Available online at http://scik.org
Adv. Fixed Point Theory, 5 (2015), No. 4, 359-368
ISSN: 1927-6303

ON QUASI-VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. Quasi-variational inclusion and fixed point problem are investigated. Strong convergence theorems of common solutions are established in the framework of Hilbert spaces.

Keywords: Fixed point; Monotone operator; Iterative algorithm; Convergence analysis; Variational inequality.

2010 AMS Subject Classification: 47H05, 90C33.

1. Introduction-preliminaries

Variational inclusion problems, which include many important problems in nonlinear functional analysis and optimization such as the Nash equilibrium problem, complementarity problems, vector optimization problems, fixed point problems, saddle point problems and game theory, recently have been studied as an effective and powerful tool for studying many real world problems which arise in economics, finance, image reconstruction, ecology, transportation, and network; see [1-8] and the references therein.

Let H be a real Hilbert space and let C be a closed convex subset of C. Let Recall that a mapping $S : C \to C$ is said to be α-contractive iff there exists a constant $\alpha \in [0,1)$ such that $\|Sx - Sy\| \leq \alpha \|x - y\|$ $\forall x, y \in C$. S is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ $\forall x, y \in H$.

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Received May 21, 2014

In this paper, we use $F(S)$ to denote the set of fixed points of S. If C is bounded, then $F(S)$ is not empty. Recall that a mapping A on H is said to be strongly positive if there is a constant $\bar{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2$, $\forall x \in H$. Recall that a mapping $B : H \to H$ is said to be inverse-strongly monotone, if there exists an $\alpha > 0$ such that $\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2$ for all $x, y \in H$. Recall that a set-valued mapping $M : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x - y, f - g\rangle \geq 0$. The domain of M is denoted by $Dom(B)$.

In this paper, we consider the following so-called quasi-variational inclusion problem: Find an $u \in H$ such that $0 \in Bu + Mu$, where $B : H \to H$ and $M : H \to 2^H$ are two nonlinear mappings. In this paper, we use $VI(H, B, M)$ to denote the solution of the problem (1). A number of problems arising in structural analysis, mechanics and economic can be studied in the framework of this class of variational inclusions. Next, we consider two special cases of the inclusion problem.

(I) If $M = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty]$ is the indicator function of C, ie.,

$$\delta_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then the quasi-variational inclusion problem is equivalent to the classical variational inequality problem, denoted by $VI(C, B)$, is to find $u \in C$ such that

$$\langle Bu, v - u\rangle \geq 0, \quad \forall v \in C.$$

(II) If $M = \partial\phi : H \to 2^H$, where $\phi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex lower semi-continuous function and $\partial\phi$ is the sub-differential of ϕ, then the quasi-variational inclusion problem is equivalent to finding $u \in H$ such that $\langle Bu, v - u\rangle + \phi(v) - \phi(u) \geq 0$, $\forall v \in H$, which is said to be the mixed quasi-variational inequality.

In this paper, we introduce an general iterative algorithm for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings and of the set of solutions to the inclusion problem. Strong convergence theorems are established in the framework of Hilbert spaces.

In order to prove our main results, we need the following conceptions and lemmas.

Lemma 1.1. [9] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that* $\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n$, *where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that* $\sum_{n=1}^{\infty} \gamma_n = \infty$ *and* $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$. *Then* $\lim_{n\to\infty} \alpha_n = 0$.

Definition 1.2. [10] Let H be a real Hilbert space and let C be convex closed subset of C. Let $\{S_i\}$ be a family of infinitely nonexpansive mappings and $\{\gamma_i\}$ be a nonnegative real sequence with $0 \leq \gamma_i < 1$, $\forall i \geq 1$. For $n \geq 1$, define a mapping W_n as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n S_n U_{n,n+1} + (1-\gamma_n)I,\\
&\vdots\\
&U_{n,2} = \gamma_2 S_2 U_{n,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1-\gamma_1)I.
\end{aligned} \tag{1.1}$$

Such a mapping W_n is nonexpansive and it is called a W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$.

