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STRONG CONVERGENCE THEOREMS OF AN INERTIAL ALGORITHM FOR APPROXIMATING COMMON FIXED POINTS OF A FAMILY OF MULTI-VALUED PSEUDOCONTRACTIVE-TYPE MAPS

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Abstract. In this paper, we construct an inertial algorithm that approximates a common fixed point of a countable family of multi-valued total asymptotically strict quasi-ϕ-pseudocontractive maps in real Banach spaces and prove strong convergence of the sequence generated by this algorithm. We provide a numerical example to illustrate the implementability of the proposed algorithm and also show that our algorithm converges faster than some algorithms recently proposed by other authors for solving this class of problem. Furthermore, we present some applications of our theorems. Finally, our theorems are significant improvement on several important recent results.

Keywords: inertial algorithm; multi-valued total asymptotically strict quasi-ϕ-pseudocontractions; equally continuous maps; strong convergence.

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1. Introduction

Let E be a smooth real Banach space. Define the Lyapunov functional $\phi : E \times E \to \mathbb{R}$ by

$$\phi(u,y) = \|u\|^2 - 2\langle u, Jy\rangle + \|y\|^2 \ \forall,\ u,y \in E.$$

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From the definition of ϕ it is easy to verify that

$$(\|u\|-\|y\|)^2 \leq \phi(u,y) \leq (\|u\|+\|y\|)^2 \ \forall,\ u,y \in E, \tag{1}$$

and

$$\phi(u,v) = \phi(u,z) + \phi(z,v) + 2\langle u-z, Jz-Jv\rangle \ \forall,\ u,v,z \in E. \tag{2}$$

Definition 0.1. *(See e.g., Zhang et al.* [3], *Chidume et al.* [11]*) A map $T : C \to 2^C$ is called*

(1) totally quasi-ϕ-asymptotically nonexpansive if $F(T) \neq \emptyset$ and there exist nonnegative real sequences $\{\gamma_n\}, \{\delta_n\}$ with $\gamma_n \to 0, \delta_n \to 0$ $(n \to \infty)$ and a strictly increasing and continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0) = 0$ such that

$$\phi(u^*, \eta_n) \leq \phi(u^*, u) + \gamma_n \rho\left[\phi(u^*, u)\right] + \delta_n,\ \forall,\ u \in C, u^* \in F(T), \eta_n \in T^n u,\ n \geq 1.$$

(2) asymptotically strict quasi-ϕ-pseudocontraction if $F(T) \neq \emptyset$ and there exist nonnegative real sequence $\{\gamma_n\} \subset [0,\infty)$ with $\gamma_n \to 0$ as $n \to \infty$ and a constant $k \in [0,1)$ such that

$$\phi(u^*, \eta_n) \leq (1-\gamma_n)\phi(u^*, u) + k\phi(u, \eta_n)\ \forall,\ u \in C, u^* \in F(T), \eta_n \in T^n u,\ n \geq 1.$$

(3) total asymptotically strict quasi-ϕ-pseudocontraction if $F(T) \neq \emptyset$ and there exist non-negative real sequences $\{\gamma_n\} \subset [0,\infty), \{\delta_n\} \subset [0,\infty)$ with $\gamma_n \to 0, \delta_n \to 0$ $(n \to \infty)$ and a constant $k \in [0,1)$ such that

$$\phi(u^*, \eta_n) \leq \phi(u^*, u) + k\phi(u, \eta_n) + \gamma_n \rho\left[\phi(u^*, u)\right] + \delta_n\ \forall,\ u \in C, u^* \in F(T), \eta_n \in T^n u,\ n \geq 1,$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing and continuous function with $\rho(0) = 0$.

Remark 1. *We remark that the class of total asymptotically strict quasi-ϕ-pseudocontractions contains as subclasses the class of totally quasi-ϕ-asymptotically nonexpansive maps, the class of asymptotically strict quasi-ϕ-pseudocontractions, the class of strict quasi-ϕ-pseudocontractions, and the class of asymptotically strict quasi-ϕ-pseudocontractions in the intermediate sense (see e.g.,* [2], [3], [5] *for definitions and comparison).*

Remark 2. *We remark also that the class of totally quasi-ϕ-asymptotically nonexpansive multi-valued maps contains as proper subclasses the class of relatively nonexpansive multi-valued* *maps, the class of quasi-ϕ-nonexpansive multi-valued maps, and the class of quasi-ϕ-asymptotically nonexpansive multi-valued maps (see e.g.,* [15], [16], [20] *for definitions and comparison).*

Iterative methods have been utilized to approximate fixed points of totally quasi-ϕ-asymptotically nonexpansive multi-valued maps (see e.g., [15], [16], [11], [20]), strict quasi-ϕ-pseudocontractions (see e.g., [6]), asymptotically strict quasi-ϕ-pseudocontractions in the intermediate sense (see e.g., [5]).

In 2011, Qin *et al.* [1] considered in uniformly convex and smooth real Banach spaces the following hybrid projection algorithm:

$$\begin{cases} u_0 \in E \text{ chosen arbitrary}, \; C_1 = C, \quad u_1 = \Pi_{C_1} u_0, \\ C_{n+1} = \{v \in C_n : \phi(u_n, T^n u_n) \le \frac{2}{1-k} \langle u_n - v, Ju_n - JT^n u_n \rangle + \mu_n \frac{M_n}{1-k}\}, \\ u_{n+1} = \Pi_{C_{n+1}} u_0, \quad n \ge 1, \end{cases}$$

where $M_n = \sup\{\phi(u^*, u_n) : u^* \in F(T)\}$. The authors prove that the sequence generated by the above algorithm converges strongly to $\Pi_{F(T)} u_0$ under the following assumptions:

$(C1)$ T is a closed and asymptotically strict quasi-ϕ-pseudocontraction;
$(C2)$ T is asymptotically regular on C;
$(C3)$ $F(T)$ is nonempty and bounded.

Zhang [4] established the results in [1] in the frame work of reflexive, smooth and strictly convex real Banach spaces in which both the space and its dual space have the Kadec-Klee property.

In 2015, Wang and Yang [2] enlarged the class of operators for which the results in the paper of Qin *et al.* [1] are applicable, by proving for the class of total asymptotically strict quasi-ϕ-pseudocontraction that the sequence generated by the following algorithm:

$$\begin{cases} u_0 \in E \text{ chosen arbitrary}, \; C_1 = C, \quad u_1 = \Pi_{C_1} u_0, \\ C_{n+1} = \{v \in C_n : \phi(u_n, T^n u_n) \le \frac{2}{1-k} \langle u_n - v, Ju_n - JT^n u_n \rangle + \theta_n\}, \\ u_{n+1} = \Pi_{C_{n+1}} u_0, \quad n \ge 1, \end{cases}$$

where $\theta_n = \mu_n \frac{M_n}{1-k} + \frac{\nu_n}{1-k}$, $M_n = \sup\{\rho(\phi(u^*, u_n)) : u^* \in F(T)\}$, converges strongly to $\Pi_{F(T)} u_0$ under conditions $(C2)$ and $(C3)$ with condition $(C1)$ replaced by condition:

$(C1^*)$ T is a closed and (k,μ_n,v_n,ρ)-total asymptotically strict quasi-ϕ-pseudocontraction.

