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FIXED POINTS OF OCCASIONALLY WEAKLY BIASED MAPPINGS

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Abstract. Common fixed point results due to Pant et al. [Pant et al., Weak reciprocal continuity and fixed point theorems, Ann Univ Ferrara, **57**(1), (2011), 181-190] are extended to a class of non commuting operators called occasionally weakly biased pair introduced by Hussain et al.[N. Hussain et al., Common fixed points for $\mathcal{JH}$-operators and occasionally weakly biased pairs under relaxed conditions, Nonlinear Analysis, **74**, (2011), 2133-2140]. We also provide illustrative examples to justify the improvements.

Keywords: compatible maps, weakly compatible mappings, occasionally weakly compatible mappings, occasionally weakly biased , coincidence points and common fixed point.

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1. Introduction and Preliminaries

In 1975, Jungck [4] proved a common fixed point theorem for commuting mappings by extending the Banach contraction principle. Sessa [14] introduced the concept of weakly commuting mappings. The study of common fixed points of compatible mappings is an active area for many authors since Jungck[5] introduced the notion of compatible mappings. Non-compatible mappings can be extended to the class of non-expansive or

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Lipshitz type of mappings pair without assuming continuity of the mappings involved or completeness of the underlying space. Work along these lines has been initiated recently by Pant[8, 10] and Pant et al.[11]. Recently, Aamri and Matouwakil[1] introduced the concept of property (E.A) (or tangential mappings, see Sastry and Murthy [13] which is contained in the compatible and non-compatible mappings. In the metric fixed point theory, the results obtained by using the notion of non-compatible mappings([8, 10]) or property (E.A)[1] are also equally interesting. Al-Thagafi and Shahzad[2] introduced the concept of occasionally weakly compatible mappings (owc) which is more general than the concept of weakly compatible mappings[7]. Hussain et al.[3] introduced the concept of occasionally weakly biased mappings in order to generalize weakly biased[6] (resp. occasionally weakly compatible mappings[2]). Recently Pant[11] introduced the concept of weakly reciprocally continuous mappings and established common fixed point theorems for non-compatible weakly reciprocally continuous mappings satisfying contractive, non-expansive and Lipschitz type conditions.

Let f and g be two self mappings of a metric space (X, d). We denote $C(f, g) = \{u \in X : fu = gu\}$, the set of coincidence points of f and g, and $cl(fX)$ is the closure of fX.

Definition 1.1[14]. *Mappings f and g are called weakly commuting if*

$$d(fgx, gfx) \leq d(fx, gx), \forall x \in X.$$

Definition 1.2[5]. *Mappings f and g are called compatible mappings if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 1.3([8],[10]). *Mappings f and g are called non-compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either non-zero or non-existent.*

Definition 1.4([1], [13]). *Mappings f and g are said to satisfy property (E.A) if there exists a sequences $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.*

Definition 1.5[9]. *Mappings f and g are called reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.*

Definition 1.6[11]. *Mappings f and g are called weakly reciprocally continuous if* $\lim_{n\to\infty} fgx_n = ft$ *or* $\lim_{n\to\infty} gfx_n = gt$*, whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$ *for some* $t \in X$.

Definition 1.7[16]. *Mappings f and g are called g- reciprocally continuous (resp. f-reciprocally continuous) if* $\lim_{n\to\infty} gfx_n = gt$ *(resp.* $\lim_{n\to\infty} fgx_n = ft$*), whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$ *for some* $t \in X$.

Note that g-reciprocally continuous (or f-reciprocally continuous) mappings are called one sided reciprocally continuous(see [16]). Further, it may be noted that property(E.A)(or tangential mappings) contains the compatible mappings(resp. non-compatible , reciprocally continuous, weakly reciprocally continuous and one sided reciprocally reciprocally continuous) as proper subclasses.

Definition 1.8[10]. *Mappings f and g are called R-weakly commuting at a point x in X if* $d(fgx, gfx) \le Rd(fx, gx)$ *for some* $R > 0$.