Lemma 1.3. [10] *Let H be a real Hilbert space and let C be a convex subset of H. Let $\{S_i\}$ be an infinite family of nonexpansive mappings with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, $\{\gamma_i\}$ be a real sequence such that $0 < \gamma_i \leq l < 1$, $\forall i \geq 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \cap_{i=1}^{\infty} F(S_i)$, for each $n \geq 1$;*
(2) *the mapping W defined by $Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$, is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$.*
(3) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}$ exists.*

Lemma 1.4. [11] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 1.5. [12] *Let H be a Hilbert space H and $\{S_i\}$ a family of infinitely nonexpansive mappings with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, $\{\gamma_i\}$ a real sequence such that $0 < \gamma_i \leq l < 1$, $\forall i \geq 1$. If K is any* *bounded subset of H, then* $\lim_{n\to\infty}\sup_{x\in K}\|Wx-W_nx\|=0$. Throughout this paper, we always assume that $0<\gamma_i\le l<1$, $\forall i\ge 1$.

2. Main results

Theorem 2.1. *Let H be a real Hilbert space and let C be a closed and convex subset of H. Let M_1 and M_2 be two maximal monotone operators on H. Let B_1 : be a δ_1-inverse-strongly monotone mapping on H and let B_2 a δ_2-inverse-strongly monotone mapping on H. Let $\{S_i\}_{i=1}^{\infty}$ be an infinitely family of nonexpansive mappings from C into itself and let $f: C\to C$ be an α-contraction. Let A be a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma}>0$. Assume that $0<\gamma<\bar{\gamma}/\alpha$, $\Omega=\cap_{i=1}^{\infty}F(S_i)\cap VI(H,B_1,M_1)\cap VI(H,B_2,M_2)\neq\emptyset$ and $Dom(B_1)\subset C$. Let $x_1\in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n=(I+\eta M_2)^{-1}(x_n-\eta B_2x_n),\\ y_n=(I+\lambda M_1)^{-1}(z_n-\lambda B_1z_n),\\ x_{n+1}=\alpha_n\gamma f(x_n)+\beta_nx_n+((1-\beta_n)I-\alpha_nA)W_ny_n, \quad \forall n\ge 1,\end{cases}$$

where $\{W_n\}$ is the sequence defined by (1.1), $\lambda\in(0,2\delta_1),\eta\in(0,2\delta_2)$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $0<a\le\beta_n\le b<1$, $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$. Then $\{x_n\}$ converges strongly to $z\in\Omega$, which solves uniquely the following variational inequality

$$\langle (A-\gamma f)z, z-x^*\rangle\le 0,\quad \forall x^*\in\Omega. \tag{2.1}$$

Equivalently, we have $z=P_{\Omega}(I-A+\gamma f)z$.

Proof. The uniqueness of the solution of the variational inequality (2.1), which is indeed a consequence of the strong monotonicity of $A-\gamma f$. Below we use z to denote the unique solution. Note that both $I-\lambda B_1$ and $I-\eta B_2$ are nonexpansive. Indeed, for $\forall x,y\in C$, from the condition $\lambda\in(0,2\delta_1]$, we have

$$\begin{aligned}\|(I-\lambda B_1)x-(I-\lambda B_1)y\|^2 &\le \|x-y\|^2-2\lambda\delta_1\|B_1x-B_1y\|^2+\lambda^2\|B_1x-B_1y\|^2\\ &\le \|x-y\|^2,\end{aligned}$$

which implies mapping $I-\lambda B_1$ is nonexpansive. For $\forall x,y\in C$, from the condition $\eta\in(0,2\delta_2]$, we have

$$\begin{aligned}\|(I-\eta B_2)x-(I-\eta B_2)y\|^2 &\le \|x-y\|^2-2\eta\delta_2\|B_2x-B_2y\|^2+\eta^2\|B_2x-B_2y\|^2\\ &\le \|x-y\|^2,\end{aligned}$$

which implies mapping $I-\eta B_2$ is nonexpansive. Taking $x^*\in\Omega$, we have

$$x^*=(I+\lambda M_1)^{-1}(x^*-\lambda B_1x^*)=(I+\eta M_2)^{-1}(x^*-\eta B_2x^*).$$

It follows that

$$\|z_n-x^*\|=\|(I+\eta M_2)^{-1}(x_n-\eta B_2x_n)-(I+\eta M_2)^{-1}(x^*-\eta B_2x^*)\|\le\|x_n-x^*\|.$$