Recently, Zhang *et al.* [3] proposed the following hybrid projection algorithm for approximating common fixed points of a *finite family of closed* total asymptotically strict quasi-ϕ-pseudocontractions:

$$(3)\qquad \begin{cases} u_0 \in E \textit{ chosen arbitrary},\ C_0^i = C,\ i = 1,2,...,N, \quad C_0 = \cap_{i=1}^N C_0^i, \\ y_n^i = J^{-1}\left[\alpha_n J u_n + (1-\alpha_n) J T_i^n u_n\right], \\ C_{n+1}^i = \{v \in C_n : \phi(u_n, y_n^i) \le \phi(v, u_n) + \frac{2}{1-k}\langle u_n - v, Ju_n - JT_i^n u_n\rangle + \theta_n\}, \\ C_{n+1} = \cap_{i=1}^N C_{n+1}^i, \\ u_{n+1} = \Pi_{C_{n+1}} u_0, \end{cases}$$

where $\theta_n = \mu_n \frac{M_n}{1-k} + \frac{v_n}{1-k}$, $M_n = \sup\{\rho(\phi(u^*, u_n)) : u^* \in \Omega\}$, $\{T_i\}_{i=1}^N$ is a finite family of closed (k,μ_n,v_n,ρ)-total asymptotically strict quasi-ϕ-pseudocontractions, $\Omega := \cap_{i=1}^N F(T_i)$ is nonempty and bounded subset of C. Assuming each T_i is *asymptotically regular on* C, the authors prove that the sequence generated by the above algorithm converges strongly to $\Pi_\Omega u_0$. *It is important to observe that the authors assumed the uniformity of the parameters* k,μ_n,v_n *and* ρ *for the family of maps they considered. This condition is restrictive and as shall been seen later, we dispense with this condition in this paper.*

Methods of speeding up the rate of convergence of iterative algorithms have attracted the attention of numerous researchers since iterative algorithms that converge faster are more desirable in any possible applications. This is probably because such algorithms minimize computational cost. One of such methods was introduced by Polyak [27] who studied the heavy ball method, a two step iterative method for minimizing a smooth convex function, g. The algorithm takes the following form:

$$(4)\qquad \begin{cases} x_0, x_1 \in E, \\ y_n = x_n + \alpha_n(x_n - x_{n-1}) \\ x_{n+1} = y_n - \lambda_n \Delta g(x_n), n \ge 1, \end{cases}$$

where $\alpha_n \in [0,1)$ is an extrapolation term and λ_n is a step-size parameter to be chosen sufficiently small. The difference compared to a standard gradient method is that in each iteration, the extrapolated term y_n is used instead of x_n. It is remarkable that this minor change greatly improves the performance of the scheme, by speeding up convergence property. The term $\alpha_n(x_n - x_{n-1})$ is called *inertial*; hence algorithm (4) is called *inertial algorithm*. Since then, the study of inertial-type algorithm have become of great interest to several researchers, see for example, inertial forward-backward splitting methods (see e.g., [23], [24], [25]), inertial Douglas-Rachford splitting method [28], inertial ADMM (see e.g., [7], [8]), and inertial forward-backward-forward method [26].

Very recently, Chidume *et al.* [10] considered the following inertial algorithm for approximating fixed point of relatively nonexpansive maps in uniformly convex and uniformly smooth real Banach spaces:

$$\begin{cases} u_0, u_1 \in E \ \textit{chosen arbitrary}, \ C_0 = E, \\ w_n = u_n + \alpha_n(u_n - u_{n-1}), \\ v_n = J^{-1}\left[(1-\beta)Jw_n + \beta JTw_n\right] \\ C_{n+1} = \{v \in C_n : \phi(v, v_n) \le \phi(v, w_n)\}, \\ u_{n+1} = \Pi_{C_{n+1}} u_0, n \ge 1, \end{cases} \tag{5}$$

Our contribution in this paper, *is to construct in certain Banach spaces an inertial iterative algorithm that approximates common fixed points of an infinite family of multi-valued* $(k_i, \gamma_n^i, \delta_n^i, \psi_i)$*-total asymptotically strict quasi-*ϕ*-pseudocontractive maps*. Using some numerical illustration, we show that our algorithm converges much more faster than some algorithms proposed by other authors. Furthermore, we apply our theorem to solve a system of generalized mixed equilibrium problem and a system of convex minimization problem. Corollaries of our theorems are significant improvement on several important recent results announced by other authors, in particular, the results of Chidume *et. al* [10], Zhang *et al.* [3], Wang and Yang [2], Zhang [4], and Qin *et al.* [1] (see concluding remark below).

2. Preliminaries

Definition 0.2. *(See e.g., Feng et al.* [14]*, Wang and Yang* [2]*, Chidume et al.* [11]*) A map* $T : C \to 2^C$ *is said to be*

- *uniformly L-Lipschitz continuous if there exists a constant* $L > 0$ *such that*

$$\|\eta_u - \eta_y\| \le L\|u - y\| \ \forall \ \eta_u \in T^n u, \eta_y \in T^n y, \ n \ge 1;$$

- *uniformly continuous if for* $u_n, y_n \in C$ *we have that*

$$\lim_{n\to\infty} \|u_n - y_n\| = 0 \implies \lim_{n\to\infty} \|\eta_{n_u} - \eta_{n_y}\| = 0 \ \forall \ \eta_{n_u} \in Tu_n, \eta_{n_y} \in Ty_n;$$

- *equally continuous if for* $u_n, y_n \in C$ *we have that*

$$\lim_{n\to\infty} \|u_n - y_n\| = 0 \implies \lim_{n\to\infty} \|\eta_{n_u} - \eta_{n_y}\| = 0 \ \forall \ \eta_{n_u} \in T^n u_n, \eta_{n_y} \in T^n y_n.$$

- *asymptotically regular on C if for any bounded subset D of C,*

$$\lim_{n\to\infty} \sup_{x\in D} \{\|\eta^{n+1} - \eta^n\|\} = 0, \ \eta^n \in T^n x, \eta^{n+1} \in T^{n+1} x.$$

Remark 3. *It is easy to see that the class of uniformly L-Lipschitz multi-valued maps is a proper subclass of the class of uniformly continuous multi-valued maps and the class of uniformly continuous multi-valued maps is a proper subclass of the class of equally continuous multi-valued maps.*

Definition 0.3. *A map* $G : D \to 2^D$ *is said to be closed if for any* $u_n \in D$ *such that* $u_n \to u$, $w_n \to y$, $w_n \in Gu_n$*, we have that* $y \in Gu$*.*

We now present some lemmas that will be used in the sequel.

Lemma 0.4 (Kamimura and Takahashi, [22])**.** *Let X be a real smooth and uniformly convex Banach space, and let* $\{x_n\}$ *and* $\{y_n\}$ *be two sequences of* X*. If either* $\{x_n\}$ *or* $\{y_n\}$ *is bounded and* $\phi(x_n, y_n) \to 0$ *as* $n \to \infty$*, then* $\|x_n - y_n\| \to 0$ *as* $n \to \infty$*.*

Lemma 0.5 (see Chang *et al.* [15])**.** *Let X be a uniformly smooth and strictly convex real Banach space with Kadec-Klee property, and D be a nonempty closed convex subset of* X*. Let* $\{u_n\}$ *and* $\{y_n\}$ *be two sequences in D such that* $u_n \to u^*$ *and* $\phi(u_n, y_n) \to 0$*, where* ϕ *is the function defined by* (1)*, then,* $y_n \to u^*$*.*

Lemma 0.6. *Let X be a uniformly smooth and strictly convex real Banach space with Kadec-Klee property, and D be a nonempty closed convex subset of X. Let $G : D \to 2^D$ be a closed and $(k, \{\gamma_n\}, \{\delta_n\}, \psi)$-total asymptotically strict quasi-ϕ-pseudocontractive multi-valued map, then the fixed point set $F(G)$ of G is a closed and convex subset of D.*

