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KRASNOSELSII'S FIXED POINT THEOREM FOR GENERAL CLASSES OF MAPS

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Abstract. A new Krasnoselskii fixed point result is presented for weakly sequentially upper semicontinuous maps. The proof is immediate from results in the literature [6, 7]. We also extend the results for a general class of maps, namely the $\mathcal{B}^{\kappa}$ maps of Park.

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1. Introduction

In [2, 5] some Schauder and Krasnoselskii fixed point results were presented for weakly sequentially continuous or weakly-strongly sequentially continuous maps. In this note we show how these results can be deduced immediately from results in the literature [6]. In this paper we establish a general Krasnoselskii fixed point result for weakly sequentially continuous maps (Theorem 2.2) and for weakly sequentially upper semicontinuous maps (Theorem 2.4). Later we show the results in this paper extend to upper semicontinuous Kakutani or acyclic or approximable or admissible with respect to Gorniewicz maps. In

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fact we will establish a result for a very general class of maps, namely the $\mathcal{B}^\kappa$ maps of Park.

We now gather together some notation and preliminary facts. Let Ω_E be the bounded subsets of a Banach space E and let K^w be the family of all weakly compact subsets of E. Also let B be the closed unit ball of E. The DeBlasi [4] measure of weak noncompactness is the map $w : \Omega_E \to [0, \infty)$ defined by

$$w(X) = \inf \{t > 0 : \quad \text{there exists} \quad Y \in K^w \quad \text{with} \quad X \subseteq Y + t\, B\} \ ;$$

here $X \in \Omega_E$. For convenience we recall some properties of w:

Let $X_1,\, X_2 \in \Omega_E$. Then

(i). $X_1 \subseteq X_2$ implies $w(X_1) \leq w(X_2)$.

(ii). $w(X_1) = 0$ iff $\overline{X_1^w} \in K^w$; here $\overline{X_1^w}$ is the weak closure of X_1 in E.

(iii). $w(\overline{X_1^w}) = w(X_1)$.

(iv). $w(X_1 \cup X_2) = \max\{w(X_1), w(X_2)\}$.

(v). $w(r\, X_1) = r\, w(X_1)$ for all $r > 0$.

(vi). $w(co(X_1)) = w(X_1)$.

(vii). $w(X_1 + X_2) \leq w(X_1) + w(X_2)$.

(viii). If $\{X_n\}_1^\infty$ is a sequence of nonempty, weakly closed subsets of E with X_1 bounded and $X_1 \supseteq X_2 \supseteq \; ..... \supseteq X_n \supseteq ...$ with $\lim_{n\to\infty} w(X_n) = 0$, then $\cap_{n=1}^\infty X_n \neq \emptyset$.

Suppose $F : Z \subseteq E \to E$. Then F is said to be (1). weakly sequentially continuous if x_n $(n \in N = \{1, 2, ...\})$, $x \in Z$ with $x_n \rightharpoonup x$ implies $F\, x_n \rightharpoonup F\, x$, (2). weakly-strongly sequentially continuous if x_n $(n \in N)$, $x \in Z$ with $x_n \rightharpoonup x$ implies $F\, x_n \to F\, x$, (3). strongly-weakly sequentially continuous if x_n $(n \in N)$, $x \in Z$ with $x_n \to x$ implies $F\, x_n \rightharpoonup F\, x$. On the other hand if $F : Z \to 2^E$ (here 2^E denotes the family of nonempty subsets of E) then F is said to be weakly sequentially upper semicontinuous if for any weakly closed set A of E, $F^{-1}(A)$ is sequentially closed for the weak topology on Z.

Let X be a nonempty, convex subset of a Hausdorff topological vector space E and Y a topological space. Recall a <u>polytope</u> P in X is any convex hull of a nonempty finite subset of X.

Definition 1.1. We say $G \in \mathcal{B}(X, Y)$ if $G : X \to 2^Y$ (the nonempty subsets of Y) is such that for any polytope P in X and any continuous function $g : G(P) \to P$, the composition $g\,(G|_P) : P \to 2^P$ has a fixed point.

Definition 1.2. $F \in \mathcal{B}^\kappa(X, Y)$ (i.e. F is $\mathcal{B}^\kappa$–admissible) if $F : X \to 2^Y$ is such that for any compact, convex subset K of X, there exists a closed map $G \in \mathcal{B}(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

A nonempty subset X of a Hausdorff topological vector space E is said to be admissible if for every compact subset K of X and every neighborhood V of 0, there exists a continuous map $h : K \to X$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E.

In [10] Park proved the following result.

