVE PROCEDURE WITH ERRORS FOR APPROXIMATING FIXED POINTS OF MULTIVALUED QUASI-NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we introduce a new three-step iterative procedures with errors for approximating a common fixed point of multivalued quasi-nonexpansive mappings. Srong and weak convergence theorems are established in a uniformly convex Banach space.

Keywords: Iterative procedure; Fixed point; Analysis; Quasi-nonexpansive mapping; Convergence theorem.

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1. Introduction

Throughout this paper, we assume that E is a Banach space with the norm $\| \ \|$ and $\mathbb{N}$ is the set of all positive integers. Let D be a nonempty subset of E. The set D is said to be proximinal if for each $x \in E$, there exists an element $y \in D$ such that $\|x-y\| = d(x,D)$, where $d(x,D) = \inf\{\|x-z\| : z \in D\}$. It is known that a weakly compact convex subset of a Banach space and closed convex subsets of a uniformly convex Banach space are proximinial [8].

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We denote $CB(D)$, $C(D)$ and $P(D)$ by the families of nonempty closed and bounded subsets, nonempty compact subsets and nonempty proximinal bounded subsets of D, respectively. Let H be the Hausdorff metric induced by the metric d of E and given by

$$H(A,B) = \max\left\{ \sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A) \right\}$$

for $A, B \in CB(E)$. A multivalued mapping $T : D \to P(D)$ is said to be contraction if there exists a constant $k \in [0,1)$ such that for all $x, y \in D$,

$$H(Tx,Ty) \leq k\|x-y\|,$$

and nonexpansive if [15] $H(Tx,Ty) \leq \|x-y\|$, $x,y \in D$ and quasi-nonexpansive if $F(T) \neq \phi$ and $H(Tx,Ty) \leq \|x-y\|$ for all $x,y \in D$ and all $p \in F(T)$ [21]. A point $x \in D$ is called a fixed point of a multivalued mapping T if $x \in Tx$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in D : Tx = x\}$. It is clear that every nonexpansive multi-valued map T with $F(T) \neq \phi$ is quasi-nonexpansive. But there exist quasi-nonexpansive mappings that are not nonexpansive. It is known that if T is a quasi-nonexpansive multi-valued map, then $F(T)$ is closed [12].

A multivalued nonexpansive mapping $T : D \to CB(D)$ where D a subset of E, is said to satisfy condition (I) if there exists a nondecreasing function $f : (0,\infty) \to 0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $d(x,Tx) \geq f(d(x,F(T))$ for all $x \in D$ (see [9]).

The mapping $T : D \to CB(D)$ is said to be hemi-compact if, for any sequence $\{x_n\}$ in D such that $\lim_{n\to\infty} d(x_n,Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{n\to\infty} x_{n_k} = x \in D$.

A multivalued mapping $T : D \to P(E)$ is called demiclosed at $y \in K$ if for any sequence $\{x_n\}$ in D weakly convergent to an element x and $y_n \in Tx_n$ strongly convergent to y, we have $y \in Tx$.

A Banach space E is said to satisfy the Opial's condition [6] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying this condition are Hilbert spaces and l^p spaces $(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opail's condition.

2. Preliminaries

Fixed point iteration processes for approximating fixed points of nonexpansive mappings in Banach spaces have been studied by various authors using the Mann iteration processes or the Ishikawa ietration process. The study of fixed points for multivalued contractions and nonexpansive mappings using the Hausdorff metric was initiated by Markin [2] (see also [1]). Later, an interesting and rich fixed point theory for such maps was developed which has applications in control theory, convex optimization, differential inclusion and economics (see [3]). Morever, the existence of fixed points for multivalued nonexpansive mappings in uniformly convex Banach spaces was proved by Lim [4].

The theory of multivalued nonexpansive mappings is harder than the corresponding theory of single valued nonexpansive mappings. Different iterative processes have been used to approximate fixed points of multivalued nonexpansive mappings; in particular, Sastry and Babu [5] considered the following:

Let D be a nonempty convex subset of E, $T : D \to P(D)$ be a multivalued mapping with $p \in Tp$.

