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FIXED POINT THEOREM ON RANDOM MEIR-KEELER CONTRACTIONS IN G-METRIC SPACE

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Abstract. In this paper, we introduce the notions of random, comparable MY_γ contraction and random, comparable Meir-Keeler contraction in the framework of complete random G-metric spaces. We examine the existence of a random fixed point for these contractions. We express illustrative examples to support the presented results.

Keywords: random fixed points; random comparable MY_γ contraction; probabilistic functional analysis; random G-metric space; Meir-Keeler contraction.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

Probabilistic functional analysis is one of the most useful and interesting research field and random fixed point theory is the main pillar of it. Random fixed point theory is one of the basis of probabilistic functional analysis. Random fixed point theory is the addition of standard fixed point theory within the structure of random analysis. So, we can random fixed point theory appear at the intersection of topology, functional analysis and stochastic analysis. The initial results in random fixed point theory were reported by Speack [1] and Hans [2,3]. After that, the series of many well-known, metric fixed point theorems have been described by different

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authors; e.g., [4-14].

the very interesting fixed point theorem. In this paper, we focus on the Meir-Keeler contraction [15] which is a generalization of Banach contraction principle [16]. Mustafa and Sims [17] introduced a new class of generalized metric space called G-metric space in 2005 which is a generalization of metric sapce (M,d).

In recent past, Chen and Chang [18] gave the concept of ”weaker Meir-Keeler function” and the ”strong Meir-Keeler function” which were noticed from the reflection of the original idea of Meir-Keeler. These access have been investigated dully by several authors; e.g., [19-25].

In what follows, we assert some basic definitions and presented our terminology needed in the series. Throughout the paper, we suppose that all considered sets are non-empty. We present $R_0^+ = [0,\infty)$, and N is used for positive integers. Let $\sum$ be a sigma-algebra of subsets of Ω. Under this supposition, the pair $(\Omega,\sum)$ is called a measurable space.

2. PRELIMINARIES

Definition 2.1. [17] Let M be a non empty set, and $G: M\times M\times M \to R^+$ be a function satisfying the following properties:

(1) $G(l,m,n) = 0$ if $l = m = n$,

(2) $0 < G(l,l,m)$, for all $l,m \in M$, with $l \neq m$,

(3) $G(l,l,m) \leqslant G(l,m,n)$, for all $l,m,n \in M$, with $n \neq m$,

(4) $G(l,m,n) = G(l,n,m) = G(m,n,l) = ...$(symmetry in all three variables),

(5) $G(l,m,n) \leq G(l,a,a)+G(a,m,n)$, for all $l,m,n,a \in M$, (rectangular inequality).

Then the function G is called a generalized metric, or, more specifically a G-metric on M, and the pair (M,G) is called a *G-metric space*.

Example 2.2. Let $M = \{l,m\}$, let $G(l,l,l) = G(m,m,m) = 0, G(l,l,m) = 1, G(l,m,m) = 2$ and extend G to all of $M\times M\times M$ by symmetry in the variables. Then G is a G-metric space. It is non-symmetric since $G(l,m,m) \neq G(l,l,m)$.

Definition 2.3. [17] Let (M,G) be a G-metric space. We say that $\{l_n\}$ is a G-convergent sequence to $l \in M$ if, for any $\varepsilon > 0$, there is $N \in N$ such that for all $n, p \geq N, G(l,l_n,l_p) < \varepsilon$.

Definition 2.4. [17] Let (M,G) be a G-metric space. A sequence $\{l_n\}$ is said to be a G-Cauchy sequence if for each $\varepsilon > 0$ there exists a positive integer N such that $G(l_n, l_p, l_x) < \varepsilon$ for all $n, p, x \geq N$.

Definition 2.5. [17] A G-metric space (M,G) is said to G-complete if every G-Cauchy sequence in (M,G) is G-convergent in (M,G).

In what follow, we presented the definition of the MY function.