Proof. Let $\{\mu_n\}$ be a sequence in $F(G)$ such that $\mu_n \to \mu$. Then, $\mu_n \in G\mu_n$. By closedness of G we have that $\mu \in G\mu$. Hence, $F(G)$ is closed. Let $\mu, \upsilon \in F(G)$, $\lambda \in (0,1)$. Set $w = \lambda\mu + (1-\lambda)\upsilon$. We want to show that $w \in F(G)$. Let $\{z_n\}$ be a sequence generated by

$$z_1 \in Gw, z_2 \in Gz_1 \subset G^2w, z_3 \in Gz_2 \subset G^3w, \dots, z_n \in Gz_{n-1} \subset G^n w, \dots. \tag{6}$$

Then,

$$\begin{aligned}
\phi(w, z_n) &= \|w\|^2 - 2\langle w, Jz_n\rangle + \|z_n\|^2 \\
&= \|w\|^2 - 2\lambda\langle \mu, Jz_n\rangle + (1-\lambda)\langle \upsilon, Jz_n\rangle + \|z_n\|^2 \\
&= \|w\|^2 + \lambda\phi(\mu, z_n) + (1-\lambda)\phi(\upsilon, z_n) - \lambda\|\mu\|^2 - (1-\lambda)\|\upsilon\|^2. \qquad (7)
\end{aligned}$$

Since, G is $(k, \{\gamma_n\}, \{\delta_n\}, \psi)$-total asymptotically strict quasi-ϕ-pseudocontractive, we have that

$$\begin{aligned}
& \lambda\phi(\mu, z_n) + (1-\lambda)\phi(\upsilon, z_n) \\
\le\ & \lambda\left[\phi(\mu, w) + k\phi(w, z_n) + \gamma_n\psi\big(\phi(\mu, w)\big) + \delta_n\right] + (1-\lambda)\left[\phi(\upsilon, w) + k\phi(w, z_n)\right. \\
& \left. + \gamma_n\psi\big(\phi(\upsilon, w)\big) + \delta_n\right] \\
=\ & \lambda[\|\mu\|^2 - 2\langle \mu, Jw\rangle + \|w\|^2] + (1-\lambda)[\|\upsilon\|^2 - \langle \upsilon, Jw\rangle + \|w\|^2] + k\phi(w, z_n) + \lambda\gamma_n\psi\big(\phi(\mu, w)\big) \\
& + (1-\lambda)\gamma_n\psi\big(\phi(\upsilon, w)\big) + \delta_n \\
=\ & \lambda\|\mu\|^2 + (1-\lambda)\|\upsilon\|^2 + \|w\|^2 - 2\langle w, Jw\rangle + k\phi(w, z_n) + \lambda\gamma_n\psi\big(\phi(\mu, w)\big) \\
& + (1-\lambda)\gamma_n\psi\big(\phi(\upsilon, w)\big) + \delta_n \\
=\ & \lambda\|\mu\|^2 + (1-\lambda)\|\upsilon\|^2 - \|w\|^2 + k\phi(w, z_n) + \lambda\gamma_n\psi\big(\phi(\mu, w)\big) \\
(8)\ & + (1-\lambda)\gamma_n\psi\big(\phi(\upsilon, w)\big) + \delta_n.
\end{aligned}$$

Substituting inequality (8) into inequality (7), we have that

$$\phi(w, z_n) \le k\phi(w, z_n) + \lambda\gamma_n\psi\big(\phi(\mu, w)\big) + (1-\lambda)\gamma_n\psi\big(\phi(\upsilon, w)\big) + \delta_n,$$

which implies that

$$\phi(w,z_n) \le \frac{\lambda\gamma_n}{1-k}\psi\big(\phi(\mu,w)\big) + \frac{(1-\lambda)\gamma_n}{1-k}\psi\big(\phi(\upsilon,w)\big) + \frac{\delta_n}{1-k}.$$

Thus, $\phi(w,z_n) \to 0$ as $n \to \infty$. By Lemma 0.5, we obtain that $z_n \to w$ as $n \to \infty$. Also, $z_{n+1} \to w$ as $n \to \infty$. Since $z_{n+1} \in Gz_n$, closedness of G implies that $w \in Gw$. Thus, $w \in F(G)$. Hence, $F(G)$ is convex. □

Lemma 0.7. *Let X be a uniformly convex and smooth real Banach space, and D be a nonempty closed convex subset of X. Let $G : D \to 2^D$ be a closed and $(k,\{\gamma_n\},\{\delta_n\},\psi)$-total asymptotically strict quasi-ϕ-pseudocontractive multi-valued map, then the fixed point set $F(G)$ of G is a closed and convex subset of D.*

Proof. The proof follows the same pattern as in the proof of Lemma 28. By using the same argument as in the proof of Lemma 28, we obtain that $F(G)$ is closed. Let $\mu,\upsilon \in F(G)$, $\lambda \in (0,1)$. Set $w = \lambda\mu + (1-\lambda)\upsilon$. Let $\{z_n\}$ be a sequence generated by

$$z_1 \in Gw, z_2 \in Gz_1 \subset G^2w, z_3 \in Gz_2 \subset G^3w, \ \dots, z_n \in Gz_{n-1} \subset G^nw, \ \dots. \tag{9}$$

Then, following the same argument as in the proof of Lemma 28, we obtain that $\phi(w,z_n) \to 0$ as $n \to \infty$. By Lemma 0.4, we obtain that $z_n \to w$ as $n \to \infty$. Also, $z_{n+1} \to w$ as $n \to \infty$. Since $z_{n+1} \in Gz_n$, closedness of G implies that $w \in Gw$. Thus, $w \in F(G)$. Hence, $F(G)$ is convex. □

Lemma 0.8. *Let X be a smooth, strictly convex and reflexive real Banach space with dual space X^* such that both X and X^* have the Kadec-Klee property and let D be a nonempty, closed, convex subset of X. Let $G : D \to 2^D$ be a closed $\{k,\gamma_n,\delta_n,\rho\}$-total asymptotically strict quasi-ϕ-pseudocontractive multi-valued map. Then, $F(G)$ is a closed and convex subset of D.*

Proof. By using the same argument as in the proof of Lemma 28, we obtain that $F(G)$ is closed. Let $\mu,\upsilon \in F(G)$, $\lambda \in (0,1)$. Set $w = \lambda\mu + (1-\lambda)\upsilon$. Let $\{z_n\}$ be a sequence generated by

$$z_1 \in Gw, z_2 \in Gz_1 \subset G^2w, z_3 \in Gz_2 \subset G^3w, \ \dots, z_n \in Gz_{n-1} \subset G^nw, \ \dots. \tag{10}$$

Then, following the same argument as in the proof of Lemma 28, we obtain that $\phi(w,z_n) \to 0$ as $n \to \infty$. Consequently, it follows from (1) that

$$\|z_n\| \to \|w\|, \tag{11}$$

hence,

$$\|Jz_n\| \to \|Jw\|. \tag{12}$$

By reflexivity of X^* we have without loss of generality that $Jz_n \rightharpoonup y \in X^*$. Again, by reflexivity of X, we have that $J(X) = X^*$. Therefore, there exists a point $z \in X$ such that $Jz = y$. Since

$$\phi(w, z_n) = \|w\|^2 - 2\langle w, Jz_n\rangle + \|z_n\|^2 = \|w\|^2 - 2\langle w, Jz_n\rangle + \|Jz_n\|^2,$$

by taking liminf in the last equation and using the weak lower semi-continuity of norm, we have that

$$0 \geq \|w\|^2 - 2\langle w, Jz\rangle + \|Jz\|^2 \quad = \quad \|w\|^2 - 2\langle w, Jz\rangle + \|z\|^2 = \phi(w, z),$$

which implies that $w = z$. Thus, $Jz_n \rightharpoonup Jw$. By (12) and the Kadec-Klee property of X^*, we have that $Jz_n \to Jw$. By norm-to-weak continuity of J^{-1}, we have that, $z_n \rightharpoonup w$. Combining this with (11) and the Kadec-Klee property X, we have that $z_n \to w$ (as $n \to \infty$). Therefore, $z_{n+1} \to w$ (as $n \to \infty$). Since $z_{n+1} \in Gz_n$, then by closedness of G, we have that $w \in Gw$. Thus, $w \in F(G)$. Hence, $F(G)$ is convex. This completes the proof. □