(i) The sequences of Mann iterates is defined by $x_1 \in D$,

$$x_{n+1} = (1 - a_n)x_n + a_n y_n, \ n \in \mathbb{N},$$

where $y_n \in Tx_n$ is such that $\|y_n - p\| = d(p, Tx_n)$, and $\{a_n\}$ is sequence in (0,1) satisfying $\sum a_n = \infty$.

(ii) The sequence of Ishikawa iterates is defined by $x_1 \in D$

$$\begin{cases} y_n = (1 - b_n)x_n + b_n z_n, \\ x_{n+1} = (1 - a_n)x_n + a_n u_n, \ n \in \mathbb{N}, \end{cases}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ are such that $\|z_n - p\| = d(p, Tx_n)$ and $\|u_n - p\| = d(p, Ty_n)$, and $\{a_n\}$, $\{b_n\}$ are real sequences of numbers with $0 \leq a_n,\ b_n < 1$ satisying $\lim_{n\to\infty} b_n = 0$ and $\sum a_n b_n = \infty$.

The following is a useful Lemma due to Nadler [1].

Lemma 2.1. *Let A, $B \in CB(E)$ and $a \in A$. If $\eta > 0$, then there exists $b \in B$ such that $d(a,b) \leq H(A,B) + \eta$.*

Based on the above lemma, Song and Wang [12] modified the iteration scheme used in [13] and improved the results presented therein. This scheme reads as follows:

(iii) The sequence of Ishikawa iterates is defined $x_1 \in D$

$$\begin{cases} y_n = (1 - b_n) x_n + b_n z_n, \\ x_{n+1} = (1 - a_n) x_n + a_n u_n,\ n \in \mathbb{N}, \end{cases}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ are such that $\|z_n - u_n\| \leq H(Tx_n, Ty_n) + \eta_n$ and $\|z_{n+1} - u_n\| \leq H(Tx_{n+1}, Ty_n) + \eta_n$, $\eta_n \in (0, \infty)$ and $\{a_n\}$, $\{b_n\}$ are real sequences of numbers with $0 \leq a_n,\ b_n \leq 1$ satisfying $\lim_{n\to\infty} b_n = 0$ and $\sum a_n b_n = \infty$.

It is to be noted that Song and Wang [12] need the condition $Tp = \{p\}$ in order to prove their Theorem 1. Actually, Panyanak [13] proved some results using Ishikawa type iteration process without this condition. Song and Wang [12] showed that without this condition his process was not well-defined. They reconstructed the process using the condition $Tp = \{p\}$ which made it well-defined. Such a condition was also used by Jung [14]. They defined $P_T(x) = \{y \in Tx : ||x - y|| = d(x, Tx)\}$ for a multivalued mapping $T : D \to P(D)$. They also proved a couple of strong convergence results using Ishikawa type iteration process

In 2000, Noor [16] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Suantai [19] defined a new three-step iterations which is an extension of Noor iterations and gave some weak and strong convergence theorems of such iterations for asymptotically nonexpansive mappings in uniformly convex Banach spaces.

Recently, Cholamjiak and Suantai [17] introduced two new iterative procedures with errors for two-quasi-nonexpansive multi-valued maps and proved strong convergence theorems of the proposed iterations in uniformly convex Banach spaces. They [18] also introduced another new two- step iterative scheme with errors for finding a common fixed point of two quasi-nonexpansive multi-valued maps in Banach spaces. The results obtained in [18] are extensions of those of Shahzad and Zegeye [9]. They estimate the following scheme:

