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# SOME RESULTS, IN THE $\alpha$-NORM, FOR NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH FINITE DELAY

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**Abstract.** In this paper, we consider, in a general Banach space, a nonlinear integro-differential equation with finite delay. The nonlinear part is assumed to be continuous with respect to a fractional power of the linear part in the second variable. Using the semigroup theory, we prove some qualitatives and quantitaves results under the alpha norm. An application is provided to illustrate our results.

**Keywords:** integro-differential; finite delay; fractionnal power; mild solution; strict solution; phase space; alpha norme.

**2010 AMS Subject Classification:** 47H10, 47G20.

## 1. INTRODUCTION

Integrodifferential equations with delay are important for investigating some problems araised from natural phenomena. They have been studied in many different aspects, see [2, 12, 19, 21, 22] for more details. In [9], Ezzinbi et al. investigated the existence and regularity

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of solutions of equation of the following integrodifferential equation

$$(1.1)\qquad \begin{cases} u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + F(t,u_t) \quad \text{for} \quad t \geq 0, \\ u_0 = \varphi \in \mathscr{C} = C\big([-r,0];\mathbb{X}\big), \end{cases}$$

where $A$ and $B$ are linear and closed operators, $r$ is the delay, $F : \mathbb{R}_+ \times \mathscr{C} \to \mathbb{X}$ is a continuous function and as usual, the history function $u_t \in \mathscr{C}$ is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in [-r,0].$$

The authors obtained the uniqueness, the representation of solutions via a variation of constant formula and other properties of the resolvent operator were studied. In [8], Ezzinbi et al. studied a local existence and regularity of equation (1.1). To achieve their goal, the authors used the variation of constant formula, the theory of resolvent operator and the principle contraction method. Ezzinbi et al. in [10] studied the local existence and global continuation for equation (1.1). For more results about integrodifferential equation, the reader can see [**?**, 14, 15, 16, 17].

In the case where the nonlinear part involves spatial derivative, the above obtained results become invalid. To overcome this difficulty, Diao et al. in [4] restrict the problem in a Banach space $\mathbb{X}_\alpha \subset \mathbb{X}$ and they consider the following equation

$$(1.2)\qquad \begin{cases} u'(t) = -Au(t) + \int_0^t B(t-s)u(s)ds + F(t,u_t) \quad \text{for} \quad t \geq 0, \\ u_0 = \varphi \in \mathscr{C}_\alpha = C\big([-r,0],D(A^\alpha)]\big), \end{cases}$$

where $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$. $B(t)$ is a closed linear operator with domain $D(B) \supset D(A)$ time-independent. For $0 < \alpha < 1$, $A^\alpha$ is the fractional power of $A$ which will be precised in the sequel. The domain $D(A^\alpha) = \mathbb{X}_\alpha \supset D(A)$, endowed with the norm $\|x\|_\alpha = \|A^\alpha x\|$, called $\alpha$-norm, is a Banach space. $\mathscr{C}_\alpha$ is the Banach space $C([-r,0],D(A^\alpha))$ of continuous functions from $[-r,0]$ to $D(A^\alpha)$ endowed with the following norm

$$\|\phi\|_{\mathscr{C}_\alpha} = \sup_{-r\leq\theta\leq 0} \|\phi(\theta)\|_\alpha \quad \text{for } \phi \in \mathscr{C}_\alpha.$$

$F:\mathbb{R}_+\times\mathscr{C}_\alpha\to\mathbb{X}$ is a continuous function and as usual, $u_t\in\mathscr{C}_\alpha$ is the history function. They obtained, used the resolvent operators theory, the existence, uniqueness, regularity and compactness properties of the so-called mild solution of equation (1.2). Their results are a genaralization of the paper of Travis et al. in [20] where they considered the case $B=0$. For the previous case, more results can be founded in [1, 7].