Definition 1.9[10]. *Mappings f and g are called pointwise R-weakly commuting on X if given x in X there exists $R > 0$ such that* $d(fgx, gfx) \le Rd(fx, gx)$.

Definition 1.10[12]. *Mappings f and g are called R-weakly commuting of type (A_f), if there exists a positive real number R such that* $d(fgx, ggx) \le Rd(fx, gx), \forall x, y \in X$.

Definition 1.11[12]. *Mappings f and g are called R-weakly commuting of type (A_g), if there exists a positive real number R such that* $d(ffx, gfx) \le Rd(fx, gx), \forall x, y \in X$.

Notice that the weakly commuting and R-weakly commuting (resp. R-weakly commuting of type (A_f) or type (A_g)) of mappings are no longer true if $d(fgx, gfx) \neq 0$, whenever $fx = gx$ for some $x \in X$.

Definition 1.12[7]. *Mappings f and g are called weakly compatible if* $fgx = gfx, \ \forall x \in C(f, g)$.

Definition 1.13[6]. *Mappings f and g are called weakly g-biased if* $d(gfx, gx) \le d(fgx, fx), \forall x \in C(f, g)$.

If the role of f and g are interchanged in above definition, then the mappings are called weakly f-biased. Note that weakly compatible mappings implies weakly biased(i.e. both f- and g-biased) but the converse is not true in general (also see [15],[17]).

Definition 1.14[2]. *Mappings f and g are called occasionally weakly compatible(owc) if $fgx = gfx$ for some $x \in C(f, g)$.*

From the above definition, one may agree that weakly compatible mappings pair implies occasionally weakly compatible but the converse may not be true in general (see [2]).

Definition 1.15[3]. *Mappings f and g are called occasionally weakly g-biased if $d(gfx, gx) \leq d(fgx, fx)$ for some $x \in C(f, g)$.*

If the role of mappings are interchanged, then the mappings pair is called occasionally weakly f-biased. Further, it may be noted that the notions of *woc* and weakly g-biased are occasionally weakly g-biased but the converse do not hold in general(see [3]).

Note that weakly commuting, R-weakly commuting, R-weakly commuting of type (A_f) or (A_g), weakly compatible mappings are commuting at their coincidences points. However, the notions of weakly biased and occasionally weakly biased may not be commuting at the coincidence points. Therefore, the study of common fixed points for weakly biased and occasionally weakly biased are also equally interesting.

Example 1.16. Let $f, g : X \to X$, where $X = [0, 1] \subset \mathbb{R}$ with usual metric d on X. Define

$$fx = \frac{(1+x)}{2},\ x < \frac{1}{2},\ f\frac{1}{2} = \frac{2}{3},\ fx = 0,\ x > \frac{1}{2}$$

$$gx = \frac{(1+2x)}{2}, x < \frac{1}{2},\ g\frac{1}{2} = \frac{2}{3},\ gx = 1,\ x > \frac{1}{2}.$$

Clearly, $f0 = \frac{1}{2} = g0$ and $f\frac{1}{2} = \frac{2}{3} = g\frac{1}{2}$. Therefore, $C(f, g) = \{0, \frac{1}{2}\}$. Also, we have $fg0 = \frac{2}{3} = gf0$ and $fg\frac{1}{2} = 0 \neq gf\frac{1}{2} = 1$. Thus, f and g are *owc* but not weakly compatible mappings. Further, we have

$$d(gf\frac{1}{2}, g\frac{1}{2}) = \frac{1}{3} \leq d(fg\frac{1}{2}, f\frac{1}{2}) = \frac{2}{3}.$$

Therefore, the mappings are weakly g-biased.