Lemma 0.9 (see Alber [29]). *Let D be a nonempty closed and convex subset of a be a reflexive strictly convex and smooth Banach space X. Then,*

$$\phi\Big(u, \Pi_D y\Big) + \phi\Big(\Pi_D y, y\Big) \leq \phi(u, y), \forall u \in D,\ y \in X. \tag{13}$$

Lemma 0.10 (Wei and Zhou, [21]). *Let E be a real reflexive, strictly convex and smooth Banach space, $A : E \to 2^{E^*}$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$, then for any $x \in E$. $y \in A^{-1}0$ and $r > 0$, we have*

$$\phi(y, Q_r^A x) + \phi(Q_r^A x, x) \leq \phi(y, x),$$

where $Q_r^A : E \to E$ is defined by $Q_r^A x := (J + rA)^{-1} Jx$.

Lemma 0.11 (Deng and Bai [9]). *The unique solutions to the positive integer equation*

$$n = i_n + \frac{(m_n - 1)m_n}{2}, m_n \geq i_n, n = 1, 2, 3, ... \tag{14}$$

are

$$i_n = n - \frac{(m_n-1)m_n}{2},\ m_n = -\left[\frac{1}{2} - \sqrt{2n+\frac{1}{4}}\right], n = 1,2,3,..., \tag{15}$$

where $[x]$ *denotes the maximal integer that is not larger than x.*

3. Main results

1. INERTIAL ALGORITHM

In what follows, i_n and m_n are the unique solutions to the positive integer equation: $n = i + \frac{(m-1)m}{2}$ $(m \geq i, n = 1,2,...)$. That is, for each $n \geq 1$, there exist unique i_n and m_n such that

$$i_1 = 1,\quad i_2 = 1,\quad i_3 = 2,\quad i_4 = 1,\quad i_5 = 2,\quad i_6 = 3,\quad i_7 = 1,\quad i_8 = 2, ...;$$

$$m_1 = 1,\quad m_2 = 2,\quad m_3 = 2,\quad m_4 = 3,\quad m_5 = 3,\quad m_6 = 3,\quad m_7 = 4,\quad m_8 = 4, ...$$

See Deng and Bai [9].

Theorem 1.1. *Let E be a uniformly convex and smooth real Banach space and* $\{T_i\}_{i=1}^{\infty}$, $T_i : E \to 2^E$ *be an infinite family of equally continuous and totally asymptotically strict quasi-*ϕ*-pseudocontractive multi-valued maps with nonnegative real sequences* $\{\gamma_n^i\}, \{\delta_n^i\}$*, a constant* $k_i \in [0,1)$ *and a sequence of strictly increasing and continuous functions* $\{\psi_i\}$, $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\gamma_n^{(i)} \to 0, \delta_n^{(i)} \to 0$ *and* $\psi_i(0) = 0$. *Suppose* $\Omega := \cap_{i=1}^{\infty} F(T_i)$ *is nonempty, then the sequence* $\{u_n\}$ *generated by the algorithm*

$$\begin{cases} u_0, u_1 \in E \text{ chosen arbitrary},\ C_1 = E, \\ w_n = u_n + \alpha_n(u_n - u_{n-1}) \\ C_{n+1} = \{v \in C_n : \phi(w_n, \eta_{i_n}^{m_n}) \leq \frac{2}{1-k_{i_n}}\langle w_n - v, Jw_n - J\eta_{i_n}^{m_n}\rangle + \zeta_n\},\ \eta_{i_n}^{m_n} \in T_{i_n}^{m_n} w_n, \\ u_{n+1} = \Pi_{C_{n+1}} u_0, \quad n \geq 1, \end{cases} \tag{16}$$

converges strongly to $\Pi_\Omega u_0$*, where* $\zeta_n = \frac{\gamma_{m_n}^{i_n}}{1-k_{i_n}}\psi_{i_n}(\phi(w, w_n)) + \frac{\delta_{m_n}^{i_n}}{1-k_{i_n}}$, $w \in \Omega$ *and* $\{\alpha_n\}$ *is a non-decreasing sequence of real numbers in* $[0,1)$.

Proof. We present the proof in a number of steps.

Step 1: $\{u_n\}_{n=1}^{\infty}$ is well defined.

It suffices to show that C_n is closed and convex for all $n \geq 1$. From the definition of C_n, it is easy to see that C_n is closed for each $n \geq 1$. Clearly, $C_1 = E$ is convex. Assume C_n is convex for some $n \geq 1$. Let $v_1, v_2 \in C_{n+1},\ \lambda \in (0,1)$, and set $w = \lambda v_1 + (1-\lambda)v_2 \in C_n$. Then,

$$\phi(w_n, \eta_{i_n}^{m_n}) \leq \frac{2}{1-k_{i_n}} \langle w_n - v_1, Jw_n - J\eta_{i_n}^{m_n} \rangle + \zeta_n, \tag{17}$$

$$\phi(w_n, \eta_{i_n}^{m_n}) \leq \frac{2}{1-k_{i_n}} \langle w_n - v_2, Jw_n - J\eta_{i_n}^{m_n} \rangle + \zeta_n. \tag{18}$$

Multiplying inequalities (17) and (18) by λ and $(1-\lambda)$, respectively, and adding the resulting inequalities we obtain that

$$\phi(w_n, \eta_{i_n}^{m_n}) \leq \frac{2}{1-k_{i_n}} \langle w_n - w, Jw_n - J\eta_{i_n}^{m_n} \rangle + \zeta_n, \tag{19}$$

which implies that $w \in C_{n+1}$. Thus, C_{n+1} is convex. Therefore, C_n is closed and convex for all $n \geq 1$. Hence, $\{u_n\}_{n=0}^{\infty}$ is well defined.

Step 2: $\Pi_{\Omega} u_0$ is well define.

It suffices to show that Ω is nonempty, closed and convex. This follows from Lemma 0.7 and the assumption that $\Omega \neq \emptyset$.

Step 3: $\Omega \subset C_n$ for all $n \geq 1$.

We proceed by induction. Clearly, $\Omega \subset C_1 = E$. Suppose $\Omega \subset C_n$ for some $n \geq 1$. Let $w \in \Omega$. Since for each $i = 1,2,3...,\ T_i$ is totally asymptotically strict quasi-ϕ-pseudocontractive, we have that

$$\phi(w, \eta_{i_n}^{m_n}) \leq \phi(w, w_n) + k_{i_n} \phi(w_n, \eta_{i_n}^{m_n}) + \gamma_{m_n}^{i_n} \psi_{i_n}\big(\phi(w, w_n)\big) + \delta_{m_n}^{i_n}. \tag{20}$$

Using inequality (2) we obtain that

$$\phi(w, \eta_{i_n}^{m_n}) = \phi(w, w_n) + \phi(w_n, \eta_{i_n}^{m_n}) + 2\langle w - w_n, Jw_n - J\eta_{i_n}^{m_n} \rangle. \tag{21}$$

From inequalities (20) and (21), we obtain that

$$\begin{aligned}\phi(w_n,\eta_{i_n}^{m_n}) &\leq \frac{\gamma_{m_n}^{i_n}}{1-k_{i_n}}\psi_{i_n}\big(\phi(w,w_n)\big)+\frac{\delta_{m_n}^{i_n}}{1-k_{i_n}}+\frac{2}{1-k_{i_n}}\langle w_n-w,Jw_n-J\eta_{i_n}^{m_n}\rangle\\ &= \frac{2}{1-k_{i_n}}\langle w_n-w,Jw_n-J\eta_{i_n}^{m_n}\rangle+\zeta_n,\end{aligned}$$

which implies that $w\in C_{n+1}$. Hence, $\Omega\subset C_n,\ \forall\ n\geq 1$.

