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COMMON FIXED POINT THEOREMS FOR A FAMILY OF MAPPINGS WITH $\mathscr{A}$-IMPLICIT CONTRACTIVE CONDITIONS ON 2-METRIC SPACES

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Abstract. In this paper, using the known class of $\mathscr{A}$-contractions, we discuss the existence problems of points of coincidence and common fixed points for four self-mappings with $\mathscr{A}$-implicit contractions on non-complete 2-metric spaces and give some particular forms, also obtain a common fixed point theorem for an infinite family of self-mappings on complete 2-metric spaces and give a more general result. The obtained results generalize Kannan type (common) fixed point theorems and its variant forms and other corresponding conclusions.

Keywords: 2-metric space; class $\mathscr{A}$; point of coincidence; common fixed point.

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1. Introduction and Preliminaries

The following result is a real generalization of Banach contraction principle, i.e., Kannan type fixed point theorem:

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Theorem 1.1 [1] Let (X,d) be a complete real metric space, $f: X \to X$ a self-mapping. If there exists $k \in [0, \frac{1}{2})$ such that

$$d(fx, fy) \leq k[d(x, fx) + f(y, fy)], \forall x, y \in X.$$

Then f has an unique fixed point $z \in X$.

The next result is a variant form of Theorem 1.1:

Theorem 1.2 [2] Let (X,d) be a complete real metric space, $f: X \to X$ a self-mapping. If there exists $k \in [0, \frac{1}{3})$ such that

$$d(fx, fy) \leq k[d(x,y) + d(x, fx) + f(y, fy)], \forall x, y \in X.$$

Then f has an unique fixed point $z \in X$.

In 2008, The authors in [3] introduced a new general class of contractions(i.e., $\mathscr{A}$-contractions) and obtained a fixed point theorem which is a generalization of Kannan type theorem and its variant fixed point theorem(Theorem 1.1-1.2). The authors in [4] gave a integral version of the corresponding result in [3] on real metric spaces and the authors in [5] generalized the corresponding results in [3] on complex valued metric space.

In this paper, we will discuss and obtain some new common fixed point theorems for a family of self-mappings with $\mathscr{A}$-implicit contractions on 2-metric spaces (see [6-9]) and further generalize the corresponding conclusions.

At first, we give some well known definitions and results.

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathscr{A}$ be the set of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$satisfying

($\alpha 1$) α is continuous on the set $\mathbb{R}_+^3$ (with respect to the Euliean metric on $\mathbb{R}_+^3$);

($\alpha 2$) $a \leq kb$ for some $k \in [0,1)$ whenever $a \leq \alpha(a,b,b)$ or $a \leq \alpha(b,a,b)$ or $a \leq \alpha(b,b,a)$ for all $a, b \in [0, \infty)$.

Definition 1.1[6-7] A 2-metric space (X,d) consists of a nonempty set X and a function d : $X \times X \times X \to [0, +\infty)$ such that

(i) for distant elements $x, y \in X$, there exists an $u \in X$ such that $d(x,y,u) \neq 0$;

(ii) $d(x,y,z) = 0$ if and only if at least two elements in $\{x,y,z\}$ are equal;

(iii) $d(x,y,z) = d(u,v,w)$, where $\{u,v,w\}$ is any permutation of $\{x,y,z\}$;

(iv) $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all $x,y,z,u \in X$.

Definition 1.2 [6-7] A sequence $\{x_n\}_{n\in\mathbb{N}}$ in 2-metric space (X,d) is said to be a cauchy sequence, if for each $\varepsilon > 0$ there exists a positive integer $N \in \mathbb{N}$ such that $d(x_n,x_m,a) < \varepsilon$ for all $a \in X$ and $n,m > N$.

Definition 1.3 [8-9] A sequence $\{x_n\}_{n\in\mathbb{N}}$ in 2-metric space (X,d) is said to be convergent to $x \in X$, if $\lim_{n\to+\infty} d(x_n,x,a) = 0$ for each $a \in X$. And write $x_n \to x$ and call x the limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 1.4 [8-9] A 2-metric space (X,d) is said to be complete, if every cauchy sequence in X is convergent.

