Available online at http://scik.org
Adv. Fixed Point Theory, 9 (2019), No. 1, 69-79
https://doi.org/10.28919/afpt/3957
ISSN: 1927-6303

EXISTENCE OF NONOSCILLATORY SOLUTIONS FOR SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

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Abstract. In this work, we consider the existence of nonoscillatory solutions of second order nonlinear neutral differential equations. Our results include as special cases some well-known results for linear and nonlinear equations. We use the Lebesgue's dominated convergence theorem and Banach contraction principle to obtain new sufficient conditions for the existence of nonoscillatory solutions.

Keywords: neutral equations; delay, non-linear; Lebesgue's dominated convergence theorem; Banach's contraction mapping principle; non-oscillatory solutions.

2010 AMS Subject Classification: 34C10, 34C15, 34K11.

1. INTRODUCTION

Consider a second order nonlinear neutral differential equations of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\big(r(t)\big(x(t)+p(t)x(t-\tau)\big)'\right]+q(t)G\big(x(t-\sigma)\big)=0, \tag{1}$$

where $\tau > 0$, $\sigma \geq 0$; $q, r \in C(\mathbb{R}_+, \mathbb{R}_+)$; $p \in PC(\mathbb{R}_+, \mathbb{R})$ and $G \in C(\mathbb{R}, \mathbb{R})$ is nondecreasing such that $xG(x) > 0$ for $x \neq 0$. The objective of this work is to study existence of positive solutions for second order nonlinear neutral delay differential equation (1) for any $|p(t)| < \infty$.

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Received November 19, 2018

In [1], Culakova et al. have considered (1) and studied existence of bounded positive solutions when $p \in C([t_0,\infty),(-\infty,0))$. In recent paper [2], Candan have considered

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+P_1(t)x(t-\tau_1)+P_2(t)x(t+\tau_2)]+Q_1(t)x(t-\sigma_1)+Q_2(t)x(t+\sigma_2)=0, \tag{2}$$

and established sufficient conditions for existence of bounded positive solution of (2), for any $P_1(t)$ and $P_2(t)$ excluding $P_1(t)\equiv +1\equiv P_2(t)$ and $P_1(t)\equiv -1\equiv P_2(t)$. In [13], Santra has consider first-order neutral delay differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+p(t)x(t-\tau)]+q(t)H\big(x(t-\sigma)\big)=f(t) \tag{3}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)+p(t)x(t-\tau)]+q(t)H\big(x(t-\sigma)\big)=0 \tag{4}$$

and studied oscillatory behaviour of the solutions of (3) and (4), under various ranges of $p(t)$. Also, sufficient conditions are obtained for existence of nonoscillatory solutions of (3). The motivation of the present work come from the above studies. The methods of the work of [1] has made unnecessarily complected to study existence of positive solution of such type of functional differential equations. Unlike the method of [1] an attempt is made here to study existence of nonoscillatory solutions of (1) for any $|p(t)|<\infty$.

Oscillation and nonoscillation of functional differential equations have been studied in recent years. In this direction, we refer the reader to [6]-[10], [19]-[22] and the references cited therein. The existence of nonoscillatory solution of functional differential equations received much less attention, which is due mainly to the technical difficulties arising in its analysis.

Let $\rho=\max\{\tau,\sigma\}$. By a solution of Eq. (1) we mean a function $x\in C([t_0-\rho,\infty),\mathbb{R})$, for some $t_0\geq 0$, such that $x(t)+p(t)x(t-\tau)$ is twice continuously differentiable and $r(t)(x(t)+p(t)x(t-\tau))'$ is continuously differentiable on $[t_0,\infty)$ and such that Eq. (1) is satisfied for $t\geq t_0$. A solution of Eq. (1) is said to be *oscillatory* if it has arbitrarily large zeros; Otherwise the solution is called *nonoscillatory*.