Step 4: $\lim_{n\to\infty}\zeta_n=0$ and $\lim_{n\to\infty}u_n=u^*\in E$.
Using Lemma 0.9, we have that

$$\phi(u_n,u_0)=\phi(\Pi_{C_n}u_0,u_0)\leq\phi(w,u_0)-\phi(w,u_n)\leq\phi(w,u_0)\ \forall\ w\in\Omega, \tag{22}$$

which implies that $\{\phi(u_n,u_0)\}$ is bounded. Thus, by inequality (1) we obtain that $\{u_n\}$ is bounded. Now, for each $i\geq 1$, define $M_i:=\{k\geq 1: k=i+\frac{(m-1)m}{2}, m\geq i, m\in\mathbb{N}\}$. Observe that if for each $i\geq 1$, $k\in M_i$, then $\gamma_{m_k}^{i_k}=\gamma_{m_k}^{i}$, $\delta_{m_k}^{i_k}=\delta_{m_k}^{i}$ and $\psi_{i_k}=\psi_i$. Also, $m_k\to\infty$ as $k\to\infty$, $k\in M_i$. Therefore, $\lim_{n\to\infty}\zeta_n=0$.

Since $u_n=\Pi_{C_n}u_0\in C_n$ and $C_{n+1}\subset C_n$, we have that

$$\phi(u_n,u_0)\leq\phi(u_{n+1},u_0)\ \forall\ n\geq 1.$$

Thus, $\{\phi(u_n,u_0)\}$ is monotone nondecreasing. Consequently, limit of $\{\phi(u_n,u_0)\}$ exists. Let $m>n$. Then, using Lemma 0.9, we obtain that

$$\phi(u_m,u_n)=\phi(u_m,\Pi_{C_n}u_0)\leq\phi(u_m,u_0)-\phi(\Pi_{C_n}u_0,u_0)=\phi(u_m,u_0)-\phi(u_n,u_0),$$

which implies that $\phi(u_m,u_n)\to 0$ as $n,m\to\infty$. Therefore, by Lemma 0.4, we obtain that $\|u_m-u_n\|\to 0$ as $n,m\to\infty$. Hence, $u_n\to u^*\in E$ as $n\to\infty$. Consequently, $w_n\to u^*$ as $n\to\infty$.

Step 5: $u^*\in\Omega$

Since $u_{n+1}=\Pi_{C_{n+1}}u_0\in C_{n+1}$, we have that

$$\phi(w_n,\eta_{i_n}^{m_n})\leq\frac{2}{1-k_{i_n}}\langle w_n-u_{n+1},Ju_n-\eta_{i_n}^{m_n}\rangle+\zeta_n. \tag{23}$$

Using the fact that $\lim_{n\to\infty}\|w_n-u_{n+1}\|=0$ and $\lim_{n\to\infty}\zeta_n=0$, we obtain from inequality (23) that $\lim_{n\to\infty}\phi(w_n,\eta_i^{m_n})=0$. Thus, by Lemma 0.4, we have that

$$\lim_{n\to\infty}\|w_n-\eta_i^{m_n}\|=0. \tag{24}$$

Therefore, for each i,

$$\lim_{n\to\infty} \eta_i^{m_n} = u^*. \tag{25}$$

Now, for each $i \geq 1$ consider the sequence $\{z_{m_k}^{(i)}\}_{k\in K_i}$ generated by

$$z_{m_{k+1}}^i \in T_i\eta_i^{m_k} \subset T_i^{m_{k+1}}u_k,\ k \in K_i.$$

By continuity of T_i we have from (25) that for each $i \geq 1$, $\lim\limits_{k\to\infty} z_{m_{k+1}}^i = z^*$, $z^* \in T_iu^*$. Using (24) and equally continuity of T_i, we obtain that for each $i \geq 1$,

$$\begin{aligned} \left\| z_{m_{k+1}}^i - \eta_i^{m_k} \right\| \quad \leq \quad & \left\| z_{m_{k+1}}^i - \eta_i^{m_{k+1}} \right\| + \left\| \eta_i^{m_{k+1}} - w_{k+1} \right\| + \| w_{k+1} - w_k \| \\ & + \left\| w_k - \eta_i^{m_k} \right\| \to 0, \quad k \to \infty. \end{aligned}$$

Therefore, for each $i \geq 1$, $\lim_{k\to\infty} z_{m_{k+1}}^i = u^*$. Hence, by uniqueness of limit we have that $u^* = z^*$. Thus, $u^* \in \Omega$.

Step 6: $u^* = \Pi_\Omega u_0$.
Let $k = \Pi_\Omega u_0$. Since $u^* \in \Omega$, we have that

$$\phi(k,u_0) \leq \phi(u^*,u_0). \tag{26}$$

Also, since $u_n = \Pi_{C_n}u_0$ and $k \in \Omega \subset C_n$, we have that $\phi(u_n,u_0) \leq \phi(k,u_0)$. Since $u_n \to u^*$, we have that

$$\phi(u^*,u_0) \leq \phi(k,u_0). \tag{27}$$

From inequalities (26) and (27) we obtain that

$$\phi(u^*,u_0) = \phi(k,u_0).$$

Thus, $u^* = k = \Pi_\Omega u_0$. This completes the proof. □

Theorem 1.2. *Let E be a uniformly smooth and strictly convex real Banach space with Kadec-Klee property and let $\{T_i\}_{i=1}^\infty$, $T_i : E \to 2^E$ be an infinite family of equally continuous and totally asymptotically strict quasi-ϕ-pseudocontractive multi-valued maps with nonnegative real sequences $\{\gamma_n^i\}, \{\delta_n^i\}$, a constant $k_i \in [0,1)$ and a sequence of strictly increasing and* *continuous functions* $\{\psi_i\}$, $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\gamma_n^{(i)} \to 0, \delta_n^{(i)} \to 0$ *and* $\psi_i(0) = 0$. *Suppose* $\Omega := \cap_{i=1}^{\infty} F(T_i)$ *is nonempty, then the sequence* $\{u_n\}$ *generated by algorithm* (16) *converges strongly to* $\Pi_\Omega u_0$*, where* $\{\alpha_n\}$ *is a nondecreasing sequence of real numbers in* $[0,1)$*,* $\zeta_n = \frac{\gamma_{m_n}^{j_n}}{1-k_{i_n}} \psi_{i_n}(\phi(w,u_n)) + \frac{\delta_{m_n}^{i_n}}{1-k_{i_n}}$*, and* $w \in \Omega$.

Proof. Just as in the proof of Theorem 1.1, we shall present the proof of this theorem in a number of steps.

Step 1: $\{u_n\}_{n=0}^{\infty}$ is well defined.

This step is the same as Step 1 of the proof of Theorem 1.1.

Step 2: $\Pi_\Omega u_0$ is well define.

This follows from Lemma 28 and the assumption that Ω is nonempty.

Step 3: $\Omega \subset C_n$ for all $n \geq 1$.

This step is the same as Step 3 in the proof of Theorem 1.1.

Step 4: $\lim\limits_{n\to\infty} \zeta_n = 0$ and $\lim\limits_{n\to\infty} u_n = u^* \in E$.