Theorem 1.1. *Let E be a Hausdorff topological vector space and X an admissible, convex subset of E. Then any closed, compact map $F \in \mathcal{B}(X, X)$ has a fixed point.*

Examples of $\mathcal{B}^\kappa$ maps can be found in [10]. An important subclass of $\mathcal{B}$ is the class $\mathcal{U}_c^\kappa$. Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(i). $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions;

(ii). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and

(iii). for any polytope P, $F \in \mathcal{U}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

Definition 1.3. $F \in \mathcal{U}_c^\kappa(X, Y)$ (i.e. F is $\mathcal{U}_c^\kappa$–admissible) if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Note $\mathcal{U}_c^\kappa$ is closed under compositions.

2. Main results

In [6, pp2] the following result was established.

Theorem 2.1. *Let C be a nonempty bounded convex closed subset of a Banach space E and assume $A : C \to C$ is weakly sequentially continuous. In addition suppose there exists a $\alpha \in [0, 1)$ with*

$$w(S_{n+1}) \le \alpha\, w(S_n) \quad \text{for} \quad n \in \{1, 2, ...\}; \tag{2.1}$$

here $S_1 = C$ and $S_{n+1} = \overline{co}\,(A(S_n))$ for $n \in \{1, 2, ...\}$. Then A has a fixed point.

Remark 2.1. Note Theorem 2.2 in [5] follows immediately from Theorem 2.1 (in fact the condition that A is continuous in [5] is not needed).

Remark 2.2. One could replace (2.1) with other weakly compactness type conditions; see Theorem's 2.7-2.9 in [1].

Of course Theorem 2.1 immediately guarantees a Krasnoselskii fixed point theorem. We prove a general result which includes the results in [2, 5].

Theorem 2.2. *Let C be a nonempty bounded convex closed subset of a Banach space E and assume $F : C \to E$ and $G : C \to E$ is such that $(I - G)^{-1}$ is well defined on $(I - G)\,(C)$. Also assume the following conditions hold:*

$$F(C) \subseteq (I - G)\,(C) \tag{2.2}$$

$$(I - G)^{-1}\,F : C \to E \quad \text{is weakly sequentially continuous} \tag{2.3}$$

and

$$\exists\ \alpha \in [0, 1) \quad \text{with} \quad w(S_{n+1}) \le \alpha\, w(S_n) \quad \text{for} \quad n \in \{1, 2, ...\}; \tag{2.4}$$

here $S_1 = C$ and $S_{n+1} = \overline{co}\,((I - G)^{-1}\,F(S_n))$ for $n \in \{1, 2, ...\}$. Then there exists $x \in C$ with $x = F(x) + G(x)$.

Proof. Notice $F(C) \subseteq (I - G)\,(C)$ from (2.2) so $(I - G)^{-1}\,F : C \to C$. The result follows from Theorem 2.1. $\square$

Remark 2.3. Suppose

$$F : C \to F(C) \quad \text{is weakly-strongly sequentially continuous} \tag{2.5}$$

and

$$\begin{cases} (I-G)^{-1} : (I-G)\,(C) \to E \text{ is strongly-weakly} \\ \text{sequentially continuous} \end{cases} \tag{2.6}$$

hold. Then clearly (2.3) is satisfied.

Remark 2.4. Suppose

$$F : C \to F(C) \quad \text{is weakly sequentially continuous} \tag{2.7}$$

and

$$(I-G)^{-1} : (I-G)\,(C) \to E \quad \text{is weakly sequentially continuous} \tag{2.8}$$

hold. Then clearly (2.3) is satisfied.

Remark 2.5. Note if $G : C \to E$ is a contraction (with contractive constant $\beta \in [0,1)$) then it is well known that $(I-G)^{-1}$ exists and is continuous (note $\|(I-G)(x) - (I-G)(y)\| \geq (1-\beta)\,\|x-y\|$ for $x,\, y \in C$) so in particular (2.6) holds. If in addition

$$F(x) + G(y) \in C \quad \text{for all} \quad x,\, y \in C \tag{2.9}$$

holds, then (2.2) is satisfied. To see this first notice (2.9) implies $F(C) + G(C) \subseteq C$. Now let $w \in F(C)$. Then $z = G(z) + w$ has a unique solution since $z \to G(z) + w$ is a contraction which maps C to C. Thus $w \in (I-G)(C)$, so (2.2) is satisfied.

Remark 2.6. If $(I-G)^{-1}$ is linear and continuous on E then $(I-G)^{-1}$ is weakly continuous on E (see [3. pp39]), so (2.8) holds. Note trivially (2.6) is satisfied (of course we dont need to assume $(I-G)^{-1}$ is linear if we are interested in (2.6)).