This implies that

$$\|y_n-x^*\|\le\|(z_n-\lambda B_1z_n)-(x^*-\lambda B_1x^*)\|\le\|x_n-x^*\|.$$

Without loss of generality, we may assume that $\alpha_n\le(1-\beta_n)\|A\|^{-1}$. Since A is a strongly positive linear bounded self-adjoint operator, we have $\|A\|=\sup\{|\langle Ax,x\rangle|:x\in H,\|x\|=1\}$, Now for $x\in C$ with $\|x\|=1$, we see that $\langle((1-\beta_n)I-\alpha_nA)x,x\rangle=1-\beta_n-\alpha_n\langle Ax,x\rangle\ge0$, that is, $(1-\beta_n)I-\alpha_nA$ is positive. Hence, we have

$$\|(1-\beta_n)I-\alpha_nA\|=\sup\{1-\beta_n-\alpha_n\langle Ax,x\rangle:x\in C,\|x\|=1\}\le1-\beta_n-\alpha_n\bar{\gamma}.$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\|&\le\alpha_n\|\gamma f(x_n)-Ax^*\|+\beta_n\|x_n-x^*\|+\|(1-\beta_n)I-\alpha_nA\|\|W_ny_n-x^*\|\\ &\le[1-\alpha_n(\bar{\gamma}-\alpha\gamma)]\|x_n-x^*\|+\alpha_n\|\gamma f(x^*)-Ax^*\|.\end{aligned}$$

By simple inductions, we obtain that $\|x_n-x^*\|\le\max\left\{\|x_1-x^*\|,\frac{\|\gamma f(x^*)-Ax^*\|}{\bar{\gamma}-\alpha\gamma}\right\}$, which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Without loss of generality, we can assume that there exists a bounded set $K\subset H$ such that

$$x_n,y_n,z_n\in K,\quad\forall n\ge1.\tag{2.2}$$

Note that

$$\|z_{n+1}-z_n\|\le\|(x_{n+1}-\eta B_2x_{n+1})-(x_n-\eta B_2x_n)\|\le\|x_{n+1}-x_n\|,$$

and $\|y_{n+1}-y_n\| \le \|z_{n+1}-z_n\| \le \|x_{n+1}-x_n\|$. Setting $x_{n+1} = (1-\beta_n)v_n + \beta_n x_n$, we see that

$$\begin{aligned}\|v_{n+1}-v_n\| \le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - AW_n y_n\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - AW_n y_n\| \\ + \|W_{n+1}y_{n+1} - W_n y_n\|.\end{aligned} \tag{2.3}$$

On the other hand, we have

$$\|W_{n+1}y_{n+1} - W_n y_n\| \le \sup_{x\in K}\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\} + \|x_{n+1}-x_n\|, \tag{2.4}$$

where K is the bounded subset of H defined by (2.3). Using (2.3) and (2.4), one finds

$$\begin{aligned}&\|v_{n+1}-v_n\| - \|x_{n+1}-x_n\| \\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - AW_n y_n\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - AW_n y_n\| \\ &\quad + \sup_{x\in K}\{\|W_{n+1}x - Wx\| + \|Wx - W_n x\|\}.\end{aligned}$$

Hence, we have $\limsup_{n\to\infty}(\|v_{n+1}-v_n\| - \|x_{n+1}-x_n\|) \le 0$. Using Lemma 1.4, we obtain that $\lim_{n\to\infty}\|v_n - x_n\| = 0$. This implies $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Putting $f_n = \gamma f(x_n) - AW_n y_n$, for $\forall n \ge 1$, we see, for any $x^* \in \Omega$, that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n(\gamma f(x_n) - AW_n y_n) + [\beta_n(x_n - x^*) + (1-\beta_n)(W_n y_n - x^*)]\|^2 \\ &\le \|\beta_n(x_n - x^*) + (1-\beta_n)(W_n y_n - x^*)\|^2 + 2\alpha_n\langle f_n, x_{n+1} - x^*\rangle \\ &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + 2\alpha_n M^2,\end{aligned} \tag{2.5}$$

where $M = \max\{\sup_{n\ge1}\|f_n\|, \sup_{n\ge1}\|x_n - x^*\|\}$. On the other hand, we have

$$\begin{aligned}\|y_n - x^*\|^2 &\le \|(I-\lambda B_1)z_n - (I-\lambda_n B_1)x^*\|^2 \\ &\le \|z_n - x^*\|^2 + \lambda(\lambda - 2\delta_1)\|B_1 z_n - B_1 x^*\|^2.\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.5), we has

$$\|x_{n+1}-x^*\|^2 \le \|x_n - x^*\|^2 + (1-\beta_n)\lambda(\lambda - 2\delta_1)\|B_1 z_n - B_1 x^*\|^2 + 2\alpha_n M^2.$$