Let D be a nonempty convex subset of a Banach space E, α_n, β_n, $\alpha_n^{'}$, $\beta_n^{'} \in [0,1]$ and $\{u_n\}$, $\{v_n\}$ are bounded sequences in D. Let T_1,T_2 be two quasi-nonexpansive multi-valued maps from D into $P(D)$ and $P_{T_i}x = \{y\varepsilon T_i x = ||x-y|| = d(x,T_i x)\}$, $i = 1,2$. Let $\{x_n\}$ be the sequence defined by $x_0 \varepsilon D$,

$$\begin{cases} y_n = \alpha_n^{'} z_n^{'} + \beta_n' x_n + \left(1-\alpha_n^{'} - \beta_n'\right) u_n, n \geq 0, \\ x_{n+1} = \alpha_n z_n + \beta_n x_n + (1-\alpha_n - \beta_n) v_n,\ n \geq 0, \end{cases}$$

where $z_n^{'} \in P_{T_1} x_n$ and $z_n \in P_{T_2} y_n$.

In this paper, we introduce and study new three-step iterative procedure with errors to approximate fixed points of a multivalued quasi-nonexpansive mappings in Banach spaces. We define our iteration scheme as follows:

$$\begin{cases} x_1 \in D, \\ x_{n+1} = \alpha_n u_n + \beta_n v_n + \gamma_n s_n, \\ y_n = \alpha_n^{'} v_n + \beta_n' w_n + \gamma_n' r_n, \\ z_n = \alpha_n^{''} x_n + \beta_n^{''} v_n + \gamma_n'' t_n, , n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $v_n \in P_T(x_n)$, $u_n \in P_T(y_n)$, $w_n \in P_T(z_n)$ and $\alpha_n, \alpha_n^{'}, \alpha_n^{''}, \beta_n, \beta_n', \beta_n^{''}, \gamma_n, \gamma_n', \gamma_n'' \in [a,b] \subset (0,1)$ and $\alpha_n + \beta_n + \gamma_n = \alpha_n^{'} + \beta_n' + \gamma_n' = \alpha_n^{''} + \beta_n^{''} + \gamma_n'' = 1$.Also $\{s_n\}$,$\{r_n\}$ and $\{t_n\}$ are bounded in D.

Lemma 2.2. [20] *Let $\{s_n\}$ and $\{t_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$s_{n+1} \leq s_n + t_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n\to\infty} s_n$ exists.

Lemma 2.3. [8] *Let E be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences of E such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r,\ \limsup_{n\to\infty} \|y_n\| \leq r \text{ and } \lim_{n\to\infty} \|t_n x_n + (1-t_n) y_n\| = r$$

hold for some $r \geq 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.4. [7] *Let* $T : K \to P(K)$ *be a multivalued mapping and* $P_T(x) = \{y \in Tx : ||x-y|| = d(x,Tx)\}$. *Then the following are equivalent.*

(1) $x \in F(T)$;

(2) $P_T(x) = \{x\}$;

(3) $x \in F(P_T)$. *Morever,* $F(T) = F(P_T)$.

3. Main results

We start with the following important lemma.

Lemma 3.1. *Let E be a uniformly convex Banach space and D a nonempty closed convex subset of E. Let* $T : D \to P(D)$ *be a quasi-nonexpansive multivalued mapping such that* $F(T) \neq \emptyset$ *and* P_T *is a nonexpansive mapping. Let* $\{x_n\}$ *be the sequence as defined in (2.1). Assume that* $0 < l \leq \alpha_n, \alpha_n^{'}, \alpha_n^{''} \leq k < 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n^{'} < \infty$, $\sum_{n=1}^{\infty} \gamma_n^{''} < \infty$. *Then*

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in F$.

(2) For any $p \in F$ $\lim_{n\to\infty} d\ (x_n, Tx_n) = 0$.