If $(I-G)^{-1}$ exists on E (an example of this is if $G : E \to E$ is a contraction) and if in addition

$$\text{if} \quad y \in C \quad \text{and} \quad x = F(y) + G(x), \quad \text{then} \quad x \in C \tag{2.10}$$

holds, then (2.2) is satisfied. To see this let $w \in F(C)$. Then there exists $y \in C$ with $w = F(y)$. Let $z = (I - G)^{-1}(w)$. Thus $z - G(z) = w$ which implies $z = G(z) + F(y)$ and so (2.10) implies $z \in C$. Thus $(I - G)^{-1}(w) \in C$ so $w \in (I - G)(C)$ and as a result (2.2) is satisfied.

Remark 2.7. Suppose there exists $\alpha \in [0, 1)$ with

$$w(F(X) + G(X)) \leq \alpha\, w(X), \ \forall\, X \subseteq C. \tag{2.11}$$

Then (2.4) holds. To see this notice for $n \in \{1, 2, ...\}$ that

$$\begin{aligned} (I - G)^{-1}\, F(S_n) &\subseteq F(S_n) + G\,(I - G)^{-1}\, F(S_n) \subseteq F(S_n) + G(S_{n+1}) \\ &\subseteq F(S_n) + G(S_n) \end{aligned}$$

since $S_{n+1} = \overline{co}\,((I - G)^{-1}\, F(S_n)) \subseteq S_n$. Thus

$$\begin{aligned} w(S_{n+1}) &= w(\overline{co}\,((I - G)^{-1}\, F(S_n))) = w((I - G)^{-1}\, F(S_n)) \\ &\leq w(F(S_n) + G(S_n)) \leq \alpha\, w(S_n). \end{aligned}$$

Remark 2.8. One could replace (2.4) with other weakly compactness type conditions; see Theorem's 2.7-2.9 in [1].

From the above remarks one can see that Theorem 2.1 of [2] and Theorem 2.3 of [5] follow immediately from Theorem 2.2.

For completeness we now discuss the multivalued situation. In [7] the following result was established.

Theorem 2.3. *Let C be a nonempty bounded convex closed subset of a Banach space E and assume $A : C \to K(C)$ is weakly sequentially upper semicontinuous; here $K(C)$ denotes the family of nonempty closed convex subsets of C. In addition suppose there exists a $\alpha \in [0, 1)$ with*

$$w(S_{n+1}) \leq \alpha\, w(S_n) \quad \textit{for} \ \ n \in \{1, 2, ...\}; \tag{2.12}$$

here $S_1 = C$ and $S_{n+1} = \overline{co}\,(A(S_n))$ for $n \in \{1, 2, ...\}$. Then A has a fixed point.

Remark 2.9. One could replace (2.12) with other weakly compactness type conditions; see Theorem's 2.7-2.9 in [1].

Theorem 2.3 immediately guarantees our next result.

Theorem 2.4. *Let C be a nonempty bounded convex closed subset of a Banach space E and assume $F : C \to 2^E$ and $G : C \to E$ is such that $(I-G)^{-1}$ is well defined on $(I-G)(C)$; here 2^E denotes the family of nonempty subsets of E. Also assume the following conditions hold:*

$$F(C) \subseteq (I-G)(C) \tag{2.13}$$

$$\begin{cases} (I-G)^{-1} F : C \to K(E) \ \ \textit{is weakly sequentially} \\ \textit{upper semicontinuous} \end{cases} \tag{2.14}$$

and

$$\exists\, \alpha \in [0,1) \ \ \textit{with} \ \ w(S_{n+1}) \leq \alpha\, w(S_n) \ \ \ \textit{for} \ \ n \in \{1,2,....\}; \tag{2.15}$$

here $S_1 = C$ and $S_{n+1} = \overline{co}\,((I-G)^{-1} F(S_n))$ for $n \in \{1,2,...\}$. Then there exists $x \in C$ with $x \in F(x) + G(x)$.

Proof. Notice $F(C) \subseteq (I-G)(C)$ from (2.13) so $(I-G)^{-1} F : C \to K(C)$. The result follows from Theorem 2.3. $\square$

We now show there is an obvious analogue of Theorem 2.4 (respectively Theorem 2.2) for upper semicontinuous Kakutani or acyclic or approximable or admissible with respect to Gorniewicz (respectively continuous) maps $(I-G)^{-1} F$. In this situation w is replaced by the Kuratowski measure of noncompactness α or the ball measure of noncompactness χ. We will write our results with the Kuratowski measure of noncompactness α.

For simplicity we will present fixed point results in Banach spaces (it is trivial to extend the ideas to topological vector spaces).