Definition 1.5 [10-11] Let f and g be two self-mappings on a set X. If $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Definition 1.6[12] Two mappings $f,g : X \to X$ are said to be weakly compatible if, for every $x \in X$, holds $fgx = gfx$ whenever $fx = gx$.

The following three lemmas are known results.

Lemma 1.1 [6-9] Let (X,d) be a 2-metric space and $\{x_n\}_{n\in\mathbb{N}}$ a sequence. If there exists $h \in [0,1)$ such that $d(x_{n+2},x_{n+1},a) \le hd(x_{n+1},x_n,a)$ for all $a \in X$ and $n \in \mathbb{N}$, then $d(x_n,x_m,x_l) = 0$ for all $n,m,l \in \mathbb{N}$, and $\{x_n\}_{n\in\mathbb{N}}$ is a cauchy sequence

Lemma 1.2 [6-9] If (X,d) is a 2-metric space and sequence $\{x_n\}_{n\in\mathbb{N}} \to x \in X$, then $\lim_{n\to+\infty} d(x_n,b,c) = d(x,b,c)$ for each $b,c \in X$.

Lemma 1.3[10-11] Let $f,g : X \to X$ be weakly compatible. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

2. Main Results

Theorem 2.1 Let (X, d) be a 2-metric space, $S,T,F,G : X \to X$ four mappings satisfying that $S(X) \subset G(X)$ and $T(X) \subset F(X)$. Suppose that for each $x,y,a \in X$,

$$d(Tx,Sy,a) \le \alpha\big(d(Gx,Fy,a),d(Gx,Tx,a),d(Fy,Sy,a)\big), \tag{2.1}$$

where $\alpha \in \mathscr{A}$. If one of $S(X), T(X), F(X)$ and $G(X)$ is complete, then T and G, S and F have an unique point of coincidence in X respectively. Further, if $\{G, T\}$ and $\{S, F\}$ are weakly compatible respectively, then S, T, F, G have an unique common fixed point in X.

Proof Take any element $x_0 \in X$, then using the conditions $S(X) \subset G(X)$ and $T(X) \subset F(X)$, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows:

$$y_{2n} = Tx_{2n} = Fx_{2n+1},\ y_{2n+1} = Sx_{2n+1} = Gx_{2n+2}, n = 0, 1, \cdots. \tag{2.2}$$

For any $n = 0, 1, \cdots$ and $a \in X$, by (2.1),

$$d(Tx_{2n}, Sx_{2n+1}, a) \leq \alpha\big(d(Gx_{2n}, Fx_{2n+1}, a), d(Gx_{2n}, Tx_{2n}, a), d(Fx_{2n+1}, Sx_{2n+1}, a)\big),$$

i.e.,

$$d(y_{2n}, y_{2n+1}, a) \leq \alpha\big(d(y_{2n-1}, y_{2n}, a), d(y_{2n-1}, y_{2n}, a), d(y_{2n}, y_{2n+1}, a)\big),$$

hence by (α2), we obtain

$$d(y_{2n}, y_{2n+1}, a) \leq k d(y_{2n-1}, y_{2n}, a), \forall n = 1, 2, \cdots, a \in X. \tag{2.3}$$

Similarly, For any $n = 0, 1, \cdots$ and $a \in X$, by (2.1),

$$d(Tx_{2n+2}, Sx_{2n+1}, a) \leq \alpha\big(d(Gx_{2n+2}, Fx_{2n+1}, a), d(Gx_{2n+2}, Tx_{2n+2}, a), d(Fx_{2n+1}, Sx_{2n+1}, a)\big),$$

i.e.,

$$d(y_{2n+2}, y_{2n+1}, a) \leq \alpha\big(d(y_{2n+1}, y_{2n}, a), d(y_{2n+1}, y_{2n+2}, a), d(y_{2n}, y_{2n+1}, a)\big),$$

hence by (α2), we obtain

$$d(y_{2n+1}, y_{2n+2}, a) \leq k d(y_{2n}, y_{2n+1}, a), \forall n = 1, 2, \cdots, a \in X. \tag{2.4}$$

Combining (2.3) and (2.4), we have

$$d(y_{n+1}, y_{n+2}, a) \leq k d(y_n, y_{n+1}, a), \forall n = 1, 2, \cdots, a \in X. \tag{2.5}$$

Hence $\{y_n\}$ is Cauchy by Lemma 1.1.