Following the same argument as in Step 4 of the proof of Theorem 1.1, we obtain that $\lim\limits_{n\to\infty} \zeta_n = 0$ and that $\{u_n\}$ is bounded. Therefore, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup u^*$ as $k \to \infty$. Since C_{n_k} is weakly closed, $u^* \in C_{n_k}$. Thus, $u_{n_k} = \Pi_{C_{n_k}} u_0$ implies that $\phi(u_{n_k}, u_0) \leq \phi(u^*, u_0) \quad \forall\, k \geq 1$. By weak lower-semi continuity of $\|.\|$, we have that

$$\begin{aligned}
\liminf_{k\to\infty} \phi(u_{n_k}, u_0) &= \liminf_{k\to\infty} \left\{ \|u_{n_k}\|^2 - 2\langle u_{n_k}, Ju_0\rangle + \|u_0\|^2 \right\} \\
&\geq \|u^*\|^2 - 2\langle u^*, Ju_0\rangle + \|u_0\|^2 = \phi(u^*, u_0),
\end{aligned}$$

which implies that $\phi(u^*, u_0) \leq \liminf_{k\to\infty} \phi(u_{n_k}, u_0) \leq \limsup_{k\to\infty} \phi(u_{n_k}, u_0) \leq \phi(u^*, u_0)$. Hence, $\lim\limits_{k\to\infty} \phi(u_{n_k}, u_0) = \phi(u^*, u_0)$. Therefore, $\lim\limits_{k\to\infty} \|u_{n_k}\| = \|u^*\|$. By Kadec-Klee property of E, we have that $\lim\limits_{k\to\infty} u_{n_k} = u^*$. Since $\{\phi(u_n, u_0)\}$ is convergent and $\lim\limits_{k\to\infty} \phi(u_{n_k}, u_0) = \phi(u^*, u_0)$, we have that $\lim\limits_{n\to\infty} \phi(u_n, u_0) = \phi(u^*, u_0)$.

Claim: $u_n \to u^*$.

Suppose not. Then, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \to p$ as $j \to \infty$. By applying Lemma 0.9, we have that

$$\begin{aligned}\phi(u^*,p) &= \lim_{j,k\to\infty}\phi(u_{n_k},u_{n_j}) = \lim_{j,k\to\infty}\phi\left(u_{n_k},\Pi_{C_{n_j}}u_0\right)\\ &\le \lim_{j,k\to\infty}\left(\phi(u_{n_k},u_0)-\phi\left(\Pi_{C_{n_j}}u_0,u_0\right)\right)\\ &= \lim_{j,k\to\infty}\left(\phi(u_{n_k},u_0)-\phi(u_{n_i},u_0)\right) = \phi(u^*,u_0)-\phi(u^*,u_0)=0,\end{aligned}$$

which implies $u^* = p$. Hence, the claim holds.

Step 5: $u^* \in \Omega$

Following the same argument as in Step 5 of the proof of Theorem 1.1, we obtain that $\lim_{n\to\infty}\phi(w_n,\eta_i^{m_n}) = 0$. Thus, by Lemma 0.5, we have that

$$\lim_{n\to\infty}\|\eta_i^{m_n}-u^*\|=0,$$

which implies that $\lim_{n\to\infty}\|w_n-\eta_i^{m_n}\|=0$. The rest of the verification of this step follows the same pattern as in Step 5 of the proof of Theorem 1.1.

Step 6: $u^* \in \Omega$

This step is the same as Step 6 of the proof of Theorem 1.1. Hence, this completes the proof of this theorem. □

Theorem 1.3. *Let E be a reflexive, smooth and strictly convex real Banach space with dual space E^* such that both E and E^* have the Kadec-Klee property and let $\{T_i\}_{i=1}^{\infty}$, $T_i : E \to 2^E$ be an infinite family of equally continuous and totally asymptotically strict quasi-ϕ-pseudocontractive multi-valued maps with nonnegative real sequences $\{\gamma_n^i\},\{\delta_n^i\}$, a constant $k_i \in [0,1)$ and a sequence of strictly increasing and continuous functions $\{\psi_i\}$, $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma_n^{(i)} \to 0, \delta_n^{(i)} \to 0$ and $\psi_i(0) = 0$. Suppose $\Omega := \cap_{i=1}^{\infty}F(T_i)$ is nonempty, then the sequence $\{u_n\}$ generated by algorithm (16) converges strongly to $\Pi_\Omega u_0$, where $\{\alpha_n\}$ is a nondecreasing sequence of real numbers in $[0,1)$, $\zeta_n = \frac{\gamma_{m_n}^{i_n}}{1-k_{i_n}}\psi_{i_n}(\phi(w,u_n)) + \frac{\delta_{m_n}^{i_n}}{1-k_{i_n}}$, and $w \in \Omega$.*

Proof. Just as in the proof of Theorem 1.1, we shall present the proof of this theorem in a number of steps.

Step 1: $\{u_n\}_{n=0}^{\infty}$ is well defined.

This step is the same as Step 1 of the proof of Theorem 1.1.

Step 2: $\Pi_\Omega u_0$ is well define.

This follows from Lemma 0.8 and the assumption that Ω is nonempty.

Step 3: $\Omega \subset C_n$ for all $n \geq 1$.

This step is the same as Step 3 in the proof of Theorem 1.1.

Step 4: $\lim\limits_{n\to\infty} \zeta_n = 0$ and $\lim\limits_{n\to\infty} u_n = u^* \in E$.

The verification of this step follows the same pattern as in the verification of Step 4 of the proof of Theorem 1.2. The rest of the proof of this theorem follows the same pattern as in the proof of Theorem 1.2. □

By setting $\alpha_n = 0$ in algorithm (16), we obtain the following corollaries which are extentions and generalizations of the works discussed in this paper (see concluding remarks below).

Corollary 1.4. *Let E be a uniformly convex and smooth real Banach space and C be a nonempty closed and convex subset of E. Let $\{T_i\}_{i=1}^{\infty}$, $T_i : C \to 2^C$ be an infinite family of equally continuous and totally asymptotically strict quasi-ϕ-pseudocontractive multi-valued maps with nonnegative real sequences $\{\gamma_n^i\}, \{\delta_n^i\}$, a constant $k_i \in [0,1)$ and a sequence of strictly increasing and continuous functions $\{\psi_i\}$, $\psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma_n^{(i)} \to 0, \delta_n^{(i)} \to 0$ and $\psi_i(0) = 0$. Suppose $\Omega := \cap_{i=1}^{\infty} F(T_i)$ is nonempty, then the sequence $\{u_n\}$ generated by the algorithm*

$$(28)\begin{cases} u_1 \in E \text{ chosen arbitrary}, \ C_1 = C, \\ C_{n+1} = \{v \in C_n : \phi(u_n, \eta_{i_n}^{m_n}) \leq \frac{2}{1-k_{i_n}} \langle u_n - v, Ju_n - J\eta_{i_n}^{m_n}\rangle + \zeta_n\}, \ \eta_{i_n}^{m_n} \in T_{i_n}^{m_n} u_n, \\ u_{n+1} = \Pi_{C_{n+1}} u_0, \quad n \geq 1, \end{cases}$$

converges strongly to $\Pi_\Omega u_0$, where $\zeta_n = \frac{\gamma_{m_n}^{i_n}}{1-k_{i_n}} \psi_{i_n}(\phi(w,u_n)) + \frac{\delta_{m_n}^{i_n}}{1-k_{i_n}}$, $w \in \Omega$ and $\{\alpha_n\}$ is a nondecreasing sequence of real numbers in $[0,1)$.