Example 1.17. Let $X = \mathbb{R}$ with usual metric d on X and $M = [0, 1] \subset X$. Define $f, g : M \to M$ as

$$f0 = \frac{2}{3} = g0,\ fx = 1 - \frac{x}{2},\ gx = \frac{x}{2}, x \neq 0.$$

Then, $C(f,g)=\{0,1\}$. Also, we have $fg0=f2/3=2/3\neq gf0=1/3$ and $fg1=3/4\neq gf1=1/4$. Consequently,

$$|gf0-g0|=1/3\nleq|f0g-f0|=0$$

and

$$|gf1-g1|=1/4\leq|fg1-f1|=1/4.$$

Therefore, the mappings f and g are occasionally weakly g-biased but neither weakly g-biased(resp. occasionally weakly compatible, R-weakly commuting) nor weakly compatible.

In this paper, our attempt is to extend the results of Pant[11] by employing occasionally weakly biased mappings pair in lieu of R-weakly commuting of type(A_f) or (A_g) without assuming completeness of the underlying space.

2. Main results

Pant et al.[11] proved the following theorem.

Theorem 2.1(Theorem 1[11]). *Let f and g be weakly reciprocally continuous selfmappings of a complete metric space (X,d) such that*

(i) $fX\subset gX$;

(ii) $d(fx,fy)\leq\ a\,d(gx,gy)+\ b\,d(fx,gx)+\ c\,d(fy,gy),\ \forall x,y\in X$,

$0\leq a,b,c<1$ and $0\leq a+b+c<1$.

If f and g are either compatible or R-weakly commuting of type (A_g) or R- weakly commuting of type (A_f) then, f and g have a unique common fixed point.

We relax weakly reciprocally continuous, completeness of the underlying space, compatibility and R-weakly commuting of type (A_g) (or type (A_f)) in above Theorem 2.1 by imposing occasionally weakly g-biased and property(E.A). Now we state as follows.

Theorem 2.2. *Let f and g be self mappings of a metric space (X,d) satisfying the following*

(i) f and g satisfy property (E.A);

(ii) $d(fx,fy)\leq\ a\,d(gx,gy)+\ b\,d(fx,gx)+\ c\,d(fy,gy),\ \forall x,y\in X$,

$0 \le a, b, c < 1$ *and* $0 \le a + 2b < 1$ *or,* $0 \le a + 2c < 1$.

If gX *is closed(resp.* $cl(fX) \subset gX$*) then,* $C(f, g) \neq \phi$*. Further, assume that* f *and* g *are occasionally weakly* g*-biased then,* f *and* g *have a unique common fixed point.*

Proof. Since f and g satisfy property (E.A), there exists a sequence $\{x_n\}$ in X such that $fx_n,\ gx_n \to t$ for some $t \in X$. As gX is closed in X then, $t \in gX$. Suppose that $cl(fX) \subset gX$ then, $t \in cl(fX) \subset gX$. Therefore, closeness of gX(resp. $cl(fX) \subset gX$) implies that $t \in gX$, so there exists $u \in X$ such that $t = gu$. We claim that $fu = gu$. For this, taking $x = x_n$, $y = u$ in (ii), we obtain

$$d(fx_n, fu) \le a\, d(gx_n, gu) + b\, d(fx_n, gx_n) + c\, d(fu, gu)$$

On letting $n \to \infty$, we obtain

$$d(gu, fu) \le c\, d(fu, gu), \text{ where } c < 1.$$

Therefore, $fu = gu$. Similarly, one can show that $fu = gu$ by interchanging x and y in (ii), provided $b < 1$. Hence, $C(f, g) \neq \phi$.

Since, f and g are occasionally weakly g-biased, there exists $v \in C(f, g)$ such that $fv = gv$ and $d(gfv, gv) \le d(fgv, fv)$. Also, $fv = gv$ yields $ffv = fgv$, $gfv = ggv$. We claim that $ffv = fv$. For this, taking $x = fv, y = v$ in (ii) and occasionally weakly g-biased mappings, we obtain

$$\begin{aligned} d(ffv, fv) &\le ad(gfv, gv) + b(ffv, gfv) + cd(fv, gv) \\ &\le a\, d(fgv, fv) + 2b\, d(ffv, fv) \\ &= (a + 2b) d(ffv, fv), \end{aligned}$$

where $a + 2b < 1$, which in turn gives $ffv = fv$. Similarly, by interchanging x and y in (ii), and occasionally weakly g-biased mappings, one can show that $ffv = fv$, if $a + 2c < 1$.

