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COMMON FIXED POINT THEOREMS IN G –METRIC SPACES

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Abstract: In this paper, we prove common fixed point theorems for a pair of compatible mappings and a pair of occasionally weakly compatible mappings in G –metric spaces.

Keywords: compatible mappings; occasionally weakly compatible; property (E.A); fixed point.

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1. Introduction and Preliminaries

In 1922, Banach proved a fixed-point theorem, "Let (X, d) be a complete metric space. If a mapping $T : X \to R^+$ satisfies $d(Tx, Ty) \leq k\, d(x, y)$ for each x, y in X where $0 < k < 1$, then T has a unique fixed point in X " which ensures under appropriate conditions, the existence and uniqueness of a fixed point. This theorem has had many applications, but suffers from one drawback - the definition requires that T be continuous throughout X. Then there was a flood of papers involving contractive definition that do not require the continuity of T.

In 1984 Dhage [3] introduced the concept of D – metric spaces. The situation for a D-metric space is quite different from 2-metric spaces. Geometrically, a D- metric D(x, y, z) represents the perimeter of the triangle with vertices x, y and z in R^2. Recently, Mustafa and Sims [5] has shown that most of the results concerning Dhage's D – metric spaces are invalid. Therefore, they introduced an improved version of the generalized metric space structure, which they called it as G – metric spaces. For more details on G – metric spaces, one can refer to the papers [5]- [8].

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In 2004, Mustafa and Sims [5] introduced the concept of G-metric spaces as follows:

Definition 1.1. Let X be a nonempty set, $G : X \times X \times X \to R^+$ a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality).

The function G is called a generalized metric or, more specifically, a G – metric on X , and the pair (X, G) is called a G – metric space.

Definition 1.2. [5]. Let (X, G) be a G – metric space, $\{x_n\}$ a sequence of points in X. We say that $\{x_n\}$ is G – convergent to x if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$; i.e., for each

$\in > 0$ there exists an N such that $G(x, x_n, x_m) < \in$ for all $m, n \geq N$.

We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1.3. [5]. Let (X, G) be a G – metric space. Then the following are equivalent:

(i) $\{x_n\}$ is G convergent to x,

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition 1.4. [5]. Let (X, G) be a G – metric space. A sequence $\{x_n\}$ is called G – Cauchy if, for each $\in > 0$ there exists an N such that $G(x_n, x_m, x_l) < \in$ for all $n, m, l \geq N$.

Proposition 1.5.[5]. In a G – metric space (X, G) the following are equivalent:

(i) The sequence $\{x_n\}$ is G – Cauchy,

(ii) for each $\in > 0$ there exists an N such that $G(x_n, x_m, x_l) < \in$ for all $n, m, l \geq N$.

Proposition 1.6. [5]. Let (X, G) be a G – metric space. Then the function G(x, y, z) is jointly continuous in all three of its variables.

Definition 1.7. [5]. A G – metric space (X, G) is called a symmetric G – metric space if $G(x, y, y) = G(y, x, x)$ for all x, y in X.

Proposition 1.8. [5]. Every G – metric space (X, G) defines a metric space (X, d_G)

(i) $d_G(x, y) = G(x, y, y) + G(y, x, x)$ for all x, y in X.

If (X, G) is a symmetric G – metric space, then

(ii) $d_G(x, y) = 2G(x, y, y)$ for all x, y in X.

However, if (X, G) is not symmetric, then it follows from the G – metric properties that

(iii) $^3/_2\, G(x, y, y) \leq d_G(x, y) \leq 3G(x, y, y)$ for all x, y in X.

Proposition 1.9.[5]. A G – metric space (X, G) is G – complete if and only if (X, d_G) is a complete metric space.

Proposition 1.10.[5]. Let (X, G) be a G – metric space. Then, for any x, y, z, a in X it follows that:

(i) if G(x, y, z) = 0, then x = y = z,

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, y, y) \leq 2G(y, x, x)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq {}^2/_3\, (G(x, a, a) + G(y, a, a) + G(z, a, a))$.