Definition 2.6. [26] Let ψ be a function that is defined from non-negative reals into the interval $[0,1)$ then ψ is called the MY function if the following are satisfied:

$$\lim_{s\to t^+} \sup \psi(s) = \inf_{a>0} \sup_{0<s-t<a} \psi(s) < 1 \text{ for all } t \in R^+.$$

Theorem 2.7. [26] *For a mapping $\psi : R^+ \to [0,1)$, the following are equivalent.*

(1) ψ is an MY function.

(2) For any non-increasing sequence $\{\delta_n\}_{n\in N}$ in R^+, we have

$$0 \leq \sup_{n\in N} \psi(\delta_n) < 1.$$

Remark 2.8. [26] Notice that in the case that $\psi : R^+ \to [0,1)$ is non-increasing or non-decreasing, then ψ is a MY function

3. Main Results

In 2020, Li, Karapinar and Chen [27] proved the random Meir-Keeler contraction results in metric space and we extend the following results in G-metric space.
The mappings $\gamma : R_0^+ \times R_0^+ \times R_0^+ \times R_0^+ \to R_0^+$ is called a comparable function, if the following three axioms are fulfilled:

(1) γ is a non-decreasing, continuous function in each coordinate;

(2) $\gamma(r,r,r,r) \leq r, \gamma(0,r,0,r) \leq r$ and $\gamma(0,0,r,r) \leq r$, for all $r > 0$;

(3) $\gamma(r_1, r_2, r_3, r_4) = 0$ if and only if $r_1 = r_2 = r_3 = r_4 = 0$.

Definition 3.1. Let X be a nonempty subset of a random G-metric space $(M,G), \psi$ be a MY function and $Y : \Omega \times X \times X \to X$ be a random operator. Then, for $l \in \Omega, Y(l,.,.)$ is called a random, comparable $MY_{-\gamma}$ contraction if the following condition holds:

$$G(Y(\zeta(l)),Y(\xi(l)),Y(\xi(l))) \leq \psi(G(\zeta(l),\xi(l),\xi(l))).\Gamma(\zeta(l),\xi(l),\xi(l)),$$

where

$$\Gamma(\zeta(l),\xi(l),\xi(l)) = \gamma(G(\zeta(l),\xi(l),\xi(l)),G(\zeta(l),Y(\zeta(l)),Y(\zeta(l)),\Gamma(\xi(l),Y(\xi(l)),Y(\xi(l))), \frac{G(\zeta(l),Y(\xi(l)),Y(\xi(l))+\Gamma(\xi(l),Y(\zeta(l)),Y(\zeta(l)))}{2}),$$

for all $\zeta,\xi \in X$.

Theorem 3.2. *Suppose (M,G) is a complete random G-metric space and $X \subset M$. If $Y(l,.) : \Omega\times X\times X \to X$ is a continuous, random, comparable $\mathsf{MY}-_{\gamma}$ contraction, then Y possesses a random fixed point in M.*

Proof. Given $\zeta_0(l) \in \Omega\times M\times M$ and defining $\zeta_1(l) = Y(\zeta_0(l))$ and $\zeta_{n+1}(l) = Y(\zeta_n(l)) = Y^{n+1}(\zeta_0(l))$ for each $n \in N$, since $Y(l,.) : \Omega\times X\times X \to X$ is a random, comparable $\mathsf{MY}-_{\gamma}$ contraction, we have

$$G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)) = G(Y(\zeta_{n-1}(l)),Y(\zeta_n(l)),Y(\zeta_n(l)))$$
$$\leq \psi(G(\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l))).\Gamma((\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l)))$$

and

$$\Gamma((\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l)) =$$
$$\gamma(G((\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l))),G((\zeta_{n-1}(l),Y(\zeta_n(l)),Y(\zeta_n(l))),G(\zeta_n(l),Y(\zeta_n(l)),Y(\zeta_n(l))), \frac{G(\zeta_{n-1}(l),Y(\zeta_n(l)),Y(\zeta_n(l)))+G(\zeta_n(l),Y(\zeta_{n-1}(l)),Y(\zeta_{n-1}(l)))}{2})$$

$$= \gamma((\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l)),G((\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l)),G((\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)), \frac{G(\zeta_{n-1}(l),\zeta_{n+1}(l),\zeta_{n+1}(l))+G(\zeta_n(l),\zeta_n(l),\zeta_n(l))}{2})$$