Recently, Koumla et al. [6] investigated the study of equation (1.1) where the kernel $B$ is nonlinear, that is they consider the following system

$$\begin{cases} u'(t)=-Au(t)+\displaystyle\int_0^t g\big(t-s,u(s)\big)ds+F(t,u_t), \text{ for } t\geq 0,\\ u_0=\varphi\in\mathscr{C}=\mathscr{C}([-r,0];D(A)), \end{cases} \tag{1.3}$$

where $g:\mathbb{R}^+\times D(A)\longrightarrow\mathbb{X}$ and $F:\mathbb{R}^+\times\mathscr{C}\longrightarrow D(A)$ are two nonlinear functions and $\mathscr{C}$ is the espace of continuous function from $[-r,0]$ to $D(A)$. In this case, the theory of resolvent operators do not work, so they used the semigroup theory to obtain existence, regularity and continuous dependance of the initial data.

The main purpose of this work is to study the existence, uniqueness, continuous dependence and regularity properties of a class for nonlinear partial functional integrodifferential equations of retarded type with deviating arguments in terms involving spatial partial derivatives in the form

$$\begin{cases} u'(t)=-Au(t)+\displaystyle\int_0^t g\big(t-s,u(s)\big)ds+F(t,u_t), \text{ for } t\geq 0,\\ u_0=\varphi\in\mathscr{C}_\alpha=\mathscr{C}([-r,0];\mathbb{X}_\alpha), \end{cases} \tag{1.4}$$

where $g:\mathbb{R}^+\times\mathbb{X}_\alpha\longrightarrow\mathbb{X}_\alpha$ and $F:\mathbb{R}^+\times\mathscr{C}_\alpha\longrightarrow\mathbb{X}$ are two nonlinear functions. We recall that $\mathbb{X}_\alpha$ is larger than $D(A)$, that is $D(A)\subset\mathbb{X}_\alpha$. As such a system, one can consider the following equation

$$\begin{cases} \dfrac{\partial}{\partial t}w(t,x)=\dfrac{\partial^2}{\partial x^2}w(t,x)+\displaystyle\int_0^t g(t-s,w(s,x))ds\\ \qquad\qquad +\displaystyle\int_{-r}^0 k(t,\frac{\partial}{\partial x}w(t+\theta,x))d\theta \quad \text{for } t\geq 0 \text{ and } x\in[0,\pi],\\ w(t,0)=w(t,\pi)=0 \quad \text{for } t\geq 0,\\ w(\theta,x)=w_0(\theta,x) \quad \text{for } \theta\in[-r,0] \text{ and } x\in[0,\pi], \end{cases} \tag{1.5}$$

where $w_0 : [-r,0] \times [0,\pi] \longrightarrow \mathbb{R}$, $k : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ and $h : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are appropriates functions. The present paper is motivated by the paper of Travis et al. in [20] and Diao et al. [4].

The paper is organized as follows. In Section 2, we recall some fundamental properties of the semigroup theory and fractional powers of closed operators. The global existence, uniqueness and continuous dependence with respect to the initial data are studied in the Section 3. In section 4 we prove under some conditions, the regularity of the mild solution. And finally we illustrate our main results in Section 5 by examining an example.

## 2. Fractional Power of Closed Operators and Semigroup Theory

In this section, we shall write $\mathbb{Y}$ for $D(A)$ endowed with the graph norm, $\mathbb{X}_\alpha$ for $D(A^\alpha)$ and $\mathscr{L}(\mathbb{E},\mathbb{F})$ will denote the space of bounded linear operators from the Banach espace $\mathbb{E}$ to the Banach espace $\mathbb{F}$ and if $\mathbb{E} = \mathbb{F}$, we write $\mathscr{L}(\mathbb{E})$ with norm $\|.\|$. We assume that $-A$ generates an analytic semigroup and, without loss of generality that $0 \in \rho(A)$, then one can define the fractional power $A^\alpha$ for $0 < \alpha < 1$, as a closed linear operator on its domain $\mathbb{X}_\alpha$ with its inverse $A^{-\alpha}$ is given by

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\, dt,$$

where $\Gamma$ is the Gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt.$$

See Pazy [18], for more details. We have the following known results.