Also, by occasionally weakly g-biased pair, we obtain

$$\begin{aligned} d(gfv, gv) &\leq d(fgv, fv) \\ &= d(ffv, fv) \\ &= 0 \end{aligned}$$

which gives $ffv = gfv = fv$. Therefore fv is a common fixed point of f and g.

For the uniqueness, suppose that $w \neq z \in X$ such that $fw = gw = w$ and $fz = gz = z$. By (ii), we obtain

$$\begin{aligned} d(w, z) &= d(fw, fz) \\ &\leq ad(gw, gz) \\ &= ad(w, z), \end{aligned}$$

where $a < 1$, which gives $w = z$. This completes the proof.

We now give the following example to illustrate the validity of above theorem.

Example 2.3. Let $X = [0, 1)$ with usual metric d on X. Define $f, g : X \to X$ as

$$fx = \frac{3}{8},\ x < \frac{1}{2},\ fx = \frac{1}{2},\ x \geq \frac{1}{2};$$
$$gx = \frac{2}{3},\ x < \frac{1}{2},\ g\frac{1}{2} = \frac{1}{2},\ gx = \frac{(1+x)}{3},\ x > \frac{1}{2}.$$

We see that $fX = \{\frac{3}{8}, \frac{1}{2}\}$ and $gX = [\frac{1}{2}, \frac{5}{6}]$, but fX is not contained in gX. However, gX is closed. Mappings f and g satisfy property(E.A), to see this let $x_n \to \frac{1}{2}$ such that $x_n > \frac{1}{2}$ for $n = 1, 2, 3, ...$, then we $fx_n, gx_n \to \frac{1}{2} = t$. Further we can verify that f and g satisfy the inequality (ii) for all $x, y \in X$ taking with $a = \frac{4}{5}, b = \frac{1}{11}, c = \frac{2}{3}$, where $a + 2b < 1$. Moreover $C(f, g) = \{\frac{1}{2}\}$, f and g are also occasionally weakly g-biased. Thus, all the conditions of the theorem are satisfied and $x = \frac{1}{2}$ is a unique common fixed point.

Corollary 2.4. *Let f and g be self mappings of a metric space (X, d) satisfying the following*

(i) f and g satisfy property (E.A);

(ii) $d(fx, fy) \leq\ a\, d(gx, gy) +\ b\, d(fx, gx) +\ c\, d(fy, gy), \forall x, y \in X,$

where $0 \leq a, b, c < 1$.

If gX *is closed(resp.* $cl(fX) \subset gX$*) then,* $C(f, g) \neq \phi$*. Further assume that* f *and* g *are owc (resp. weakly compatible, R-weakly commuting, R-weakly commuting of type* (A_f) *or type* (A_g)*) then,* f *and* g *have a unique common fixed point.*

The following theorem is proved in Pant et al.[11] for non-compatible pair of mappings satisfying a nonexpansive type condition.

Theorem 2.5(Theorem 2[11]**).** *Let* f *and* g *be weakly reciprocally non-compatible self mappings of a metric space* (X, d) *satisfying*

(i) $fX \subseteq gX$;

(ii) $d(fx, fy) \leq d(gx, gy) + b\, d(fx, gx) + c\, d(fy, gy), \forall x, y \in X$

where $0 \leq b, c < 1$;

(iii) $d(fx, f^2x) < d(gx, g^2x)$*, whenever* $gx \neq g^2x$.