If $G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)) > G(\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l))$ for some n, then by the conditions of the function γ we have that

$\Gamma(\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l))\gamma((\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l)),G((\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l)),G((\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)), \frac{G(\zeta_{n-1}(l),\zeta_{n+1}(l),\zeta_{n+1}(l))+G(\zeta_n(l),\zeta_n(l),\zeta_n(l))}{2})$

$\leq (G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)),G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)),G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)),G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)))$

$\leq G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)).$

In a different order pair of γ

$$G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)) = G(Y(\zeta_n(l)),Y(\zeta_{n-1}(l)),Y(\zeta_{n-1}(l)))$$
$$\leq \psi(G(\zeta_n(l),\zeta_{n-1}(l),\zeta_{n-1}(l))).\Gamma(\zeta_n(l),\zeta_{n-1}(l),\zeta_{n-1}(l)),$$

and

$\Gamma(\zeta_n(l),\zeta_{n-1}(l),\zeta_{n-1}(l))$
$= \gamma(G(\zeta_n(l),\zeta_{n-1}(l),\zeta_{n-1}(l)), G(\zeta_n(l)),Y(\zeta_n(l)),Y(\zeta_n(l)), G(\zeta_{n-1}(l)),Y(\zeta_{n-1}(l)),Y(\zeta_{n-1}(l)),$
$\frac{G(\zeta_n(l)),Y(\zeta_{n-1}(l)),Y(\zeta_{n-1}(l))+G(\zeta_{n-1}(l)),Y(\zeta_n(l)),Y(\zeta_n(l))}{2})$
$\leq \gamma(G(\zeta_n(l),\zeta_{n-1}(l),\zeta_{n-1}(l)), G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l), G(\zeta_{n-1}(l)),\zeta_n(l)),\zeta_n(l))),$
$\frac{G(\zeta_n(l)),\zeta_n(l)),\zeta_n(l)+G(\zeta_{n-1}(l)),\zeta_{n+1}(l),\zeta_{n+1}(l)}{2}).$

If $G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)) > G(\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l))$ for some n, then by the conditions of the comparable function γ we have that

$\Gamma(\zeta_n(l),\zeta_{n-1}(l),\zeta_{n-1}(l))$
$= \gamma(G(\zeta_n(l),\zeta_{n-1}(l),\zeta_{n-1}(l)), G(\zeta_n(l)),\zeta_{n+1}(l),\zeta_{n+1}(l)), G(\zeta_{n-1}(l)),\zeta_n(l),\zeta_n(l)),$
$\frac{G(\zeta_n(l)),\zeta_n(l),\zeta_n(l))+G(\zeta_{n-1}(l)),\zeta_{n+1}(l),\zeta_{n+1}(l))}{2})$
$\leq \gamma(G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)), G(\zeta_n(l)),\zeta_{n+1}(l),\zeta_{n+1}(l)), G(\zeta_n(l)),\zeta_{n+1}(l),\zeta_{n+1}(l)),$
$G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)))$
$\leq G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)).$

Since ψ is a MY function, we conclude that

$$G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)) \leq \psi(G(\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l))).G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)))$$
$$< G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)),$$

which implies a contradiction. So, we conclude that

$$G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)) \leq G(\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l)), \text{ for each } n \in N.$$

From above argument, then sequence $\{G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l))\}_{n\in N\cup\{0\}}$ is non-increasing in R_0^+. Since ψ is an MY function, by Theorem 1 we conclude that

$$0 \leq \sup_{n\in N} \psi(G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l))) < 1.$$