Theorem 2.5. *Let C be a nonempty bounded convex closed subset of a Banach space E and assume $A \in \mathcal{B}^{\kappa}(C,C)$. In addition suppose there exists a $r \in [0,1)$ with*

$$\alpha(S_{n+1}) \leq r\, \alpha(S_n) \ \ \ \textit{for} \ \ n \in \{1,2,....\}; \tag{2.16}$$

here $S_1 = C$ *and* $S_{n+1} = \overline{co}\,(A(S_n))$ *for* $n \in \{1, 2, ...\}$. *Then* A *has a fixed point.*

Proof. Notice

$$\alpha(S_2) = \alpha(\overline{co}\,(A(S_1))) = \alpha(A(S_1)) \le r\,\alpha(S_1) \quad \text{and} \quad S_2 \subseteq \overline{co}\,(C) = C = S_1.$$

It is now easy to see that

$$S_{n+1} \subseteq S_n \quad \text{and} \quad \alpha(S_{n+1}) \le r^n\,\alpha(S_1) \quad \text{for} \quad n \in \{1, 2, ...\}.$$

Thus $\alpha(S_n) \to 0$ as $n \to \infty$ so $S_\infty = \cap_1^\infty S_n$ is nonempty, convex and closed. Notice S_∞ is compact since $\alpha(S_\infty) = 0$. Also since

$$A(S_n) \subseteq A(S_{n-1}) \subseteq \overline{co}\,(A(S_{n-1})) = S_n, \ \forall\, n,$$

we have $A(S_\infty) \subseteq S_\infty$. Now $A \in \mathcal{B}^\kappa(C, C)$ so there exists a closed map $G \in \mathcal{B}(S_\infty, S_\infty)$ with $G(x) \subseteq F(x)$ for $x \in S_\infty$. Now Theorem 1.1 guarantees that there exists a $x_0 \in S_\infty$ with $x_0 \in G(x_0) \subseteq F(x_0)$. $\square$

Remark 2.10. One could replace (2.16) with other compactness type conditions; see Theorem's 2.1-2.5 in [8] and Theorem's 2.1-2.2 of [9] (all these compactness conditions are in topological vector spaces). Also the boundedness assumption on C can be removed in certain situations; see Theorem's 2.1-2.5 in [8] and Theorem 2.2 of [9].

Theorem 2.6. *Let* C *be a nonempty bounded convex closed subset of a Banach space* E *and assume* $F : C \to 2^E$ *and* $G : C \to E$ *is such that* $(I - G)^{-1}$ *is well defined on* $(I - G)\,(C)$. *Also assume the following conditions hold:*

$$F(C) \subseteq (I - G)\,(C) \tag{2.17}$$

$$(I - G)^{-1}\,F \in \mathcal{B}^\kappa(C, E) \tag{2.18}$$

and

$$\exists\, r \in [0, 1) \ \ \textit{with} \ \ \alpha(S_{n+1}) \le r\,\alpha(S_n) \quad \textit{for} \ \ n \in \{1, 2, ...\}; \tag{2.19}$$

here $S_1 = C$ *and* $S_{n+1} = \overline{co}\,((I - G)^{-1}\,F(S_n))$ *for* $n \in \{1, 2, ...\}$. *Then there exists* $x \in C$ *with* $x \in F(x) + G(x)$.

Proof. Notice $F(C) \subseteq (I-G)\,(C)$ from (2.17) so $(I-G)^{-1}\,F \in \mathcal{B}^{\kappa}(C,C)$. The result follows from Theorem 2.5. $\square$

Remark 2.11. Suppose

$$F \in \mathcal{U}_c^{\kappa}(C,E) \tag{2.20}$$

and

$$(I-G)^{-1} : (I-G)\,(C) \to E \text{ is continuous} \tag{2.21}$$

hold. Then $(I-G)^{-1}\,F \in \mathcal{U}_c^{\kappa}(C,E)$ since $\mathcal{U}_c^{\kappa}$ is closed under compositions. As a result $(I-G)^{-1}\,F \in \mathcal{B}^{\kappa}(C,E)$, so (2.18) holds.

Remark 2.12. Note if $G : C \to E$ is a contraction then it is well known that $(I-G)^{-1}$ exists and is continuous. If in addition suppose

$$F(x) + G(y) \subseteq C \quad \text{for all} \quad x,\, y \in C \tag{2.22}$$

holds. Fix $z \in C$ and let $w \in F(z)$. Then as in Remark 2.5, $x = G(x) + w$ has a unique solution so $w \in (I-G)(C)$. As a result $F(z) \subseteq (I-G)(C)$. We can do this for each $z \in C$ so (2.17) is satisfied.

Remark 2.13. If $(I-G)^{-1}$ exists on E and if in addition

$$\text{if} \quad y \in C \quad \text{and} \quad x \in F(y) + G(x), \quad \text{then} \quad x \in C \tag{2.23}$$

holds, then as in Remark 2.6 it is easy to see that (2.17) is satisfied.

Remark 2.14. One could replace (2.19) with other compactness type conditions; see Theorem's 2.1-2.5 in [8] and Theorem's 2.1-2.2 of [9]. Also the boundedness assumption on C can be removed in certain situations; see Theorem's 2.1-2.5 in [8] and Theorem 2.2 of [9].

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