This implies

$$\lim_{n\to\infty}\|B_1 z_n - B_1 x^*\| = 0. \tag{2.7}$$

Further, one has

$$\|x_{n+1}-x^*\|^2 \le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|z_n - x^*\|^2 + 2\alpha_n M^2. \tag{2.8}$$

In a similar way, we can obtain

$$\lim_{n\to\infty} \|B_2x_n - B_2x^*\| = 0. \tag{2.9}$$

On the other hand, we see that

$$\begin{aligned}\|z_n - x^*\|^2 \le \frac{1}{2}\{&\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 \\ &+ 2\eta\langle x_n - z_n, B_2x_n - B_2x^*\rangle - \eta^2\|B_2x_n - B_2x^*\|^2\},\end{aligned}$$

which implies that

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\eta\|x_n - z_n\|\|B_2x_n - B_2x^*\|. \tag{2.10}$$

Substituting (2.10) into (2.8), we arrive at

$$\begin{aligned}&(1-\beta_n)\|x_n - z_n\|^2 \\ &\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + 2\alpha_n M^2 + 2\eta\|x_n - z_n\|\|B_2x_n - B_2x^*\|.\end{aligned}$$

It follows from $0 < a \le \beta_n \le b < 1$ that $\lim_{n\to\infty}\|x_n - z_n\| = 0$. In a similar way, we can obtain that $\lim_{n\to\infty}\|y_n - z_n\| = 0$. On the other hand, we have $(1-\beta_n)\|W_ny_n - x_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - BW_ny_n\|$. From the assumptions imposed on the control sequences, we obtain that $\lim_{n\to\infty}\|W_ny_n - x_n\| = 0$. Notice that $\|W_ny_n - y_n\| \le \|y_n - z_n\| + \|z_n - x_n\| + \|x_n - W_ny_n\|$. It follows that

$$\lim_{n\to\infty}\|W_ny_n - y_n\| = 0. \tag{2.11}$$

Next, we prove that $\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle \le 0$, where $z = P_\Omega[I - (A - \gamma f)]z$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f - A)z, x_n - z\rangle = \lim_{i\to\infty}\langle(\gamma f - A)z, x_{n_i} - z\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. On the other hand, we have $\|x_n - y_n\| \le (\|x_n - z_n\| + \|y_n - z_n\|) \to 0$ as $n \to \infty$. Therefore, we see that $y_{n_i} \rightharpoonup w$.

First, we prove that $w \in VI(H, B_1, M_1)$. From [13], we see that $M_1 + B_1$ is maximal monotone. Let $(e_1, e_2) \in \text{Graph}(M_1 + B_1)$. On the other hand, we have $y_{n_i} = (I + \lambda M_1)^{-1}(z_{n_i} - \lambda B_1 z_{n_i})$. It follows that $\frac{1}{\lambda}(z_{n_i} - y_{n_i} - \lambda B_1 z_{n_i}) \in M_1(y_{n_i})$. By virtue of the maximal monotonicity of $M_1 + B_1$, we see that $\langle e_1 - y_{n_i}, e_2 - B_1 e_1 - \frac{1}{\lambda}(z_{n_i} - y_{n_i} - \lambda B_1 z_{n_i})\rangle \geq 0$, which yields that

$$\langle e_1 - y_{n_i}, e_2\rangle \geq \langle e_1 - y_{n_i}, B_1 y_{n_i} - B_1 z_{n_i}\rangle + \frac{1}{\lambda}\langle e_1 - y_{n_i}, z_{n_i} - y_{n_i}\rangle.$$

Using $y_n - z_n \to 0$ and $y_{n_i} \rightharpoonup w$, one has $\langle e_1 - w, e_2\rangle \geq 0$. Since $M_1 + B_1$ is maximal monotone, we see that $\theta \in (M_1 + B_1)(w)$, i.e., $w \in VI(H, B_1, M_1)$. From $x_n - z_n \to 0$, we see that $z_{n_i} \rightharpoonup w$. In the same way, we can obtain $w \in VI(H, B_2, M_2)$.