Corollary 1.5. *Let E be a uniformly smooth and strictly convex real Banach space with Kadec-Klee property and let C be a nonempty closed and convex subset of E. Let $\{T_i\}_{i=1}^{\infty}$, $T_i : C \to 2^C$ be an infinite family of equally continuous and totally asymptotically strict quasi-ϕ-pseudocontractive multi-valued maps with nonnegative real sequences $\{\gamma_n^i\}, \{\delta_n^i\}$, a constant*

$k_i \in [0,1)$ and a sequence of strictly increasing and continuous functions $\{\psi_i\},\ \psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma_n^{(i)} \to 0, \delta_n^{(i)} \to 0$ and $\psi_i(0) = 0$. Suppose $\Omega := \cap_{i=1}^{\infty} F(T_i)$ is nonempty, then the sequence $\{u_n\}$ generated by algorithm (28) converges strongly to $\Pi_\Omega u_0$, where $\{\alpha_n\}$ is a nondecreasing sequence of real numbers in $[0,1)$, $\zeta_n = \frac{\gamma_{m_n}^{j_n}}{1-k_{i_n}}\psi_{i_n}(\phi(w,u_n)) + \frac{\delta_{m_n}^{i_n}}{1-k_{i_n}}$, and $w \in \Omega$.

Corollary 1.6. *Let E be a reflexive, smooth and strictly convex real Banach space with dual space E^* such that both E and E^* have the Kadec-Klee property. Let C be a nonempty, closed and convex subset of E and let $\{T_i\}_{i=1}^{\infty}$, $T_i : C \to 2^C$ be an infinite family of equally continuous and totally asymptotically strict quasi-ϕ-pseudocontractive multi-valued maps with nonnegative real sequences $\{\gamma_n^i\}, \{\delta_n^i\}$, a constant $k_i \in [0,1)$ and a sequence of strictly increasing and continuous functions $\{\psi_i\},\ \psi_i : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\gamma_n^{(i)} \to 0, \delta_n^{(i)} \to 0$ and $\psi_i(0) = 0$. Suppose $\Omega := \cap_{i=1}^{\infty} F(T_i)$ is nonempty, then the sequence $\{u_n\}$ generated by algorithm (28) converges strongly to $\Pi_\Omega u_0$, where $\{\alpha_n\}$ is a nondecreasing sequence of real numbers in $[0,1)$, $\zeta_n = \frac{\gamma_{m_n}^{j_n}}{1-k_{i_n}}\psi_{i_n}(\phi(w,u_n)) + \frac{\delta_{m_n}^{i_n}}{1-k_{i_n}}$, and $w \in \Omega$.*

2. APPLICATIONS

2.1. **System of generalized mixed equilibrium problem.** Let C be a nonempty, closed and convex subset of a real Banach space E, $\zeta : C \to \mathbb{R}$ be a real-valued function, $A : C \to X^*$ be a nonlinear map and $h : C \times C \to \mathbb{R}$ be a bifunction. The *generalized mixed equilibrium problem* is to find $u^* \in C$ such that

$$h(u^*, y) + \zeta(y) - \zeta(u^*) + \langle y - u^*, Au^* \rangle \geq 0 \ \forall\, y \in C. \tag{29}$$

The set of solutions of the generalized mixed equilibrium problem is denoted by **GMEP**(h, ζ, A).

The class of generalized mixed equilibrium problem includes, as special cases, the class of *mixed equilibrium problem* ($A \equiv 0$, see e.g., Ceng and Yao [22] and the references contained therein); the class of *generalized equilibrium problem* ($\zeta \equiv 0$, see e.g., Takahashi and Takahashi [13]); the class of *equilibrium problem* ($A \equiv 0,\ \zeta \equiv 0$, see e.g., Fan [18], Blum and Oettli [19] and the references contained in them); the class of *variational inequality problem* ($h \equiv 0,\ \zeta \equiv 0$, see e.g., Stampacchia [12]) and the class of *convex minimization problem* ($A \equiv 0,\ h \equiv 0$).

In the sequel, we shall assume that $f : C \times C \to \mathbb{R}$ is a bifunction satisfying the following conditions:

$(B1)$ $f(u,u) = 0$, $\forall u \in C$,

$(B2)$ f is monotone, that is, $f(u,y) + f(y,u) \leq 0$, $\forall u, y \in C$,

$(B3)$ for all $u, y, z \in E$, $\limsup_{t\downarrow 0} f(tz + (1-t)u, y) \leq f(u,y)$,

$(B4)$ for all $u \in C$, $y \longmapsto f(u,y)$ is convex and lower semicontinuous.

The following lemma will be needed in what follows.

Lemma 2.1 (see Zhang [17])**.** *Let E be a smooth, strictly convex and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $A : C \to X^*$ be a continuous and monotone mapping, $\zeta : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and $h : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $(B1)-(B4)$. Let $r > 0$ be any given number and $u \in E$ be any given point. Then, the followings hold:*

(1) There exists $z \in C$ such that

$$h(z,y) + \zeta(y) - \zeta(z) + \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Ju\rangle \geq 0, \quad \forall\, y \in C$$

(2) If we define a mapping $G_r : C \to C$ by

$$G_r(u) = \Big\{ z \in C : h(z,y) + \zeta(y) - \zeta(z) + \langle y - z, Az\rangle + \frac{1}{r}\langle y - z, Jz - Ju\rangle \geq 0, \quad \forall\, y \in C \Big\},\ u \in C,$$

the mapping G_r has the following properties:

(a) G_r is single-valued;

(b) G_r is a firmly nonexpansive-type mapping, that is, for all $u, y \in E$,

$$\langle G_r u - G_r y, JG_r u - JG_r y\rangle \leq \langle G_r u - G_r y, Ju - Jy\rangle;$$

(c) $F(G_r) = GMEP(h, A, \zeta) = \hat{F}(G_r)$;

(d) $GMEP(h, A, \zeta)$ a is closed convex set of C;

(e) $\phi(q, G_r u) + \phi(G_r u, u) \leq \phi(q, u)$ $\forall q \in F(G_r), u \in E$.

We now prove the following important theorem.

Theorem 2.2. *Let E be a reflexive, smooth and strictly convex real Banach space with dual space E^* such that both E and E^* have the Kadec-Klee property. Let C be a nonempty, closed and convex subset of E and let $\{f_i\}_{i=1}^{\infty}$, $f_i : C \times C \to \mathbb{R}$ be an infinite family of bifunctions satisfying condition $B1 - B2$. Let $\{A_i\}_{i=1}^{\infty}$, $A_i : C \to X^*$ be a sequence of continuous monotone maps and $\{\zeta_i\}_{i=1}^{\infty}$, $\zeta_i : C \to \mathbb{R}$ be a sequence of convex and lower-semi continuous functions. Suppose $\Lambda := \cap_{i=1}^{\infty} GMEP(f_i, A_i, \zeta_i)$ is nonempty, then the sequence $\{u_n\}$ generated by*

$$\begin{cases} u_0, u_1 \in E \text{ chosen arbitrary},\ C_1 = C, \\ w_n = u_n + \alpha_n(u_n - u_{n-1}) \\ C_{n+1} = \{v \in C_n : \phi(w_n, G_{r_{i_n}} w_n) \leq \frac{2}{1-k_{i_n}} \langle w_n - v, Jw_n - JG_{r_{i_n}} w_n \rangle\}, \\ u_{n+1} = \Pi_{C_{n+1}} u_0, \quad n \geq 1, \end{cases}$$

converges strongly to $\Pi_\Lambda u_0$, where G_r is as defined in Lemma 2.1 and $\{\alpha_n\}$ is a nondecreasing sequence of real numbers.