If f *and* gX *are R-weakly commuting of type* (A_g) *or R- weakly commuting of type* (A_f) *then,* f *and* g *have a common fixed point.*

Now we prove the following theorem which extends the above one to occasionally weakly g-biased mappings.

Theorem 2.6. *Let* f *and* g *be self mappings of a metric space* (X, d) *satisfying the following*

(i) f *and* g *satisfy property (E.A);*

(ii) $d(fx, fy) \leq d(gx, gy) + b\, d(fx, gx) + c\, d(fy, gy), \forall x, y \in X$

where $b, c \geq 0$ *and* $b < 1$ *or,* $c < 1$;

(iii) $d(fx, f^2x) < max\{d(fx, fgx), d(fx, gx), d(gx, g^2x)\}$*, whenever the right hand side is non-zero.*

If gX *is closed (resp.* $cl(fX) \subseteq gX$*) then,* $C(f, g) \neq \phi$*. Further assume that* f *and* g *are occasionally weakly* g*-biased then,* f *and* g *have a common fixed point.*

Proof. Since, f and g satisfy property (E.A), there exists a sequence $\{x_n\}$ in X such that $fx_n, gx_n \to t$ for some $t \in X$. As in Theorem 2.2, one can show that $t \in gX$, so there exists $u \in X$ such that $t = gu$. We claim that $fu = gu$, otherwise by (ii), we obtain

$$d(fx_n, fu) \leq d(gx_n, gu) + b\,d(fx_n, gx_n) + c\,d(fu, gu)$$

On letting $n \to \infty$, we obtain

$$d(gu, fu) \leq c\,d(fu, gu), \text{ where } c < 1.$$

Therefore, $fu = gu$. Similarly, one can prove that $fu = gu$, when $b < 1$. Hence, $C(f, g) \neq \phi$. Since, f and g are occasionally weakly g-biased, there exists $v \in C(f, g)$ such that $fv = gv$ and $d(gfv, gv) \leq d(fgv, fv)$. Also, $fv = gv$ yields $ffv = fgv$, $gfv = ggv$. If $ffv \neq fv$, then by (iii) and occasionally weakly g-biased of mappings pair, we obtain

$$\begin{aligned} d(fv, f^2v) &< max\{d(fv, fgv), d(fv, gv), d(gv, g^2v)\} \\ &= max\{d(fv, f^2v), 0, d(gv, gfv)\} \\ &\leq max\{d(fv, f^2v), d(fv, fgv)\} \\ &= d(fv, f^2v) \end{aligned}$$

which is a contradiction. Therefore $ffv = fv$. Also by occasionally weakly g-biased of mappings pair, we obtain

$$d(gfv, gv) \leq d(fgv, fv) = d(ffv, fv) = 0$$

which gives $gfv = fv$ and hence $ffv = fgv = fv$. Therefore, fv is a common fixed point of f and g. This completes the proof.

The following example shows the validity of above Theorem 2.6.

Example 2.7. Let $X = [0, 1)$ with usual metric d on X. Define $f, g : X \to X$ as

$$\begin{aligned} &fx = 0, x < \frac{1}{2}, f\frac{1}{2} = \frac{1}{2}, fx = \frac{3}{4}, x > \frac{1}{2} \\ &gx = \frac{(1+x)}{2}, x < \frac{1}{2}, g\frac{1}{2} = \frac{1}{2}, gx = 3/4, x > \frac{1}{2}. \end{aligned}$$