**Theorem 2.1.** [18] The following properties are true.

- **(i)** $\mathbb{X}_\alpha = D(A^\alpha)$ is a Banach space with the norm $\|x\|_\alpha = \|A^\alpha x\|$ for $x \in \mathbb{X}_\alpha$,
- **(ii)** $A^\alpha$ is a closed linear operator with domain $\mathbb{X}_\alpha = \mathrm{Im}(A^{-\alpha})$ and $A^\alpha = (A^{-\alpha})^{-1}$,
- **(iii)** $A^{-\alpha}$ is a bounded linear operator in $\mathbb{X}$,
- **(iv)** If $0 < \alpha \leq \beta$ then $D(A^\beta) \hookrightarrow D(A^\alpha)$. Moreover the injection is compact if $T(t)$ is compact for $t > 0$.

Now, we collect the definition and basic results about the theory of semigroup.

**Definition 2.1.** [18] A fammily of bounded linear operators $T(z)$, $z \in \Delta$ where

$$\Delta = \{z \in \mathbb{C} : \varphi_1 < argz < \varphi_2, \varphi_1 < 0 < \varphi_2\}$$

is called analytic semigroup in $\Delta$ if the following properties hold:

**(i)** $z \longrightarrow T(z)$ is analytic in $\Delta$,

**(ii)** $T(0) = I$ and $\lim\limits_{\substack{z\in\Delta \\ z\to 0}} T(z)x = x$ for $x \in \mathbb{X}$,

**(iii)** $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.

We say that a semigroup is analytic if it is analytic in some sector $\Delta$ containing the nonnegative real axis.

In this paper, we assume that the operator $-A$ is an infinitesimal generator of an analytic semigroup. See [5, 13, 11] for more informations.

**Theorem 2.2.** [18] The following properties are true:

**(i)** $T(t) : \mathbb{X} \longrightarrow \mathbb{X}_\alpha$ for $t \geq 0$ and $\alpha \geq 0$.

**(ii)** $T(t)A^\alpha x = A^\alpha T(t)x$ for $x \in \mathbb{X}_\alpha$.

**(iii)** For $t > 0$, $A^\alpha T(t)$ is a bounded operator and there exists $M_\alpha, w \in \mathbb{R}$ such that

$$\|A^\alpha T(t)\| \leq M_\alpha t^{-\alpha} e^{-\omega t}.$$

We gives in next, the definition of the so-called strict and mild solutions. Consider the following nonhomogeneous equation

$$\begin{cases} u'(t) = -Au(t) + f(t) \quad \text{for} \quad t \in [0,b], \\ u(0) = u_0 \in \mathbb{X}. \end{cases} \tag{2.1}$$

**Definition 2.2.** [18] A continuous function $u : [0,b] \to \mathbb{X}$ is called a strict solution of the equation (2.1) if

**(i)** $t \to u(t)$ is continuously differentiable on $[0,b]$,

**(ii)** $u(t) \in \mathbb{Y}$ for $t \in [0,b]$,

**(iii)** $u$ satisfies equation (2.1) on $[0,b]$.

**Theorem 2.3.** [18] If $u$ is a strict solution of equation (2.1), then $u$ is given by the following formula

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds \quad \text{for} \quad t \in [0,b]. \tag{2.2}$$

**Remark 2.1.** If $u$ satisfies formula (2.2), $u$ is not generally a strict solution of equation (2.1). This motive the following definition.

**Definition 2.3.** [18] A continuous function $u : [0,b] \to \mathbb{X}$ is called a mild solution of equation (2.1) if $u$ satisfies equation (2.2).

## 3. Global Existence and Continuous Dependance of Initial Data

This section is asserted to the results of global existence, uniqueness and continuous dependence with respect to the initials data. We give the definitions of the so-called mild and strict solutions of equation (1.4).