Proof. From Lemma 2.1 (c), (e) and the assumption that Λ is nonempty, we obtain that G_{r_i} is quasi-ϕ-nonexpansive for each $i = 1, 2, \ldots$. Observe that every quasi-ϕ-nonexpansive map is totally asymptotically strict quasi-ϕ-pseudocontractive. Hence, by Theorem 1.3 and Lemma 2.1 (c) we have that $\{u_n\}$ converges strongly to $\Pi_\Lambda u_0$. □

2.2. **System of convex minimization problem.** Let E be a real Banach space with dual space E^* and let $f : E \to \mathbb{R} \cup \{\infty\}$ be a proper, lower semicontinuous and convex function. The subdifferential of f, $\partial f : E \to 2^{E^*}$ is defined by

$$\partial f(u) = \{u^* \in E^* : f(y) - f(u) \geq \langle y - u, u^* \rangle,\ \forall\, y \in E\}.$$

It is well known that ∂f is maximal monotone and that $0 \in \partial f(u)$ if and only if u is a minimizer of f.

Suppose E is a reflexive, smooth and strictly convex real Banach space with dual space E^* and $T_r u := (J + r\partial f)^{-1} Ju$. Then, from Lemma 0.10, we obtain that T_r is quasi-ϕ-nonexpansive if $F(T_r)$ is nonempty. Also observe that u is a fixed point of T_r if and only if $0 \in \partial f(u)$. Hence, the following theorem is an immediate application of Theorem 1.3.

Theorem 2.3. *Let E be a reflexive, smooth and strictly convex real Banach space with dual space E^* such that both E and E^* have the Kadec-Klee property. Let C be a nonempty, closed and convex subset of E and let $\{f_i\}_{i=1}^{\infty}$, $f_i : C \to \mathbb{R} \cup \{\infty\}$ be an infinite family of proper, lower semicontinuous and convex functions such that $\Lambda := \{u \in C : f_i(u) = min_{v \in C} f_i(v), \forall\, i = 1,2,...\}$ is nonempty. Then the sequence $\{u_n\}$ generated by*

$$\begin{cases} u_0, u_1 \in E \text{ chosen arbitrary},\ C_1 = C, \\ w_n = u_n + \alpha_n(u_n - u_{n-1}) \\ C_{n+1} = \{v \in C_n : \phi(w_n, T_{r_{i_n}} w_n) \leq \frac{2}{1-k_{i_n}} \langle w_n - v, Jw_n - JG_{r_{i_n}} w_n \rangle\}, \\ u_{n+1} = \Pi_{C_{n+1}} u_0, \quad n \geq 1, \end{cases}$$

converges strongly to $\Pi_\Lambda u_0$, where $\{\alpha_n\}$ is a nondecreasing sequence of real numbers.

3. NUMERICAL ILLUSTRATION

In this section, we give an examlpe to show that algorithm (16) converges faster than both algorithm (5) proposed by Chidume *et al.* [10] and the standard algorithms studied by other authors for solving the problem considered in this paper.

Example 1. *Let $E = \mathbb{R}$, $C = [-1,1]$ and $Tu = \sin u$. It is easy to see that T is relatively nonexpansive with $F(T) = \{0\}$ and thus, totally asymptotically strict quasi-ϕ-pseudocontractive. By taking $\alpha_n = \frac{n}{n+\alpha-1}$ in algorithm (16) and algorithm (5), where $\alpha = \frac{7}{8}$. Then under the above setting for E, C and T, by Theorem (1.1) and Corollary (1.4) $\{u_n\}$ converges to 0. Furthermore, the graph of $|u_n|$ against number of iterations for algorithm (16), algorithm (5) and algorithm (28) is shown below.*

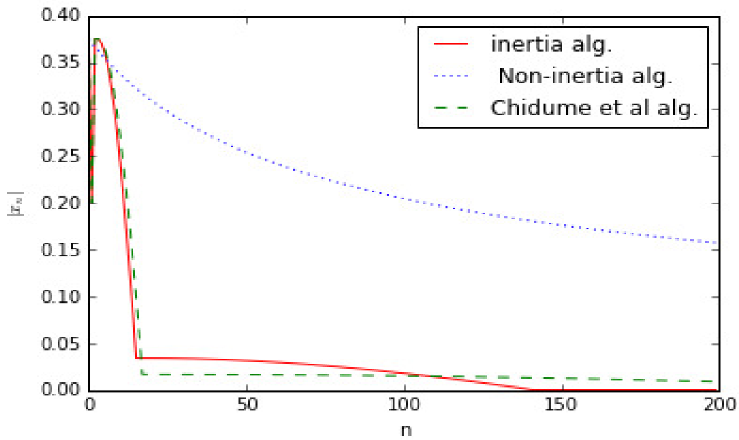

fig 1.

From the graph, we see that algorithm (16) labelled *inertia alg.* converges in less than 150 iterations whereas, algorithm (5) and algorithm (28) labelled *Chidume et al alg. and Non-inertia alg.*, respectively, are yet to converge even after 200 iterations. Hence, algorithm (16) studied in this paper converges faster than algorithm (5) and algorithm (28).

Remark 4. *All the computations and graph in Example 1 were done using Python 2.7 on hp Desktop Corei5-6500CPU@3.20GHz×4 with 64-bit OS.*

4. CONCLUDING REMARKS

Remark 5. *From Example 1, one will observe that algorithm 16 studied in Theorem 1.1 is more efficient than algorithm 5 of Chidume et al.* [10]. *Moreover, in Theorem 1.1 an infinite family of multi-valued total asymptotically strict quasi-ϕ-pseudocontractions which contain as a proper subclass, the class of relatively nonexpansive maps is considered. Hence, Theorem 1.1 improved and generalized the result of Chidume et al.* [10].

Remark 6. *In Corollary 1.4, the finite family of single-valued closed $(k, \gamma_n, \delta_n, \psi)$-total asymptotically strict quasi-ϕ-pseudocontractions studied by Zhang et al.* [3] *is extended to an infinite family of multi-valued total asymptotically strict quasi-ϕ-pseudocontractions. Furthermore, in the paper of Zhang et al.* [3], *it is required that the contractive parameters k, γ_n, δ_n, and ψ must work for each of the maps, however, in Corollary 1.4, using a special way of choosing integers this condition is dispensed with. In Corollary 1.4, we also dispense with the boundedness of the* *set of common fixed points of the maps considered, as required in their paper. As in algorithm* (3) *studied in the paper of Zhang et al.* [3], *algorithm* (28) *studied in Corollary 1.4, does not involve* y_n^i, *which in turn involves a control parameter,* α_n *to be computed at each iteration process. Thus from computational point of view, algorithm* (28) *is more efficient than algorithm 3. Hence, Corollary 1.4 improved, extend and generalized the results of Zhang et al.* [3].

Remark 7. *Corollary 1.4 improves the results of Wang and Yang* [2] *in the following ways.*

- *The closedness condition imposed on the map studied in Wang and Yang* [2] *is dispensed with in Corollary 1.4.*
- *The class of maps considered in the paper of Wang and Yang* [2] *is extended from the class of single-valued* $(k, \gamma_n, \delta_n, \psi)$*-total asymptotically strict quasi-*ϕ*-pseudocontraction to the class of infinite family of multi-valued* $(k_i, \gamma_n^i, \delta_n^i, \psi_i)$*-total asymptotically strict quasi-*ϕ*-pseudocontractions.*
- *The boundedness condition imposed on the set of common fixed points of the maps considered in the paper of Wang and Yang* [2] *is dispensed with in Corollary 1.4.*

Remark 8. *Corollary 1.6 improves the results in the paper of Zhang* [4] *in several ways.*

Conflict of Interests

The authors declare that they have no conflict of interest.

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