2. Main results

Theorem 1. *Let $p \in C(\mathbb{R}_+,[0,1))$. Assume that G is Lipschitzian on the intervals of the form $[a,b]$, $0 < a < b < \infty$. If*

$$\int_0^\infty \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)\, d\zeta \right] d\eta < +\infty, \tag{5}$$

then (1) *has a bounded nonoscillatory solution.*

Proof. Let $0 \le p(t) \le p < 1$, $t \in \mathbb{R}_+$ and $p > 0$. Due to (5), it is possible to find $T > \rho$ such that

$$\int_T^\infty \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)\, d\zeta \right] d\eta < \frac{1-p}{5L},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $\left[\frac{7}{10}(1-p), 1\right]$ for $t \ge T$. Let $Y = BC([T,\infty),\mathbb{R})$ be the space of real valued continuous functions on $[T,\infty]$. Indeed, Y is a Banach space with respect to supremum norm defined by

$$\|x\| = \sup\{|x(t)| : t \ge T\}.$$

Define

$$S = \{v \in Y : \frac{7}{10}(1-p) \le v(t) \le 1,\ t \ge T\}.$$

We notice that S is a closed and convex subspace of Y. Let $\Phi : S \to S$ be such that

$$(\Phi x)(t) = \begin{cases} (\Phi x)(T+\rho), & t \in [T, T+\rho] \\ -p(t)x(t-\tau) + \frac{9+p}{10} - \int_t^\infty \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta) G\big(x(\zeta-\sigma)\big)\, d\zeta \right] d\eta, & t \ge T+\rho. \end{cases}$$

For every $x \in S$, $(\Phi x)(t) \le \frac{9+p}{10} < 1$ and

$$(\Phi x)(t) \ge -p + \frac{9+p}{10} - \frac{1-p}{5} = \frac{7}{10}(1-p)$$

implies that $\Phi x \in S$. Now for $x_1, x_2 \in S$, we have

$$\begin{aligned} |(\Phi x_1)(t) - (\Phi x_2)(t)| &\le p|x_1(t-\tau) - x_2(t-\tau)| \\ &\quad + \int_t^\infty \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta) |G\big(x_1(\zeta-\sigma)\big) - G\big(x_2(\zeta-\sigma)\big)|\, d\zeta \right] d\eta, \end{aligned}$$

that is,

$$\begin{aligned}|(\Phi x_1)(t)-(\Phi x_2)(t)| &\le p\|x_1-x_2\|+\|x_1-x_2\|L_1\int_t^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)\,d\zeta\right]d\eta\\ &\le \left(p+\frac{1-p}{5}\right)\|x_1-x_2\|\\ &= \frac{4p+1}{5}\|x_1-x_2\|.\end{aligned}$$

Therefore, $\|\Phi x_1-\Phi x_2\|\le \frac{4p+1}{5}\|x_1-x_2\|$ implies that Φ is a contraction. By using Banach's contraction mapping principle, it follows that Φ has a unique fixed point $x(t)$ in $\left[\frac{7}{10}(1-p),1\right]$. This completes the proof of the theorem. □

Theorem 2. *Let* $1<p_1\le p(t)\le p_2<\infty$, $p_1^2\ge p_2$ *for* $t\in\mathbb{R}_+$. *Let* G *be Lipschitzian on the intervals of the form* $[a,b]$, $0<a<b<\infty$. *If* (5) *hold, then* (1) *admits a positive bounded solution.*

Proof. Due to (5), it is possible to find $T>\rho$ such that

$$\int_T^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)\,d\zeta\right]d\eta<\frac{p_1-1}{3L},$$

where $L=\max\{L_1,L_2\}$ and L_1 is the Lipschitz constant of G on $[\alpha,\beta]$, $L_2=G(\beta)$ with

$$\alpha=\frac{3\lambda(p_1^2-p_2)-p_2(p_1-1)}{3{p_1}^2p_2}$$

$$\beta=\frac{p_1-1+3\lambda}{3p_1}\quad\text{and}\quad \lambda>\frac{p_2(p_1-1)}{3({p_1}^2-p_2)}>0.$$

Let $Y=BC([T,\infty),\mathbb{R})$ be the space of real valued functions defined on $[T,\infty)$. Indeed, Y is a Banach space with respect to supremum norm defined by

$$||x||=\sup\{|x(t)|:t\ge T\}.$$

Define

$$S=\{u\in Y:\ \alpha\le u(t)\le\beta,\ t\ge T\}.$$

Let $\Phi : S \to S$ be such that

$$(\Phi x)(t) = \begin{cases} (\Phi x)(T+\rho), & t \in [T, T+\rho] \\ -\frac{x(t+\tau)}{p(t+\tau)} + \frac{\lambda}{p(t+\tau)} + \frac{1}{p(t+\tau)} \int_T^{t+\tau} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta) G\big(x(\zeta-\sigma)\big)\, d\zeta\right] d\eta, & t \geq T+\rho. \end{cases}$$