Proof. Let $p \in F(T)$. Then $p \in P_T(p) = \{p\}$ by Lemma 2.4. Since s_n, r_n and t_n are bounded, therefore exists $M > 0$ such that $\max\{\sup_{n\in N} \|t_n - p\|, \sup_{n\in N} \|s_n - p\|, \sup_{n\in N} \|r_n - p\|\} \leq M$. It follows from 2.1 that

$$\begin{aligned} \|z_n - p\| &\leq \alpha_n^{''} \|x_n - p\| + \beta_n^{''} \|v_n - p\| + \gamma_n^{''} \|t_n - p\| \\ &\leq \alpha_n^{''} \|x_n - p\| + \beta_n^{''} d(v_n, P_T(p)) + \gamma_n^{''} M \\ &\leq \alpha_n^{''} \|x_n - p\| + \beta_n^{''} H(P_T(x_n), P_T(p)) + \gamma_n^{''} M \\ &\leq \alpha_n^{''} \|x_n - p\| + \beta_n^{''} \|x_n - p\| + \gamma_n^{''} M \\ &= \left(\alpha_n^{''} + \beta_n^{''}\right) \|x_n - p\| + \gamma_n^{''} M \\ &\leq \|x_n - p\| + \gamma_n^{''} M, \end{aligned} \tag{3.1}$$

$$\begin{aligned}\|y_n-p\| &\le \alpha_n^{'}\|v_n-p\|+\beta_n^{'}\|w_n-p\|+\gamma_n^{'}\|r_n-p\| \\ &\le \alpha_n^{'}d\left(v_n,P_T\left(p\right)\right)+\beta_n^{'}d\left(w_n,P_T\left(p\right)\right)+\gamma_n^{'}M \\ &\le \alpha_n^{'}H\left(P_T\left(x_n\right),P_T\left(p\right)\right)+\beta_n^{'}H\left(P_T\left(z_n\right),P_T\left(p\right)\right)+\gamma_n^{'}M \\ &\le \alpha_n^{'}\|x_n-p\|+\beta_n^{'}\|z_n-p\|+\gamma_n^{'}M \\ &\le \alpha_n^{'}\|x_n-p\|+\beta_n^{'}\|x_n-p\|+\gamma_n^{'}M+\beta_n^{'}\gamma_n^{''}M \\ &\le \|x_n-p\|+\gamma_n^{'}M+\beta_n^{'}\gamma_n^{''}M\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\|x_{n+1}-p\| &\le \alpha_n\|u_n-p\|+\beta_n\|v_n-p\|+\gamma_n\|s_n-p\| \\ &\le \alpha_n d\left(u_n,P_T\left(p\right)\right)+\beta_n d\left(v_n,P_T\left(p\right)\right)+\gamma_n M \\ &\le \alpha_n H\left(P_T\left(y_n\right),P_T\left(p\right)\right)+\beta_n H\left(P_T\left(x_n\right),P_T\left(p\right)\right)+\gamma_n M \\ &\le \alpha_n\|y_n-p\|+\beta_n\|x_n-p\|+\gamma_n M \\ &\le \alpha_n\|x_n-p\|+\beta_n\|x_n-p\|+\gamma_n M+\alpha_n\gamma_n^{'}M+\alpha_n\beta_n^{'}\gamma_n^{''}M \\ &\le \|x_n-p\|+\left(\gamma_n+\alpha_n\gamma_n^{'}+\alpha_n\beta_n^{'}\gamma_n^{''}\right)M \\ &= \|x_n-p\|+\in_n,\end{aligned} \tag{3.3}$$

where $\in_n=\left(\gamma_n+\alpha_n\gamma_n^{'}+\alpha_n\beta_n^{'}\gamma_n^{''}\right)M$. From hipotez (ii), $\sum_{n=1}^{\infty}\in_n<\infty$. By Lemma 2.2, we have $\lim_{n\to\infty}\|x_n-p\|$ exists for each $p\in F$.