Next, we show that $w \in \cap_{i=1}^{\infty} F(S_i) = F(W)$. Suppose the contrary, $w \notin F(W)$, i.e., $Ww \neq w$. Note that $y_{n_i} \rightharpoonup w$. Using the Opial's condition [14], we see that

$$\liminf_{i\to\infty} \|y_{n_i} - w\| < \liminf_{i\to\infty} \|y_{n_i} - Ww\| \leq \liminf_{i\to\infty} \{\|y_{n_i} - Wy_{n_i}\| + \|y_{n_i} - w\|\}.$$

On the other hand, we have

$$\|Wy_n - y_n\| \leq \|Wy_n - W_n y_n\| + \|W_n y_n - y_n\| \leq \sup_{x\in K} \|Wx - W_n x\| + \|W_n y_n - y_n\|.$$

From Lemma 1.5, we obtain that $\lim_{n\to\infty} \|Wy_n - y_n\| = 0$, which further yields that $\liminf_{i\to\infty} \|y_{n_i} - w\| < \liminf_{i\to\infty} \|y_{n_i} - w\|$. This derives a contradiction. Thus, we have $w \in F(W) = \cap_{i=1}^{\infty} F(S_i)$. It follows that

$$\limsup_{n\to\infty} \langle(\gamma f - A)z, x_n - z\rangle = \langle(\gamma f - A)z, w - z\rangle \leq 0.$$

Finally, we show that $x_n \to z$, as $n \to \infty$. Note that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \alpha_n \gamma\langle f(x_n) - f(z), x_{n+1} - z\rangle + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle \\ &\quad + \beta_n \|x_n - z\|\|x_{n+1} - z\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - z\|\|x_{n+1} - z\| \\ &\leq \frac{1 - \alpha_n(\bar{\gamma} - \alpha\gamma)}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle,\end{aligned}$$

which implies that $\|x_{n+1} - z\|^2 \leq [1 - \alpha_n(\bar{\gamma} - \alpha\gamma)]\|x_n - z\|^2 + 2\alpha_n\langle\gamma f(z) - Az, x_{n+1} - z\rangle$. In view of Lemma 1.1, we see that $\lim_{n\to\infty} \|x_n - z\| = 0$. This completes the proof.

Letting $\gamma = 1$ and $A = I$, the identity mapping, we have the following result.

Corollary 2.2. *Let H be a real Hilbert space and let C be a closed and convex subset of H. Let M_1 and M_2 be two maximal monotone operators on H. Let B_1 : be a δ_1-inverse-strongly monotone mapping on H and let B_2 a δ_2-inverse-strongly monotone mapping on H. Let $\{S_i\}_{i=1}^{\infty}$*

be an infinitely family of nonexpansive mappings from C into itself and let $f : C \to C$ be an α-contraction. Let A be a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$, $\Omega = \cap_{i=1}^{\infty} F(S_i) \cap VI(H, B_1, M_1) \cap VI(H, B_2, M_2) \neq \emptyset$ and $Dom(B_1) \subset C$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = (I + \eta M_2)^{-1}(x_n - \eta B_2 x_n), \\ y_n = (I + \lambda M_1)^{-1}(z_n - \lambda B_1 z_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n) W_n y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ is the sequence defined by (1.1), $\lambda \in (0, 2\delta_1), \eta \in (0, 2\delta_2)$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $0 < a \leq \beta_n \leq b < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $z \in \Omega$, which solves uniquely the following variational inequality

$$\langle (A - \gamma f) z, z - x^* \rangle \leq 0, \quad \forall x^* \in \Omega.$$

Equivalently, we have $z = P_\Omega f(z)$.

Corollary 2.3. *Let H be a real Hilbert space and let C be a closed and convex subset of H. Let B_2 a δ_2-inverse-strongly monotone mapping on H. Let $\{S_i\}_{i=1}^{\infty}$ be an infinitely family of nonexpansive mappings from C into itself and let $f : C \to C$ be an α-contraction. Let A be a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$, $\Omega = \cap_{i=1}^{\infty} F(S_i) \cap VI(C, B_1) \cap VI(C, B_2) \neq \emptyset$. Let $x_1 \in C$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = P_C(x_n - \eta B_2 x_n), \\ y_n = P_C(z_n - \lambda B_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \alpha_n A) W_n y_n, \quad \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ is the sequence defined by (1.1), $\lambda \in (0, 2\delta_1), \eta \in (0, 2\delta_2)$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $0 < a \leq \beta_n \leq b < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to $z \in \Omega$, which solves uniquely the following variational inequality

$$\langle (A - \gamma f) z, z - x^* \rangle \leq 0, \quad \forall x^* \in \Omega.$$

Equivalently, we have $z = P_\Omega (I - A + \gamma f) z$.

Conflict of Interests

The author declares that there is no conflict of interests.

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