In 1976, Jungck [4] gave the notion of commutativity to obtain common fixed point theorems. This result was further generalized and extended in various ways by many authors. In 2012**,** Manro *et al.* [9] introduced the concept of compatible maps in G – metric space as follows: Let f and g be maps from a G – metric space (X, G) into itself. The maps f and g are said to be compatible map if there exists a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ or $\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$ whenever $\{x_n\}$ is sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

2. Main Results

Now we prove our main result using compatible maps as follows:

Theorem 2.1. Let f and g be self -maps of a G-metric space (X, G) satisfying

(2.1) $f(X) \subseteq g(X)$;

(2.2) $G(fx, fy, fz) \leq \alpha \max\{G(fx, gy, gz), G(gx, fy, gz), G(gx, gy, fz)\}$,

where $\alpha \in [0, \frac{1}{2})$;

(2.3) one of f or g is continuous.

Then f and g have a unique common fixed point in X, provided f and g are compatible maps.

Proof. Let x_0 be an arbitrary point in X. By (2.1), one can choose a point x_1 in X such that $fx_0 = gx_1$, In general one can choose x_{n+1} such that

$$y_n = fx_n = gx_{n+1},\ n = 0,1,2,\ldots.$$

From (2.2), we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq \alpha \max \begin{Bmatrix} G(fx_n, gx_{n+1}, gx_{n+1}), \\ G(gx_n, fx_{n+1}, gx_{n+1}), G(gx_n, gx_{n+1}, fx_{n+1}) \end{Bmatrix}$$

$$= \alpha \max \begin{Bmatrix} G(fx_n, fx_n, fx_n), \\ G(fx_{n-1}, fx_{n+1}, fx_n), G(fx_{n-1}, fx_n, fx_{n+1}) \end{Bmatrix}$$

$$= \alpha \max\{0, G(fx_{n-1}, fx_{n+1}, fx_n), G(fx_{n-1}, fx_n, fx_{n+1})\}$$

$$= \alpha\, G(fx_{n-1}, fx_n, fx_{n+1}).$$

By rectangular inequality of G-metric space, we have

$$G(fx_{n-1}, fx_n, fx_{n+1}) \leq G(fx_{n-1}, fx_n, fx_n) + G(fx_n, fx_n, fx_{n+1})$$

$$\leq G(fx_{n-1}, fx_n, fx_n) + 2\, G(fx_n, fx_{n+1}, fx_{n+1}), \text{ by Proposition 1.10.}$$

Therefore from above inequality, we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq \frac{\alpha}{(1-2\alpha)}\, G(fx_{n-1}, fx_n, fx_n).$$

i.e., $G(fx_n, fx_{n+1}, fx_{n+1}) \leq q\, G(fx_{n-1}, fx_n, fx_n)$, where $q = \frac{\alpha}{(1-2\alpha)} < 1$.

Continuing in the same way, we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \leq q^n\, G(fx_0, fx_1, fx_1).$$

Therefore, for all n, m $\in$ N, n < m, we have by rectangular inequality that

$$G(y_n, y_m, y_m) \leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + G(y_{n+2}, y_{n+3}, y_{n+3}) + \cdots + G(y_{m-1}, y_m, y_m)$$

$$\leq (q^n + q^{n+1} + \cdots + q^{m-1})\, G(y_0, y_1, y_1).$$

$$\leq \frac{q^n}{(1-q)}\, G(y_0, y_1, y_1).$$

Letting as n, m $\to\infty$, we have $\lim_{n\to\infty} G(y_n, y_m, y_m) = 0$.

Thus $\{y_n\}$ is a G–Cauchy sequence in X. Since (X, G) is complete G-metric space, therefore, there exists a point z ∈ X such that $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$.

Since the mapping f or g is continuous, for definiteness one can assume that g is continuous, therefore $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_{n+1}$ = gz. Further, f and g are compatible, therefore, $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ implies that $\lim_{n\to\infty} fgx_n = gz$.

On setting $x = gx_n$, $y = x_n$ and $z = x_n$, in (2.2), we have

$$G(fgx_n, fx_n, fx_n) \leq \alpha \max\{G(fgx_n, gx_n, gx_n), G(ggx_n, fx_n, gx_n), G(ggx_n, gx_n, fx_n)\}.$$

Letting as $n \to \infty$, we have

$$G(gz, z, z) \leq \alpha \max\{G(gz, z, z), G(gz, z, z), 0\} \text{implies, } gz = z.$$

Again from (2.2), we have

$$G(fx_n, fz, fz) \leq \alpha \max\{G(fx_n, gz, gz), G(gx_n, fz, gz), G(gx_n, gz, fz)\}$$

Letting as $n \to \infty$, we have fz = z.