Now, we show that $\lim_{n\to\infty}d\ (x_n,Tx_n)=0$. To this end, we show that

$$\lim_{n\to\infty}d(x_n,T(x_n))\le\lim_{n\to\infty}d(x_n,P_T(x_n))\le\lim_{n\to\infty}\|x_n-v_n\|=0.$$

By hipotez (1), since $\lim_{n\to\infty}\|x_n-p\|$ exists and therefore $\{x_n\}$, $\{y_n\}$ ve $\{z_n\}$ sequences are bounded. We suppose that $\lim_{n\to\infty}\|x_n-p\|=c$ for some $c\ge 0$. Let

$$S=\left\{\sup_{n\in N}\|s_n-v_n\|,\sup_{n\in N}\|r_n-w_n\|,\sup_{n\in N}\|t_n-v_n\|\right\}$$

$\limsup_{n\to\infty}\|v_n-p\|\le\limsup_{n\to\infty}H\left(P_T\left(x_n\right),P_T(p)\right)\le\limsup_{n\to\infty}\|x_n-p\|=c$. So

$$\limsup_{n\to\infty}\|v_n-p\|\le c. \tag{3.4}$$

Similarly, we have $\limsup_{n\to\infty}\|u_n-p\|\le c$, $\limsup_{n\to\infty}\|w_n-p\|\le c$. Taking lim sup on both sides of (3.1) and (3.2), respectively, we obtain

$$\limsup_{n\to\infty}\|z_n-p\|\le c \tag{3.5}$$

$\limsup_{n\to\infty}\|y_n-p\|\le c$. Next, we consider

$$\begin{aligned}\|u_n-p+\gamma_n(s_n-v_n)\| &\le \|u_n-p\|+\gamma_n\|s_n-v_n\|\\ &\le d(u_n,P_T(p))+\gamma_n S\\ &\le H(P_T(y_n),P_T(p))+\gamma_n S\\ &\le \|y_n-p\|+\gamma_n S.\end{aligned}$$

It follows that

$$\limsup_{n\to\infty}\|u_n-p+\gamma_n(s_n-v_n)\|\le c. \tag{3.6}$$

Similarly, we have

$$\begin{aligned}\|v_n-p+\gamma_n(s_n-v_n)\| &\le \|v_n-p\|+\gamma_n\|s_n-v_n\|\\ &\le d(v_n,P_T(p))+\gamma_n S\\ &\le H(P_T(x_n),P_T(p))+\gamma_n S\\ &\le \|x_n-p\|+\gamma_n S.\end{aligned}$$

This implies that $\limsup_{n\to\infty}\|v_n-p+\gamma_n(s_n-v_n)\|\le c$. Since

$$\lim_{n\to\infty}\|\alpha_n(u_n-p+\gamma_n(s_n-v_n))+(1-\alpha_n)(v_n-p+\gamma_n(s_n-v_n))\|=\lim_{n\to\infty}\|x_{n+1}-p\|=c,$$

we obtain from Lemma 2.3 that

$$\lim_{n\to\infty}\|u_n-v_n\|=0 \tag{3.7}$$

$$\begin{aligned}\|x_{n+1}-p\| &= \|v_n-p+\alpha_n(u_n-v_n)+\gamma_n(s_n-v_n)\|\\ &\le \|v_n-p\|+\alpha_n\|u_n-v_n\|+\gamma_n\|s_n-v_n\|\\ &\le \|v_n-p\|+\alpha_n\|u_n-v_n\|+\gamma_n S.\end{aligned}$$

This implies that $c \leq \liminf_{n\to\infty} \|v_n - p\|$ and thus together with (3.4) inequality

$$c \leq \liminf_{n\to\infty} \|v_n - p\| \leq \limsup_{n\to\infty} \|v_n - p\| \leq c.$$

We get $\lim_{n\to\infty} \|v_n - p\| = c$. Also,

$$\begin{aligned} \|v_n - p\| &\leq \|v_n - u_n\| + \|u_n - p\| \\ &\leq \|v_n - u_n\| + d(u_n, P_T(p)) \\ &\leq \|v_n - u_n\| + H(P_T(y_n), P_T(p)) \\ &\leq \|v_n - u_n\| + \|y_n - p\|. \end{aligned}$$