Suppose that FX or TX is complete. Since $y_{2n} \in TX \subset FX$ for all $n = 1, 2, \cdots$ and $\{y_n\}$ is Cauchy, there exist $u, v \in X$ such that $y_{2n} \to u = Fv$ as $n \to \infty$. We easily know

$$d(y_{2n+1}, u, a) \leq d(y_{2n}, u, a) + d(y_{2n+1}, y_{2n}, a) + d(y_{2n+1}, u, y_{2n}), \forall n = 1, 2, \cdots, a \in X$$

implies that $y_{2n+1} \to u$ as $n \to \infty$ since $y_{2n} \to u$ and $\{y_n\}$ is Cauchy.

By (2.1), for each $n \in \mathbb{N}$ and $a \in X$,

$$d(Tx_{2n}, Sv, a) \leq \alpha\big(d(Gx_{2n}, Fv, a), d(Gx_{2n}, Tx_{2n}, a), d(Fv, Sv, a)\big),$$

i.e.,

$$d(y_{2n}, Sv, a) \leq \alpha\big(d(y_{2n-1}, u, a), d(y_{2n-1}, y_{2n}, a), d(u, Sv, a)\big). \tag{2.6}$$

Letting $n \to \infty$ in (2.6) and using (α1) and Lemma 1.2, we obtain

$$d(u, Sv, a) \leq \alpha\big(0, 0, d(u, Sv, a)\big), \forall a \in X.$$

Hence $d(u, Sv, a) = 0$ for all $a \in X$ by (α2), so $Fv = u = Sv$. This shows that u is a point of coincidence of S and F.

Since $u = Sv \in SX \subset GX$, there exists $w \in X$ such that $u = Gw$. By (2.1), for all $n = 1, 2, \cdots$ and $a \in X$,

$$d(Tw, Sx_{2n+1}, a) \leq \alpha\big(d(Gw, Fx_{2n+1}, a), d(Gw, Fw, a), d(Fx_{2n+1}, Sx_{2n+1}, a)\big)$$

i.e.,

$$d(Tw, y_{2n+1}, a) \leq \alpha\big(d(u, y_{2n}, a), d(u, Fw, a), d(y_{2n}, y_{2n+1}, a)\big). \tag{2.7}$$

Letting $n \to \infty$ in (2.7), we obtain

$$d(Tw, u, a) \leq \alpha\big(0, d(u, Fw, a), 0)\big).$$

Hence we have

$$Tw = u = Gw,$$

that is, u ia also a point of coincidence of T and G.

If $z = Tx = Gx$ is also a point of coincidence of T and G, then using (2.1), we obtain

$$d(Tx, Sv, a) \leq \alpha\big(d(Gx, Fv, a), d(Gx, Tx, a), d(Fv, Sv, a)\big),$$

i.e.,

$$d(z, u, a) \leq \alpha\big(d(z, u, a), 0, 0\big),$$

hence $d(z, u, a) = 0$ for all $a \in X$ by (α2). So $u = z$, this means that u is the unique point of coincidence of T and G. Similarly, u is also the unique point of coincidence of S and F.

If $\{G,T\}$ and $\{S,F\}$ are weakly compatible respectively, then u is the unique common fixed point of $\{G,T\}$ and $\{S,F\}$ respectively by Lemma 1.3. Hence we easily know that u is the unique common fixed point of $\{S,T,F,G\}$. Similarly, we can obtain the same conclusion for SX or GX being complete.

Using Theorem 2.1, we are easy to obtain the following common fixed point theorems.