For every $x \in S$,

$$\begin{aligned} (\Phi x)(t) &\leq \frac{L}{p(t+\tau)} \int_T^{t+\tau} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)\, d\zeta\right] d\eta + \frac{\lambda}{p(t+\tau)} \\ &\leq \frac{L}{p(t+\tau)} \int_T^{\infty} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)\, d\zeta\right] d\eta + \frac{\lambda}{p(t+\tau)} \\ &\leq \frac{1}{p_1} \left[\frac{p_1 - 1}{3} + \lambda\right] = \beta \end{aligned}$$

and

$$\begin{aligned} (\Phi x)(t) &\geq -\frac{x(t+\tau)}{p(t+\tau)} + \frac{\lambda}{p(t+\tau)} \\ &> -\frac{\beta}{p_1} + \frac{\lambda}{p_2} = \alpha \end{aligned}$$

implies that $\Phi x \in S$. Again, for $x_1, x_2 \in S$

$$\begin{aligned} |(\Phi x_1)(t) - (\Phi x_2)(t)| \leq{}& \frac{1}{|p(t+\tau)|} |x_1(t+\tau) - x_2(t+\tau)| \\ &+ \frac{L}{|p(t+\tau)|} \int_T^{t+\tau} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta) |x_1(\zeta-\sigma) - x_2(\zeta-\sigma)|\, d\zeta\right] d\eta, \end{aligned}$$

that is,

$$\begin{aligned} |(\Phi x_1)(t) - (\Phi x_2)(t)| &\leq \frac{1}{p_1} ||x_1 - x_2|| + \frac{L}{p_1} ||x_1 - x_2|| \int_T^{t+\tau} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)\, d\zeta\right] d\eta \\ &< \frac{1}{p_1} ||x_1 - x_2|| \left(1 + \frac{p_1 - 1}{3}\right) \end{aligned}$$

implies that

$$||\Phi x_1 - \Phi x_2|| \leq \left(\frac{1}{p_1} + \frac{p_1 - 1}{3p_1}\right) ||x_1 - x_2||.$$

Since $\left(\frac{1}{p_1} + \frac{p_1-1}{3p_1}\right) < 1$, then Φ is a contraction mapping of S into S. We notice that S is a closed convex subset of Y and hence we apply Banach's fixed point to S. So, we conclude that Φ has a unique fixed point on $[\alpha,\beta]$. It is easy to verify that

$$x(t)=\begin{cases}(\Phi x)(T+\rho), & t\in[T,T+\rho]\\ -\frac{x(t+\tau)}{p(t+\tau)}+\frac{\lambda}{p(t+\tau)}+\frac{1}{p(t+\tau)}\int_T^{t+\tau}\frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G\big(x(\zeta-\sigma)\big)\,d\zeta\right]d\eta, & t\geq T+\rho.\end{cases}$$

is a positive bounded solution of (1) on $[\alpha,\beta]$. The the proof of the theorem is complete. ☐

Theorem 3. *Let $-1<-p\leq p(t)\leq 0$, $p>0$ for $t\in\mathbb{R}_+$. Assume that*

$$R(t)=\int_0^t\frac{ds}{r(s)}\quad and\quad \lim_{t\to\infty}R(t)=+\infty \tag{6}$$

$$\int_0^\infty q(\eta)G\big(\varepsilon R(\eta-\sigma)\big)\,d\eta<+\infty \ for\ every\ \varepsilon>0 \tag{7}$$

hold, then (1) *has a unbounded positive solution.*

Proof. Due to (7), we can find $\varepsilon>0$ such that

$$\int_T^\infty q(\eta)G\big(\varepsilon R(\eta-\sigma)\big)\,d\eta\leq\frac{\varepsilon}{3}.$$

Let's consider

$$M=\{x:x\in C([T-\rho,+\infty),\mathbb{R}),x(t)=0\ for\ t\in[T-\rho,T]\ and$$
$$\frac{\varepsilon}{3}[R(t)-R(T)]\leq x(t)\leq\varepsilon[R(t)-R(T)]\ \}$$

and define $\Phi:M\to C([T-\rho,+\infty),\mathbb{R})$ such that

$$(\Phi x)(t)=\begin{cases}0, & t\in[T-\rho,T)\\ -p(t)x(t-\tau)+\int_T^t\frac{1}{r(\eta)}[\frac{\varepsilon}{3}+\int_\eta^\infty q(\zeta)G\big(x(\zeta-\sigma)\big)\,d\zeta]\,d\eta, & t\geq T.\end{cases}$$