It implies that $c \leq \liminf_{n\to\infty} \|y_n - p\|$. Thus $c \leq \liminf_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq c$, it follows that $\lim_{n\to\infty} \|y_n - p\| = c$. Note that

$$y_n - p = \alpha_n' \left(v_n - p + \gamma_n' (r_n - w_n)) + \left(1 - \alpha_n'\right) (w_n - p + \gamma_n' (r_n - w_n)) \right).$$

It follows that

$$\lim_{n\to\infty} \left\| \alpha_n' \left(v_n - p + \gamma_n' (r_n - w_n)) + \left(1 - \alpha_n'\right) (w_n - p + \gamma_n' (r_n - w_n)) \right) \right\| = c.$$

Moreover, we get

$$\begin{aligned} \left\| v_n - p + \gamma_n' (r_n - w_n) \right\| &\leq \|v_n - p\| + \gamma_n' \|r_n - w_n\| \\ &\leq d(v_n, P_T(p)) + \gamma_n' S \\ &\leq H(P_T(x_n), P_T(p)) + \gamma_n' S \\ &\leq \|x_n - p\| + \gamma_n' S. \end{aligned}$$

This yields that $\limsup_{n\to\infty} \|v_n - p + \gamma_n' (r_n - w_n)\| \leq c$. Similarly, we have

$$\begin{aligned} \left\| w_n - p + \gamma_n' (r_n - w_n) \right\| &\leq \|w_n - p\| + \gamma_n' \|r_n - w_n\| \\ &\leq d(w_n, P_T(p)) + \gamma_n' S \\ &\leq H(P_T(z_n), P_T(p)) + \gamma_n' S \\ &\leq \|z_n - p\| + \gamma_n' S. \end{aligned}$$

This implies that $\limsup_{n\to\infty}\|w_n - p + \gamma_n'(r_n - w_n)\| \le c$. Again by Lemma 2.3, we have

$$\lim_{n\to\infty}\|v_n - w_n\| = 0. \tag{3.8}$$

It follows from (3.8) that

$$\begin{aligned}
\|v_n - p\| &\le \|v_n - p\| +' \|w_n - p\| \\
&\le \|v_n - w_n\| + d(w_n, P_T(p)) \\
&\le \|v_n - w_n\| + H(P_T(z_n), P_T(p)) \\
&\le \|v_n - w_n\| + \|z_n - p\|.
\end{aligned}$$

It implies that $c \le \liminf_{n\to\infty}\|z_n - p\|$ and thus together with (3.5)

$$c \le \liminf_{n\to\infty}\|z_n - p\| \le \limsup_{n\to\infty}\|z_n - p\| \le c.$$

This implies that $\lim_{n\to\infty}\|z_n - p\| = c$. Moreover,

$$z_n - p = \alpha_n''\left(x_n - p + \gamma_n''(t_n - v_n)\right) + \left(1 - \alpha_n''\right)\left(v_n - p + \gamma_n''(t_n - v_n)\right).$$

It follows that

$$\lim_{n\to\infty}\left\|\alpha_n''\left(x_n - p + \gamma_n''(t_n - v_n)\right) + \left(1 - \alpha_n''\right)\left(v_n - p + \gamma_n''(t_n - v_n)\right)\right\| = c.$$

Also

$$\begin{aligned}
\left\|x_n - p + \gamma_n''(t_n - w_n)\right\| &\le \|x_n - p\| + \gamma_n''\|t_n - v_n\| \\
&\le \|x_n - p\| + \gamma_n'' S,
\end{aligned}$$

which implies that $\limsup_{n\to\infty}\left\|x_n - p + \gamma_n''(t_n - v_n)\right\| \le c$. Similarly, we get

$$\begin{aligned}
\left\|v_n - p + \gamma_n''(t_n - v_n)\right\| &\le \|v_n - p\| + \gamma_n''\|t_n - v_n\| \\
&\le d(v_n, P_T(p)) + \gamma_n'' S \\
&\le H(P_T(x_n), P_T(p)) + \gamma_n'' S \\
&\le \|x_n - p\| + \gamma_n' S.
\end{aligned}$$

This implies that

$$\limsup_{n\to\infty}\left\|x_n - p + \gamma_n^{''}(t_n - v_n)\right\| \le c.$$

Hence by Lemma 2.3, we have $\lim_{n\to\infty}\|x_n - v_n\| = 0$. So, the proof is completed.