Let $\lambda = sup_{n\in N}\psi(G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l))) < 1$; then

$$0 \leq \psi(G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l))) \leq \lambda, \text{ for all } n \in N.$$

Following from the above arguments and by Y being a random, comparable MY contraction, we conclude that for each n

$$(G(\zeta_n(l), \zeta_{n+1}(l), \zeta_{n+1}(l)))$$

$$\leq \psi(G(\zeta_{n-1}(l), \zeta_n(l), \zeta_n(l))).G(\zeta_{n-1}(l), \zeta_n(l), \zeta_n(l))$$

$$\leq \lambda.G(\zeta_{n-1}(l), \zeta_n(l), \zeta_n(l)).$$

Therefore, we also conclude that

$$G(\zeta_n(l), \zeta_{n+1}(l), \zeta_{n+1}(l))$$

$$= G(Y\zeta_{n-1}(l), Y\zeta_n(l), Y\zeta_n(l))$$

$$\leq \lambda.G(\zeta_{n-1}(l), \zeta_n(l), \zeta_n(l))$$

$$\leq \lambda^2.G(\zeta_{n-2}(l), \zeta_{n-1}(l), \zeta_{n-1}(l))$$

$$\leq ...$$

$$\leq \lambda^n.G(\zeta_0(l), \zeta_1(l), \zeta_1(l)).$$

So we have that $\lim_{n\to\infty} G(\zeta_n(l), \zeta_{n+1}(l), \zeta_{n+1}(l)) = 0$, since $\lambda < 1$, and for $n > m$,

$$G(\zeta_m(l), \zeta_n(l), \zeta_n(l))$$

$$\leq (\lambda^m + \lambda^{m+1} + ... + \lambda^{n-1}).G(\zeta_0(l), \zeta_1(l), \zeta_1(l))$$

$$\leq \tfrac{\lambda^m}{1-\lambda}.G(\zeta_0(l), \zeta_1(l), \zeta_1(l)).$$

Let $0 \leq \delta$ be given. Then we can choose a natural number M such that

$$\tfrac{\lambda^m}{1-\lambda}.G(\zeta_0(l), \zeta_1(l), \zeta_1(l)) \leq \delta, \text{ for all } m \geq M,$$

and we also conclude that

$$G(\zeta_m(l), \zeta_n(l), \zeta_n(l)) < \delta, \text{ for all } m \geq M.$$

So $\{\zeta_n(l)\}$ is a Cauchy sequence in $\Omega \times M \times M$. On account of the fact that (M, G) is complete, there exists a $\zeta^*(l) \in \Omega \times M \times M$ such that $\zeta_n(l)$ converges to $\zeta^*(l)$; that is,

$$\lim_{n\to\infty} \zeta_n(l) = \zeta^*(l).$$

Thus, we have

$$G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l)))$$
$$\leq G(\zeta^*(l),\zeta_{n+1}(l),\zeta_{n+1}(l))+G(\zeta_{n+1}(l),Y(\zeta^*(l)),Y(\zeta^*(l)))$$
$$\leq G(\zeta^*(l),\zeta_{n+1}(l),\zeta_{n+1}(l))+G(Y(\zeta_n(l)),Y(\zeta^*(l)),Y(\zeta^*(l)))$$
$$\leq G(\zeta^*(l),\zeta_{n+1}(l),\zeta_{n+1}(l))+\psi(G(\zeta_n(l),Y(\zeta^*(l)),Y(\zeta^*(l))).\Gamma(\zeta_n(l),\zeta^*(l),\zeta^*(l))$$
$$< G(\zeta^*(l),\zeta_{n+1}(l),\zeta_{n+1}(l))+\Gamma(\zeta_n(l),\zeta^*(l),\zeta^*(l)),$$