For every $x\in M$,

$$\begin{aligned}(\Phi x)(t)&\geq\int_T^t\frac{1}{r(\eta)}\left[\frac{\varepsilon}{3}+\int_\eta^\infty q(\zeta)G\big(x(\zeta-\sigma)\big)\,d\zeta\right]d\eta\\ &\geq\frac{\varepsilon}{3}\int_T^t\frac{d\eta}{r(\eta)}=\frac{\varepsilon}{3}[R(t)-R(T)],\end{aligned}$$

and $x(t) \leq \varepsilon R(t)$ and definition of the set M implies that

$$\begin{aligned}(\Phi x)(t) &\leq -p(t)x(t-\tau) + \frac{2\varepsilon}{3}\int_T^t \frac{d\eta}{r(\eta)} \\ &\leq p\varepsilon[R(t-\tau)-R(T)] + \frac{2\varepsilon}{3}[R(t)-R(T)] \\ &\leq p\varepsilon[R(t)-R(T)] + \frac{2\varepsilon}{3}[R(t)-R(T)] \\ &= \left(p+\frac{2}{3}\right)\varepsilon[R(t)-R(T)] \\ &\leq \varepsilon[R(t)-R(T)]\end{aligned}$$

implies that $\Phi x \in M$. Define $v_n : [T-\rho, +\infty) \to \mathbb{R}$ by the recursive formula

$$v_n(t) = (\Phi v_{n-1})(t), \quad n \geq 1,$$

with the initial condition

$$v_0(t) = \begin{cases} 0, & t \in [T-\rho, T) \\ \frac{\varepsilon}{3}[R(t)-R(T)], & t \geq T. \end{cases}$$

Inductively it is easy to verify that

$$\frac{\varepsilon}{3}[R(t)-R(T)] \leq v_{n-1}(t) \leq v_n(t) \leq \varepsilon[R(t)-R(T)].$$

for $t \geq T$. Therefore for $t \geq T-\rho$, $\lim_{n\to\infty} v_n(t)$ exists. Let $\lim_{n\to\infty} v_n(t) = v(t)$ for $t \geq T-\rho$. By the Lebesgue's dominated convergence theorem $v \in M$ and $(\Phi v)(t) = v(t)$, where $v(t)$ is a solution of (1) on $[T-\rho, \infty)$ such that $v(t) > 0$. We may note that $\lim_{t\to\infty} \frac{z(t)}{R(t)} = \frac{\varepsilon}{3}$, where $z(t) = x(t) + p(t)x(t-\tau)$. Thus the proof is complete. □

Theorem 4. *Let $p \in C(\mathbb{R}_+, (-1, 0])$. Assume that* (5) *hold, then* (1) *admits a bounded positive solutions.*

Proof. Let $-1 < -p \leq p(t) \leq 0$, $p > 0$ for $t \in \mathbb{R}_+$. Due to (5),

$$G(\varepsilon)\int_T^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \leq \frac{\varepsilon}{3},\ T \geq \rho,$$

where $\varepsilon > 0$ is a constant. Consider

$$M = \{x \in C([T-\sigma, +\infty), \mathbb{R}) : x(t) = \frac{\varepsilon}{3},\ t \in [T-\rho, T]; \frac{\varepsilon}{3} \leq x(t) \leq \varepsilon,\ for\, t \geq T\}$$

and let $\Phi : M \to M$ be defined by

$$(\Phi x)(t) = \begin{cases} \frac{\varepsilon}{3}, & t-\rho \le t \le T \\ -p(t)x(t-\tau) + \frac{\varepsilon}{3} + \int_T^t \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)G\big(x(\zeta-\sigma)\big)d\zeta\right] d\eta, & t \ge T. \end{cases}$$

For every $x \in M$, $(\Phi x)(t) \ge \frac{\varepsilon}{3}$ and

$$\begin{aligned}(\Phi x)(t) &\le p\varepsilon + \frac{\varepsilon}{3} + G(\varepsilon)\int_T^t \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)d\zeta\right] d\eta \\ &\le p\varepsilon + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \left(p + \frac{2}{3}\right)\varepsilon \le \varepsilon\end{aligned}$$

implies that $\Phi x \in M$. The rest of the proof follows from Theorem 3. Thus the theorem is proved. □