Now, we have $fX = \{0, \frac{1}{2}, \frac{3}{4}\}$ and $gX = [\frac{1}{2}, \frac{3}{4}]$ but fX is not contained in gX, where gX is closed. Moreover, $C(f, g) = \{\frac{1}{2}, \frac{3}{4}\}$. The mappings f and g satisfy property (E.A), to verify this, let $\{x_n\}$ be a sequence in X such that $fx_n, gx_n \to t$ for some $t \in X$. Then $x_n = \frac{1}{2}$ for $n = 1, 2, 3, ...$, or $x_n \to \frac{1}{2}, x_n > \frac{1}{2}$ for $n = 1, 2, 3, ...$ If $x_n = \frac{1}{2}$ for $n = 1, 2, 3, ...$

then $fx_n, gx_n \to 1/2 = t$ and $fx_n, gx_n \to \frac{3}{4} = t$ as $x_n \to \frac{1}{2}$. One can verify that f and g satisfy the inequality (ii) for all $x, y \in X$ with $b = \frac{3}{2}, c = \frac{2}{3}$ and they are occasionally weakly g-biased. Thus, all the conditions of theorem are satisfied and $x = \frac{1}{2}, \frac{3}{4}$ are the common fixed points.

Note that in Theorem 2.6, we may obtain many common fixed points that means the common fixed points of the pair of mappings thus obtained under the non-expansive type condition may not be unique.

Corollary 2.8. *Let f and g be self mappings of a metric space (X, d) satisfying the conditions (i), (ii) and (iii) of Theorem 2.6. where, $b, c \geq 0$ and $b < 1$ or, $c < 1$.*

If gX is closed(resp. $cl(fX) \subseteq gX$) then, $C(f, g) \neq \phi$. Further assume that f and g are owc (resp. weakly compatible, R-weakly commuting, R-weakly commuting of type (A_f) or type (A_g), weakly g-biased) then, f and g have a unique common fixed point.

In place of condition (ii) of Theorem 2.5, Pant et al.[11] also used the following Lipschitz type condition

$$d(fx, fy) \leq ad(gx, gy) + b\, d(fx, gx) + c\, d(fy, gy), \forall x, y \in X \tag{1}$$

where $a \geq 0$ and $0 \leq b, c < 1$.

In the following we prove common fixed point theorem under Lipschitz condition which extends Theorem 3 of Pant[11].

Theorem 2.9. *Let f and g be self mappings of a metric space (X, d) satisfying the inequality (1) in lieu of condition (ii) of Theorem 2.5 taking with a, b, $c \geq 0$ and $b < 1$ or, $c < 1$ besides retaining the condition (i), and*

(iii) $d(fx, f^2x) < max\{d(fx, fgx), d(gx, fgx), d(fx, gx), d(fx, gfx), d(gx, g^2x)\}$, whenever the right hand side is non-zero.

If gX is closed (resp. $cl(fX) \subseteq gX$) then, $C(f, g) \neq \phi$. Further, assume that f and g are occasionally weakly g-biased then, f and g have a common fixed point.

Proof. As in Theorem 2.6, one can prove that $C(f, g) \neq \phi$, so there exists $v \in C(f, g)$ such that $fv = gv$ and $d(gfv, gv) \leq d(fgv, fv)$. Also, $fv = gv$ yields $ffv = fgv$, $gfv = ggv$. If $ffv \neq fv$, then by (iii) and occasionally weakly g-biased of mappings, we obtain

$$\begin{aligned} d(fv, f^2v) &< max\{d(fv, fgv), d(gv, fgv), d(fv, gv), d(fv, gfv), d(gv, g^2v)\} \\ &= max\{d(fv, f^2v), d(fv, f^2v), 0, d(gv, gfv), d(gv, gfv)\} \\ &\leq max\{d(fv, f^2v), d(fv, fgv)\} \\ &= d(fv, f^2v) \end{aligned}$$

which is a contradiction. Therefore, $ffv = fv$. Also by occasionally weakly g-biased of mappings pair, we obtain

$$d(gfv, gv) \leq d(fgv, fv) = 0$$

which gives $gfv = fv$ and hence $ffv = fgv = fv$. Therefore, fv is a common fixed point of f and g. This completes the proof.

One may also verify the validity of above theorem with Example 2.7 by letting $a = b = \frac{3}{2}$ and $c = \frac{2}{3}$, that $x = \frac{1}{2}, \frac{3}{4}$ are the common fixed points.

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