and

$$\Gamma(\zeta_n(l),\zeta^*(l),\zeta^*(l))$$
$$=\gamma(G(\zeta_n(l),\zeta^*(l),\zeta^*(l)),G(\zeta_n(l),Y(\zeta_n(l)),Y(\zeta_n(l))),G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l))),$$
$$\frac{G(\zeta_n(l),Y(\zeta^*(l)),Y(\zeta^*(l)))+G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l)))}{2}$$
$$\leq \gamma(G(\zeta_n(l),\zeta^*(l),\zeta^*(l)),G(\zeta_n(l),Y(\zeta_{n+1}(l)),Y(\zeta_{n+1}(l))),G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l))),$$
$$\frac{G(\zeta_n(l),Y(\zeta^*(l)),Y(\zeta^*(l)))+G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*)+G(\zeta^*(l),\zeta_{n+1}(l),\zeta_{n+1}(l)))}{2})$$

Taking $n\to\infty$, we have

$$\lim_{n\to\infty}\Gamma(\zeta_n(l),\zeta^*(l),\zeta^*(l))=\gamma(0,0,G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l)),\frac{G^*(l),Y(\zeta^*(l)),Y(\zeta^*(l))}{2})).$$

In a different order pair of γ

$$\Gamma(\zeta^*(l),\zeta_n(l),\zeta_n(l))$$
$$=\gamma(G(\zeta^*(l),\zeta_n(l),\zeta_n(l)),G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l))),G(\zeta_n(l),Y(\zeta_n(l)),Y(\zeta_n(l))),$$
$$\frac{G(\zeta^*(l),Y(\zeta_n(l)),Y(\zeta_n(l)))+G(\zeta_n(l),Y(\zeta^*(l)),Y(\zeta^*(l)))}{2})$$
$$\leq \gamma(G(\zeta^*(l),\zeta_n(l),\zeta_n(l)),G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l))),G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l))$$
$$\frac{G(\zeta^*(l),\zeta_{n+1}(l),\zeta_{n+1}(l))+G(\zeta_n(l),\zeta^*(l),\zeta^*(l)+G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l)))}{2})$$

Taking $n\to\infty$, we have

$$\lim_{n\to\infty}\Gamma(\zeta^*(l),\zeta_n(l),\zeta_n(l))=\gamma(0,G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l))),0,\frac{G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l)))}{2})$$

By the condition of the mapping γ, we conclude that

$$G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l)))<G(\zeta^*(l),Y(\zeta^*(l)),Y(\zeta^*(l))),$$

and this is a contradiction unless $G(\zeta^*(l), Y(\zeta^*(l)), Y(\zeta^*(l))) = 0$.
Therefore, $\zeta^*(l) = Y(\zeta^*(l))$, that is $\zeta^*(l)$ is a random fixed point of Y in M. □

Example 3.3. Let $M = X = R_0^+\Omega = [0,1]$ and Σ be the sigma algebra of Lebegue's measurable subset of $[0,1]$. We define mapping as $G : (\Omega \times M \times M) \times (\Omega \times M \times M) \to X$ by $G(\zeta(l), \xi(l), \eta(l)) = max\{|\zeta(l) - \xi(l)| + |\xi(l) - \eta(l)| + |\zeta(l) - \eta(l)|\}$. Then (M, G) is a random G- metric space. Define random operator $Y : \Omega \times M \times M \to M$ as

$$Y(\zeta(l)) = \frac{\zeta(l)+1-l^2}{2}.$$

Let $\psi(y) = \frac{y}{2y+1} + \frac{1}{2}$ and $\gamma(y_1, y_2, y_3, y_4) = max\{y_1, y_2, y3, y_4\}$; then