Theorem 5. *Let $-\infty < -p_1 \le p(t) \le -p_2 < -1$ for $t \in \mathbb{R}_+$, where $p_1, p_2 > 0$ such that $3p_2 > p_1$. Assume that* (5) *hold. Furthermore assume that G is Lipschitzian on the interval of the form $[a,b]$, $0 < a < b < \infty$. Then equation* (1) *admits a positive bounded solution.*

Proof. Due to (5). it is possible to find $T > \rho$ such that

$$\int_T^\infty \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)\,d\zeta\right] d\eta < \frac{p_2-1}{3L},$$

where $L = \max\{L_1, G(1)\}$ and L_1 is the Lipschitz constant of G on $(\alpha, 1)$, $\alpha = \frac{(p_2-1)(3p_2-p_1)}{3p_1p_2}$. Let $Y = BC([T,\infty),\mathbb{R})$ be the space of real valued continuous functions defined on $[T,\infty)$. Indeed, Y is a Banach space with the supremum norm defined by

$$||x|| = \sup\{|x(t)| : t \ge T\}.$$

Define

$$S = \{v \in Y : \ \alpha \le v(t) \le 1,\, t \ge T\}.$$

and we may note that S is a closed and convex subspace of Y. Let $\Psi : S \to S$ be such that

$$(\Psi x)(t) = \begin{cases} (\Psi x)(T+\rho), & t \in [T, T+\rho] \\ -\frac{x(t+\tau)}{p(t+\tau)} - \frac{p_2-1}{p(t+\tau)} + \frac{1}{p(t+\tau)} \int_T^{t+\tau} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)G\big(x(\zeta-\sigma)\big)\,d\zeta\right] d\eta, & t \ge T+\rho. \end{cases}$$

For every $x \in S$,

$$\begin{aligned}(\Psi x)(t) &\leq -\frac{x(t+\tau)}{p(t+\tau)} - \frac{p_2-1}{p(t+\tau)} \\ &\leq \frac{1}{p_2} + \frac{p_2-1}{p_2} = 1\end{aligned}$$

and

$$\begin{aligned}(\Psi x)(t) &\geq -\frac{p_2-1}{p(t+\tau)} + \frac{1}{p(t+\tau)} \int_T^{t+\tau} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta) G\big(x(\zeta-\sigma)\big)\, d\zeta\right] d\eta \\ &\geq \frac{p_2-1}{p_1} + \frac{G(1)}{p(t+\tau)} \int_T^{t+\tau} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)\, d\zeta\right] d\eta \\ &\geq \frac{p_2-1}{p_1} - \frac{G(1)}{p_2} \int_T^{\infty} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)\, d\zeta\right] d\eta \\ &\geq \frac{p_2-1}{p_1} - \frac{p_2-1}{3p_2} = \alpha\end{aligned}$$

implies that $\Psi x \in S$. Now for $x_1, x_2 \in S$, we have

$$\begin{aligned}|(\Psi x_1)(t) - (\Psi x_2)(t)| &\leq \frac{1}{|p(t+\tau)|} |x_1(t+\tau) - x_2(t+\tau)| \\ &\quad + \frac{L}{|p(t+\tau)|} \int_T^{t+\tau} \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta) |x_1(\zeta-\sigma) - x_2(\zeta-\sigma)|\, d\zeta\right] d\eta,\end{aligned}$$

that is,

$$\begin{aligned}|(\Psi x_1)(t) - (\Phi x_2)(t)| &\leq \frac{1}{p_2} ||x_1 - x_2|| + \frac{p_2-1}{3p_2} ||x_1 - x_2|| \\ &= \mu ||x_1 - x_2||\end{aligned}$$

implies that

$$||\Psi x_1 - \Psi x_2|| \leq \mu ||x_1 - x_2||,$$

where $\mu = \frac{1}{p_2}\left(1 + \frac{p_2-1}{3}\right) < 1$. Therefore, Ψ is a contraction. Hence by Banach's contraction mapping principle, Ψ has a unique fixed point $x \in S$. It is easy to see that $\lim_{t\to\infty} x(t) \neq 0$. This completes the proof of the theorem. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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