**Definition 3.1.** A function $u : [0,b] \to \mathbb{X}_\alpha$ is called a strict solution of equation (1.4), if

- **(i)** $t \to u(t)$ is continuously differentiable on $[0,b]$,
- **(ii)** $u(t) \in \mathbb{Y}$ for $t \in [0,b]$,
- **(iii)** $u$ satisfies equation (1.4) on $[0,b]$.

**Definition 3.2.** If $u$ is a mild solution of equation (1.4), then $u$ is given by

(3.1)
$$\begin{cases} u(t) = T(t)\varphi(0) + \int_0^t T(t-s) \int_0^s g\big(s-\tau, u(\tau)\big) d\tau ds + \int_0^t T(t-s) F(s, u_s) ds, \forall\, t > 0. \\ u_0 = \varphi \in \mathscr{C}_\alpha. \end{cases}$$

**Definition 3.3.** A continuous function $u : [-r, +\infty[ \longrightarrow \mathbb{X}_\alpha$ is called a mild solution of equation (1.4) if $u$ satisfies the equation (3.1).

Now to obtain our first result, we take the following assumptions

**(H1)** $F : \mathbb{R}^+ \times \mathscr{C}_\alpha \longrightarrow \mathbb{X}_\alpha$ is continuous and there exists $L_f \geq 0$ such that

$$|F(t,\varphi) - F(t,\psi)| \leq L_f |\varphi - \psi|_{\mathscr{C}_\alpha} \text{ for } t \geq 0 \text{ and } \varphi, \psi \in \mathscr{C}_\alpha.$$

**(H2)** $g:\mathbb{R}^+\times\mathbb{X}_\alpha\longrightarrow\mathbb{X}$ is continuous and there exists $L_g\geq 0$ such that

$$|g(t,x)-g(t,y)|\leq L_g|x-y|_\alpha \text{ for } t\geq 0 \text{ and } x,y\in\mathbb{X}_\alpha.$$

**Theorem 3.1.** Assume that **(H1)-(H2)** hold. Then for $\varphi\in\mathscr{C}_\alpha$, equation (1.4) has an unique mild solution which is defined for all $t\geq 0$.

**Proof.** Let $t_1>0$. For $\varphi\in\mathscr{C}_\alpha$ we define the set $M_{t_1}(\varphi)$ by

$$M_{t_1}(\varphi):=\{u\in\mathscr{C}([0,t_1];\mathbb{X}_\alpha):u(0)=\varphi(0)\}.$$

We claim that $M_{t_1}(\varphi)$ is a closed set of $\mathscr{C}\big([0,t_1];\mathbb{X}_\alpha\big)$, where $\mathscr{C}\big([0,t_1];\mathbb{X}_\alpha\big)$ is the set of continuous functions define from $[0,t_1]$ to $\mathbb{X}_\alpha$ endowed with the uniforme norm topology. Inded, let a sequence $(u_n)_{n\geq 0}$ of $M_{t_1}(\varphi)$ such that $\lim\limits_{n\to+\infty}u_n=u$. By uniform convergence, $u$ is continuous and $u(0)=\varphi(0)$. For $u\in M_{t_1}(\varphi)$, we define it extension on $[-r,t_1]$ by $\widetilde{u}=\varphi(t)$ if $t\in[-r,0]$ and $\widetilde{u}=u(t)$ if $t\in[0,t_1]$. Let $L$ the operator define on $M_{t_1}(\varphi)$ by

$$(Lu)(t)=T(t)\varphi(0)+\int_0^t T(t-s)\left[\int_0^s g\left(s-\tau,\widetilde{u}(\tau)\right)d\tau+F\left(s,\widetilde{u}_s\right)\right]ds.$$