$$G(Y(\zeta(l)), Y(\xi(l)), Y(\eta(l))) = G(\frac{\zeta(l)+-l^2}{2}, \frac{\xi(l)+-l^2}{2}, \frac{\eta(l)+-l^2}{2})$$
$$= \max\{|\frac{\zeta(l)+-l^2}{2} - \frac{\xi(l)+-l^2}{2}| + |\frac{\xi(l)+-l^2}{2}| + |\frac{\eta(l)+-l^2}{2}| + |\frac{\zeta(l)+-l^2}{2} - \frac{\eta(l)+-l^2}{2}|\}$$
$$= \max\{|\frac{\zeta(l)-\xi(l)}{2}| + |\frac{\xi(l)-\eta(l)}{2}| + |\frac{\zeta(l)-\eta(l)}{2}|\}$$
$$= \frac{1}{2}G(\zeta(l), \xi(l), \eta(l))$$

and

$$\psi(y).\gamma(G(\zeta(l), \xi(l), \eta(l)), G(\zeta(l), Y(\xi(l)), Y(\eta(l))), G(\xi(l), Y(\zeta(l)), Y(\eta(l))), G(\eta(l), Y(\zeta(l)), Y(\xi(l)))$$
$$\frac{G)\zeta(l),Y(\xi(l)),Y(\eta(l)))+G(\xi(l),Y(\zeta(l)),Y(\eta(l)))+G(\eta(l),Y(\zeta(l)),Y(\xi(l)))}{2})$$
$$\geq \frac{1}{2}.\max\{|\frac{\zeta(l)-\xi(l)}{2}| + |\frac{\xi(l)-\eta(l)}{2}| + |\frac{\zeta(l)-\eta(l)}{2}|\}$$
$$\geq \frac{1}{2}G(\zeta(l), \xi(l), \eta(l))$$

and then Y is a continuous, random, comparable $MY_{-\gamma}$ contraction.
Take the measurable mapping $m : \Omega \to M$ as $\zeta(l) = \{1 - l^2\}$, then for every $l \in \Omega$,

$$Y(\zeta(l) = \frac{\zeta(l)+1-l^2}{2}) = \frac{1-l^2+1-l^2}{2} = 1 - l^2 = \zeta(l).$$

$(1 - l^2)$ is a random fixed point of Y.

Definition 3.4. Let X be a nonempty subset of a random G-metric space (M, G), and let $Y : \Omega \times X \times X \to X$ be a random operator. Then, for $l \in \Omega, Y(l, .)$ is called a random Meir-Keeler contraction if for any real number $\xi > 0$, there exists $\delta > 0$ such that for each $\zeta(l), \xi(l), \eta(l) \in \Omega \times X \times X$.

$$\xi \leq G(\zeta(l), \xi(l), \eta(l)) < \xi + \delta \Rightarrow G(Y(\zeta(l)), Y(\xi(l)), Y(\eta(l))) < \xi$$

Remark 3.5. Note that if Y is a random Meir-Keeler contraction, then we have

$$G(Y(\zeta(l)),Y(\xi(l)),Y(\eta(l))) < G(\zeta(l),\xi(l),\eta(l)).$$

Further, if $G(\zeta(l),\xi(l),\eta(l)) = 0$, then $G(Y(\zeta(l)),Y(\xi(l)),Y(\eta(l))) = 0$. On the other hand, if $G(\zeta(l),\xi(l),\eta(l)) > 0$, then

$$G(Y(\zeta(l)),Y(\xi(l)),Y(\eta(l))) < G(\zeta(l),\xi(l),\eta(l)).$$

Theorem 3.6. *Suppose (M,G) is a complete random G-metric space and $X \subset M$. If $Y(l,.) : \Omega \times X \times X \to X$ is continuous, random, comparable Meir-Keeler contraction, then Y possesses a random fixed point in M.*

Proof. Given $\zeta_0(l) \in \Omega \times M \times M$ and defining $\zeta_1(l) = Y(\zeta_0(l))$, and $\zeta_{n+1}(l) = Y(\zeta_n(l)) = Y^n(\zeta_0(l))$ for each $n \in N$, since $Y(l,.) : \Omega \times X \times X \to X$ is a random Meir-Keeler contraction, by Remark 2, we have

$$G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)) = G(Y(\zeta_{n-1}(l)),Y(\zeta_n(l)),Y(\zeta_n(l))) < G(\zeta_{n-1}(l),\zeta_n(l),\zeta_n(l)).$$