Theorem 2.2 Let (X, d) be a 2-metric space, $S,T,F : X \to X$ three mappings satisfying that $S(X) \cup T(X) \subset F(X)$. Suppose that for each $x,y,a \in X$,

$$d(Tx,Sy,a) \leq \alpha\big(d(Fx,Fy,a),d(Fx,Tx,a),d(Fy,Sy,a)\big),$$

where $\alpha \in \mathscr{A}$. If one of $S(X),T(X)$ and $F(X)$ is complete, $\{F,T\}$ and $\{S,F\}$ are weakly compatible respectively, then S,T,F have an unique common fixed point in X.

Theorem 2.3 Let (X, d) be a 2-metric space, $T,F,G : X \to X$ three mappings satisfying that $T(X) \subset F(X) \cap G(X)$. Suppose that for each $x,y,a \in X$,

$$d(Tx,Ty,a) \leq \alpha\big(d(Gx,Fy,a),d(Gx,Tx,a),d(Fy,Ty,a)\big),$$

where $\alpha \in \mathscr{A}$. If one of $T(X),F(X)$ and $G(X)$ is complete, $\{G,T\}$ and $\{T,F\}$ are weakly compatible respectively, then T,F,G have an unique common fixed point in X.

Theorem 2.4 Let (X, d) be a 2-metric space, $S,T : X \to X$ two mappings. Suppose that for each $x,y,a \in X$,

$$d(Tx,Sy,a) \leq \alpha\big(d(x,y,a),d(x,Tx,a),d(y,Sy,a)\big),$$

where $\alpha \in \mathscr{A}$. If $S(X)$ or $T(X)$ is complete, then S,T have an unique common fixed point.

Theorem 2.5 Let (X, d) be a complete 2-metric space, $F,G : X \to X$ two surjective mappings. Suppose that for each $x,y,a \in X$,

$$d(x,y,a) \leq \alpha\big(d(Gx,Fy,a),d(Gx,x,a),d(Fy,y,a)\big),$$

where $\alpha \in \mathscr{A}$.Then F,G have an unique common fixed point in X.

Remark 2.1 If $T = S$ and $G = F$ in Theorem 2.4 and Theorem 2.5 respectively, then we obtain two fixed point theorems. The first case is the version of the corresponding conclusion in [3] on 2-metric spaces, the second case is a more generalization of a known fixed point theorem for a mappings with a quasi-contractive condition on 2-metric spaces.

Next, we obtain common fixed point theorems for an infinite family of self-mappings on complete 2-metric spaces.

Theorem 2.6 Let (X,d) be a complete 2-metric space, $\{T_i\}_{i=1}^{\infty}$ a family of self-mappings on X. Suppose that for each $i,j\in\mathbb{N}$ with $i\neq j$ and $a\in X$,

$$d(T_ix,T_jy,a)\leq\alpha\big(d(x,y,a),d(x,T_ix,a),d(y,T_iy,a)\big), \tag{2.8}$$

where $\alpha\in\mathscr{A}$. Then $\{T_i\}_{i=1}^{\infty}$ have an unique common fixed point $z\in X$.

Proof Take an element $x_1\in X$ and construct a sequence $\{x_n\}_{n=1}^{\infty}$ as follows

$$Tx_n=x_{n+1}, n=1,2,\cdots. \tag{2.9}$$

For any $n\in\mathbb{N}$ and $a\in X$, using (2.8) and (2.9), we have

$$d(T_nx_n,T_{n+1}x_{n+1},a)\leq\alpha\big(d(x_n,x_{n+1},a),d(x_n,T_nx_n,a),d(x_{n+1},T_{n+1}x_{n+1},a)\big),$$

i.e.,

$$d(x_{n+1},x_{n+2},a)\leq\alpha\big(d(x_n,x_{n+1},a),d(x_n,x_{n+1},a),d(x_{n+1},x_{n+2},a)\big),$$

using (α2), we obtain

$$d(x_{n+1},x_{n+2},a)\leq k\,d(x_n,x_{n+1},a),\forall n\in\mathbb{N}, a\in X.$$

Hence $\{x_n\}$ is a Cauchy sequence by Lemma 1.1. Let $x_n\to u$ as $n\to\infty$ by the completeness of X.