Theorem 3.2. *Let E be a uniformly convex Banach space and D a nonempty closed convex subset of E . Let $T : D \to P(D)$ be a quasi-nonexpansive multivalued mapping such that $F(T) \neq \emptyset$ and P_T is a nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (2.1). Assume that*

(1) T satisfys Condition(I);
(2) $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n^{'} < \infty$ ve $\sum_{n=1}^{\infty}\gamma_n^{''} < \infty$;
(3) $0 < l \le \alpha_n, \alpha_n^{'}, \alpha_n^{''} \le k < 1$.

Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to a fixed point of F.

Proof. Let $p \in F(T)$. Then as in the proof Lemma 3.1, $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p \in F(T)$ and it can be shown that $d(x_n, Tx_n) \to 0$ as $n \to \infty$. Show that $\{x_n\}$ converges strongly to q for some $q \in F$. Since that T satisfies the condition (I), we have $d(x_n, F) = 0$. Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\} \subset F$ such that $\|x_{n_k} - p_k\| < \frac{1}{2^k}$ for all k. From (3.3), we obtain

$$\begin{aligned}\left\|x_{n_{k+1}} - p\right\| &\le \left\|x_{n_{k+1}-1} - p\right\| + \in_{n_{k+1}-1}\\ &\le \left\|x_{n_{k+1}-2} - p\right\| + \in_{n_{k+1}-2} + \in_{n_{k+1}-1}\\ &\;\;\vdots\\ &\le \|x_{n_k} - p\| + \sum_{i=0}^{n_{k+1}-n_k-1} \in_{n_k+i}\end{aligned}$$

for all $p \in F$. This implies that

$$\left\|x_{n_{k+1}} - p\right\| < \|x_{n_k} - p\| + \sum_{i=0}^{n_{k+1}-n_k-1} \in_{n_k+i} < \frac{1}{2^k} + \sum_{i=0}^{n_{k+1}-n_k-1} \in_{n_k+i}.$$

Now, we shall show that $\{p_k\}$ is a Cauchy sequence in D. Noted that

$$\begin{aligned} \|p_{k+1}-p_k\| &\leq \left\|p_{k+1}-x_{n_{k+1}}\right\|+\left\|x_{n_{k+1}}-p_k\right\| \\ &< \frac{1}{2^{k+1}}+\frac{1}{2^k}+\sum_{i=0}^{n_{k+1}-n_k-1} \in_{n_k+i} \\ &< \frac{1}{2^{k-1}}+\sum_{i=0}^{n_{k+1}-n_k-1} \in_{n_k+i} . \end{aligned}$$

This implies that $\{p_k\}$ is a Cauchy sequence in D and thus $q \in D$. then we show that $q \in F$. therefore

$$dist\,(p_k,T(q)) \leq dist(p_k,P_T(q)) \leq H(P_T(p_k),P_T(q)) \leq \|p_k-q\|$$

$p_k \to q$ as $n \to \infty$, it follows that $dist(q,Tq)=0$ and thus $q \in F$. P_T is a nonexpansive mapping so $F(P_T)$ is closed. Therefore, $q \in F(T)=F(P_T)$. It implies by $\|x_{n_k}-p_k\| < \frac{1}{2^k}$ that $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty}\|x_n-q\|$ exists, it follows $\{x_n\}$ converges strongly to q. This completes the proof.