Therefore, fz = gz = z. i.e., z is a common fixed point of f and g.

Uniqueness: We assume that $z_1 (\neq z)$ be another common fixed point of f and g . Then $G(z, z_1, z_1) > 0$ and

$$G(z, z_1, z_1) = G(fz, fz_1, fz_1)$$
$$\leq \alpha \max\{G(fz, gz_1, gz_1), G(gz, fz_1, gz_1), G(gz, gz_1, fz_1)\}$$
$$= \alpha G(z, z_1, z_1) < G(z, z_1, z_1) \text{ , a contradiction,}$$

which shows that $z = z_1$.

3. Property (E.A.) in G-metric Spaces.

Recently, Amari and Moutawakil [1] introduced a generalization of non-compatible maps as property (E.A.) in metric spaces as follows:

Definition 3.1. Let A and S be two self-maps of a metric space (X,d) .The pair (A,S) is said to satisfy property (E.A.), if there exists a sequence$\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

In similar mode we use property (E.A.) in G - metric spaces.

Example 3.2. [1] Let X =[0,+∞). Define S, T : X→X by

$$Tx = \frac{x}{4} \text{ and } Sx = \frac{3x}{4} \text{ , for all x in X. Consider the sequence } x_n = \frac{1}{n}.$$

Clearly $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = 0$. Then S and T satisfy property (E.A.).

Example 3.3. [1] Let X =[2,+∞). Define S, T : X→X by

Tx= x+1 and Sx= 2x+1, for all x in X. Suppose that the property (E.A.) holds. Then, there exists in X a sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X.

Therefore, $\lim_{n\to\infty} x_n = z-1$ and $\lim_{n\to\infty} x_n = \frac{z-1}{2}$.Thus, z = 1, which is a contradiction, since 1 is not contained in X. Hence S and T do not satisfy property (E.A.).

Remark 3.4. Property (E.A.) buys containment of maps without any continuity requirement. So above Theorem 2.1 can be rewritten in terms of property (E.A).

Theorem 3.5. Let f and g be self -maps of a G-metric space (X, G) satisfying (2.2)
and f and g satisfy property (E.A.).
Then f and g have a unique common fixed point in X, provided f and g are compatible maps.

4. Occasionally Weakly Compatible (owc)

Definition 4.1[2]. Two self -mappings f and g of a symmetric G-metric space (X, G) are said to be occasionally weakly compatible (o w c) iff there is a
point x in X which is coincidence point of f and g at which f and g commute.

Lemma 4.2[2]. Let (X, G) be a symmetric G-metric space. f and g be self maps on X and let f and g have a unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.

Theorem 4.3. Let (X, G) be a symmetric G-metric space. If f and g are o w c self -maps on X satisfying (2.2) . Then f and g have a unique common fixed point in X.

Proof. Since f and g are o w c, there exist a point u in X such that fu = gu and fgu = gfu. We first claim that fu is a fixed point of f.

For, if ffu ≠ fu, then from equation (2.2), we get

$$\begin{aligned} G(fu, ffu, ffu) &\le \alpha \max\{G(fu, gfu, gfu), G(gu, ffu, gfu), G(gu, gfu, ffu)\} \\ &= \alpha \max\{G(fu, ffu, ffu), G(fu, ffu, ffu), G(fu, ffu, ffu)\} \\ &= \alpha\, G(fu, ffu, ffu). \end{aligned}$$

This implies that ffu = fu and ffu = fgu = gfu = fu.

Hence fu is a common fixed point of f and g.

Uniqueness:

Suppose that u, v in X such that fu = gu = u and fv = gv = v and u ≠ v.

Then from equation (2.2),

$$G(u,v,v) = G(fu,fv,fv) \leq \alpha \max\{G(fu,gv,gv), G(gu,fv,gv), G(gu,gv,fv)\}$$
$$= \alpha \max\{G(u,v,v), G(u,v,v), G(u,v,v)\}.$$
$$= \alpha\, G(u,v,v), \text{ a contradiction.}$$

Thus u = v. Therefore, the common fixed point of f and g is unique.

Conflict of Interest

The authors declare that there is no conflict of interests.

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