For any fixed $n\in\mathbb{N}$ and any $i\in\mathbb{N}$ with $i>n$ and $a\in X$, by (2.8) and (2.9),

$$d(T_nu,T_ix_i,a)\leq\alpha\big(d(u,x_i,a),d(u,T_nu,a),d(x_i,T_ix_i,a)\big),$$

i.e.,

$$d(T_nu,x_{i+1},a)\leq\alpha\big(d(u,x_i,a),d(u,T_nu,a),d(x_i,x_{i+1},a)\big). \tag{2.10}$$

Let $i\to\infty$ in (2.10) and using (α1) and Lemma 1.2 and the Cauchy property of $\{x_i\}$, we obtain

$$d(T_nu,u,a)\leq\alpha\big(0,d(u,T_nu,a),0)\big),\forall a\in X.$$

Hence $T_nu=u\,(\forall n\in\mathbb{N})$ by (α2), i.e., u is a common fixed point of $\{T_i\}_{i=1}^{\infty}$.

Suppose that v is also a common fixed point of $\{T_i\}_{i=1}^{\infty}$, then for any $a \in X$,

$$d(u,v,a) = d(T_1u, T_2v, a) \le \alpha\big(d(u,v,a), d(u,T_1u,a), d(v,T_2v,a)\big) = \alpha\big(d(u,v,a),0,0\big),$$

hence $d(u,v,a) = 0$ for all $a \in X$, therefore $u = v$. This shows that u is the unique common fixed point of $\{T_i\}_{i=1}^{\infty}$.

Using Theorem 2.6, we obtain the following more general common fixed point theorem.

Theorem 2.7 Let (X,d) be a complete 2-metric space, $\{T_i\}_{i=1}^{\infty}$ a family of self-mappings on X and $\{m_i\}_{i=1}^{\infty}$ a family of natural numbers. Suppose that for each $i, j \in \mathbb{N}$ with $i \neq j$ and $a \in X$,

$$d(T_i^{m_i}x, T_j^{m_j}y, a) \le \alpha\big(d(x,y,a), d(x,T_i^{m_i}x,a), d(y,T_i^{m_j}y,a)\big), \tag{2.11}$$

where $\alpha \in \mathscr{A}$. Then $\{T_i\}_{i=1}^{\infty}$ have an unique common fixed point $u \in X$.

Proof Let $f_i = T_i^{m_i}$ for all $i = 1,2,\cdots$, then $\{f_i\}_{i=1}^{\infty}$ satisfies all conditions of Theorem 2.6, hence $\{f_i\}_{i=1}^{\infty}$ have an unique common fixed point $u \in X$.

Fix any $i \in \mathbb{N}$. Since $f_iT_iu = T_if_iu = T_iu$, T_iu is a fixed point of f_i. For any $j \in \mathbb{N}$ with $j \neq i$, by (2.11)

$$d(f_iT_iu, f_jT_iu, a) \le \alpha\big(d(T_iu,T_iu,a), d(T_iu, f_iT_iu,a), d(T_iu, f_jT_u,a)\big), \forall a \in X,$$

hence

$$d(T_iu, f_jT_iu, a) \le \alpha\big(0,0,d(T_iu, f_jT_u,a)\big), \forall a \in X.$$

This implies $d(T_iu, f_jT_iu, a) = 0$ for all $a \in X$ by (α2), hence T_iu is a fixed point of f_j for $j \neq i$, further T_iu is a common fixed point of $\{f_k\}_{k=1}^{\infty}$ for any $i \in \mathbb{N}$. Therefore $T_iu = u$ by the uniqueness of common fixed points of $\{f_k\}_{k=1}^{\infty}$ for any $i \in \mathbb{N}$, hence u is a common fixed point of $\{T_i\}_{i=1}^{\infty}$. Obviously, u is the unique common fixed point of $\{T_i\}_{i=1}^{\infty}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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