Theorem 3.3. *Let E be a uniformly convex Banach space and D a nonempty closed convex subset of E . Let $T : D \to P(D)$ be a quasi-nonexpansive multivalued mapping such that $F(T) \neq \emptyset$ and P_T is a nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (2.1). Assume that*

(1) T hemicompact;
(2) $\sum_{n=1}^{\infty}\gamma_n < \infty$, $\sum_{n=1}^{\infty}\gamma_n^{'} < \infty$ ve $\sum_{n=1}^{\infty}\gamma_n^{''} < \infty$;
(3) $0 < l \leq \alpha_n, \alpha_n^{'}, \alpha_n^{''} \leq k < 1$.

Then the sequence $\{x_n\}$ generated by (2.1) converges strongly to a fixed point of F.

Proof. Let $p \in F(T)$. Then as in the proof Lemma 3.1 $\lim_{n\to\infty}\|x_n-p\|$ exists for any $p \in F(T)$ and it can be shown that $d(x_n,Tx_n) \to 0$ as $n \to \infty$. Since T is hemicompact, we may assume that $x_{n_k} \to q$ for some $q \in D$. Note that

$$\begin{aligned} d(q,T(q)) &\leq d(q,P_T(q)) \leq \|q-x_{n_k}\|+\|x_{n_k}-z_{n_k}\|+d(z_{n_k},P_T(q)) \\ &\leq \|q-x_{n_k}\|+\|x_{n_k}-z_{n_k}\|+H(P_T(x_{n_k}),P_T(q)) \\ &\leq 2\,\|q-x_{n_k}\|+\|x_{n_k}-z_{n_k}\| \to 0. \end{aligned}$$

This implies that $d(q,T(q))=0$ and thus $q\in F(T)$. Since $\lim_{n\to\infty}\|x_n-p\|$ exists for each $p\in F(T)$, it follows that $x_n\to q$ as $n\to\infty$. This completes the proof.

Now we approximate fixed points of the mapping T through weak convergence of the sequence $\{x_n\}$ defined in (2.1).

Theorem 3.4. *Let E be a uniformly convex Banach space satisfying Opial's condition and D a nonempty closed convex subset of E. Let $T:D\to P(D)$ be a quasi-nonexpansive multivalued mapping such that $F(T)\neq\emptyset$ and P_T is a nonexpansive mapping. Let $\{x_n\}$ be the sequence as defined in (2.1). Assume that $\sum_{n=1}^{\infty}\gamma_n<\infty$, $\sum_{n=1}^{\infty}\gamma_n^{'}<\infty$ and $\sum_{n=1}^{\infty}\gamma_n^{''}<\infty$ and $0<l\leq\alpha_n$, $\alpha_n^{'}$, $\alpha_n^{''}\leq k<1$. Let $I-P_T$ be demiclosed with respect to zero, then $\{x_n\}$ converges weakly to fixed point of T.*

Proof. Let $p\in F(T)=F(P_T)$. As in the proof of Lemma 3.1, $\lim_{n\to\infty}\|x_n-p\|$ exists. Now we prove that$\{x_n\}$ has a unique weak subsequently limit in $F(T)$. To prove this, let z_1 and z_2 be weak ences limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By (3.9), there exists $v_n\in Tx_n$ such that $\lim_{n\to\infty}\|x_n-v_n\|=0$. Since $I-P_T$ is demiclosed with respect to zero, therefore we obtain $z_1\in F(P_T)=F(T)$. In the same way, we can prove that $z_2\in F(T)$.

Next, we prove the uniqueness. To his end, we suppose that $z_1\neq z_2$. By the Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n-z_1\| &= \lim_{n_i\to\infty}\|x_{n_i}-z_1\| \\ &< \lim_{n_i\to\infty}\|x_{n_i}-z_2\| \\ &= \lim_{n\to\infty}\|x_n-z_2\| \\ &= \lim_{n_j\to\infty}\left\|x_{n_j}-z_2\right\| \\ &< \lim_{n_i\to\infty}\left\|x_{n_j}-z_1\right\| \\ &= \lim_{n\to\infty}\|x_n-z_1\|.\end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a point in F. This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

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