Therefore, $\{G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l))\}_{n\in N\cup\{0\}}$ is decreasing and bounded below; it must converge to some real number $\mu \geq 0$; that is,

$$G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l)) \searrow \mu, \text{ as } n \to \infty.$$

Note that

$$\mu = \inf\{G(\zeta_n(l),\zeta_{n+1}(l),\zeta_{n+1}(l))\}_{n\in N\cup\{0\}}. \tag{3.1}$$

We assert that $\mu = 0$. Suppose, on the contrary, that $\mu > 0$. Since $Y(l,.) : \Omega \times X \times X \to X$ is a continuous, random Meir-Keeler contraction, corresponding to this μ, there exist $\delta > 0$ and $m \in N$ such that

$$\mu \leq G(\zeta_m(l),\zeta_{m+1}(l),\zeta_{m+1}(l)) < \mu + \delta$$

$$\Longrightarrow G(\zeta_{m+1}(l),\zeta_{m+2}(l),\zeta_{m+2}(l)) = G(Y(\zeta_m(l)),Y(\zeta_{m+1}(l)),Y(\zeta_{m+1}(l))) < \mu,$$

a contradiction. Attendantly, we find that $\mu = 0$.

We next show that $\{\zeta_n(w)\}$ is Cauchy sequence in (M,G). We shall use the method of reduction absurdum. Suppose, on the contrary, that there exists a real number $\varepsilon > 0$ such that for any $k \in N$, there are $m_k, n_k \in N$ with $n_k \geq m_k > k$ satisfying

$$G(\zeta_{m_k}(l), \zeta_{n_k}(l), \zeta_{n_k}(l)) \geq \varepsilon.$$

In addition, comparable to $m_k \geq k$, we can choose n_k so that $n_k > m_k \geq k$ and $G(\zeta_{m_k}(l), \zeta_{n_k}(l), \zeta_{n_k}(l)) \geq \varepsilon$.

Therefore, we also have $G(\zeta_{m_k}(l), \zeta_{n_k-2}(l), \zeta_{n_k-2}(l)) \geq \varepsilon$. So, we have that for all $k \in N$,

$$\varepsilon \leq G(\zeta_{m_k}(l), \zeta_{n_k}(l), \zeta_{n_k}(l))$$

$$\leq G(\zeta_{m_k}(l), \zeta_{n_k-2}(l), \zeta_{n_k-2}(l)) + G(\zeta_{n_k-2}(l), \zeta_{n_k-1}(l), \zeta_{n_k-1}(l)) + G(\zeta_{n_k-1}(l), \zeta_{n_k}(l), \zeta_{n_k}(l))$$

$$< \varepsilon + G(\zeta_{n_k-2}(l), \zeta_{n_k-1}(l), \zeta_{n_k-1}(l)) + G(\zeta_{n_k-1}(l), \zeta_{n_k}(l), \zeta_{n_k}(l)).$$

Letting $k \to \infty$, we have that

$$\lim_{k\to\infty} G(\zeta_{m_k}(l), \zeta_{n_k}(l), \zeta_{n_k}(l)) = \varepsilon.$$

On the other hand, we have that

$$\varepsilon \leq G(\zeta_{m_k}(l), \zeta_{n_k}(l), \zeta_{n_k}(l))$$

$$\leq G(\zeta_{m_k}(l), \zeta_{m_k+1}(l), \zeta_{m_k+1}(l)) + G(\zeta_{m_k+1}(l), \zeta_{n_k+1}(l), \zeta_{n_k+1}(l)) + G(\zeta_{n_k+1}(l), \zeta_{n_k}(l), \zeta_{n_k}(l))$$

$$\leq G(\zeta_{m_k}(l), \zeta_{m_k+1}(l), \zeta_{m_k+1}(l)) + G(\zeta_{m_k+1}(l), \zeta_{m_k}(l), \zeta_{m_k}(l)) + G(\zeta_{m_k}(l), \zeta_{n_k}(l), \zeta_{n_k}(l)) +$$
$$G(\zeta_{n_k}(l), \zeta_{n_k+1}(l), \zeta_{n_k+1}(l)) + G(\zeta_{n_k+1}(l), \zeta_{n_k}(l), \zeta_{n_k}(l)).$$

Letting $n \to \infty$, we have

$$G(Y(\zeta^*(l)), \zeta^*(l), \zeta^*)l)) = 0.$$

This implies that $Y(\zeta^*(l)) = \zeta^*(l)$; that is $\zeta^*(l)$ is a random fixed point of Y. □

Example 3.7. Let $M = X = R_0^+ \cup 0$, also $\Omega = [0,1]$ and $\sum$ be the sigma algebr4as of Lebegue's measurable subset of $[0,1]$. Let $M = [0,\infty)$, and define mappings as $G : (\Omega \times M \times M) \times (\Omega \times M \times M) \times (\Omega \times M \times M) \to X$ by

$$G(\zeta(l), \xi(l), \eta(l)) = \max\{|\zeta(l) - \xi(l)| + |\xi(l) - \eta(l)| + |\zeta(l) - \eta(l)|\}.$$

Then, (M, G) is a cone random G metric space. Define random operator $Y : \Omega \times M \times M \to M$ as

$$Y(\zeta(l)) = \tfrac{\zeta(l)+-l^2}{2}.$$

For any $\xi > 0$, take $\delta = \xi$, if

$$\xi \leq G(\zeta(l),\xi(l),\eta(l)) < \xi+\delta,$$

then

$$G(Y(\zeta(l)),Y(\xi(l)),Y(\eta(l))) = G(\tfrac{\zeta(l)+-l^2}{2},\tfrac{\xi(l)+-l^2}{2},\tfrac{\eta(l)+-l^2}{2})$$

$$= \max\{|\tfrac{\zeta(l)+-l^2}{2}-\tfrac{\xi(l)+-l^2}{2}|+|\tfrac{\xi(l)+-l^2}{2}|+|\tfrac{\eta(l)+-l^2}{2}|+|\tfrac{\zeta(l)+-l^2}{2}-\tfrac{\eta(l)+-l^2}{2}|\}$$

$$= \max\{|\tfrac{\zeta(l)-\xi(l)}{2}|+|\tfrac{\xi(l)-\eta(l)}{2}|+|\tfrac{\zeta(l)-\eta(l)}{2}|\}$$

$$\leq G(\zeta(l),\xi(l),\eta(l))$$

$$\tfrac{1}{2}(\xi+\delta) = \xi.$$

This implies that Y is a continuous, random Meir-Keeler contraction.

Take the measurable mapping $m:\Omega\to M$ as $\zeta(l)=\{1-l^2\}$, then for every $l\in\Omega$,

$$Y(\zeta(l) = \tfrac{\zeta(l)+1-l^2}{2}) = \tfrac{1-l^2+1-l^2}{2} = 1-l^2 = \zeta(l).$$

$(1-l^2)$ is a random fixed point of Y.

4. Conclusions

One of the most interesting and useful field of research is probabilistic functional analysis due to its extensive potential of application to probabilistic models in applied problems. In the theory of random operators many different classes of random equations were investigated. More specifically, we need some mathematical models or equations when explaining many different phenomena in different quantitative disciplines (for e.g., engineering, biology and physics). These models and equations contain some parameters or coefficients with unknown values, but they have specific interpretations. Random fixed point results play an important role in differential / integral equations solution [28]. Accordingly, the random fixed point theory is an important tool for solutions to real-world problems whenever they are realistically modeled. Our results help to expand the theory of random fixed-points.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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