Let $u\in M_{t_1}(\varphi)$. Then we have $(Lu)(0)=\varphi(0)$ and by continuity of $F$ and $g$ we deduce that $Lu\in M_{t_1}(\varphi)$, whiche imply that $\Gamma\big(M_{t_1}(\varphi)\big)\subseteq M_{t_1}(\varphi)$. Now we prove that $L$ is a contraction on $M_{t_1}(\varphi)$. To sow this, let $u,v\in M_{t_1}(\varphi)$, $t\geq 0$. Then

$$\begin{aligned}A^\alpha(Lu)(t)-A^\alpha(Lv)(t)=&\int_0^t A^\alpha T(t-s)\int_0^s\Big(g(s-\tau,\widetilde{u}(\tau))-g(s-\tau,\widetilde{v}(\tau))\Big)d\tau ds\\&+\int_0^t A^\alpha T(t-s)\Big(F(s,\widetilde{u}_s)-F(s,\widetilde{v}_s)\Big)ds.\end{aligned}$$

Taking the norm and using the hypothesis **(H1)-(H2)**, we obtain

$$\begin{aligned}\|A^\alpha(Lu)(t)-A^\alpha(Lv)(t)\|\leq& L_g t_1\int_0^t M_\alpha(t-s)^{-\alpha}e^{-w(t-s)}\sup_{0\leq s\leq t_1}\|u(s)-v(s)\|_\alpha ds\\&+L_f\int_0^t M_\alpha(t-s)^{-\alpha}e^{-w(t-s)}\left\|\widetilde{u}_s-\widetilde{v}_s\right\|_{\mathscr{C}_\alpha}.\end{aligned}$$

We deduce that

$$\|(Lu)-(Lv)\|_{\mathscr{C}_\alpha}\leq\big(t_1L_g+L_f\big)M_\alpha\left(\int_0^{t_1}s^{-\alpha}e^{-ws}ds\right)\|u-v\|_{\mathscr{C}_\alpha}.$$

Now we choose $t_1$ such that

$$\left(t_1 L_g + L_f\right) M_\alpha \left(\int_0^{t_1} s^{-\alpha} e^{-ws} ds\right) < 1.$$

Then $L$ is a contraction on $M_{t_1}(\varphi)$ and it has an unique fixed point $u$ which is the unique mild solution of equation (1.4) on $[0,t_1]$. To extend the solution of equation (1.4) in $[t_1, 2t_1]$, we show the existence and uniqueness of the following equation

$$\begin{cases} z'(t) = -Az(t) + \int_{t_1}^{t} g(t-s, z(s))\, ds + F(t, z_t) \quad \text{for} \quad t \in [t_1, 2t_1], \\ z_{t_1} = u_{t_1} \in C\Big([-r, t_1], \mathbb{X}_\alpha\Big). \end{cases} \tag{3.2}$$

Notice that the solution of equation (3.2) is given by

$$z(t) = T(t-t_1)z(t_1) + \int_{t_1}^{t} T(t-s) \int_{t_1}^{s} g\big(s-\tau, z(\tau)\big) d\tau ds + \int_{t_1}^{t} T(t-s) F(s, z_s) ds,\ t \in [t_1, 2t_1]$$

Let $\overline{z}$ be the function define by $\overline{z}(t) = z(t)$ for $t \in [t_1, 2t_1]$ et $\overline{z}(t) = u(t)$ for $t \in ]-r, t_1]$. Consider the set $M_{2t_1}(\varphi) = \big\{z \in \mathscr{C}_{t_1} = C\big([t_1, 2t_1]; \mathbb{X}_\alpha\big); z(t_1) = u(t_1)\big\}$ provided with the induced topological norm. We define the operator $H$ on $M_{2t_1}(\varphi)$ by

$$(Hz)(t) = T(t-t_1)z(t_1) + \int_{t_1}^{t} T(t-s) \int_{t_1}^{s} g\big(s-\tau, z(\tau)\big) d\tau ds + \int_{t_1}^{t} T(t-s) f(s, z_s) ds, t \in [t_1, 2t_1]$$

We have $(Hz)(t_1) = u(t_1)$ and $Hz$ is continuous. Then we deduce the following inclusion $H\big(M_{2t_1}(\varphi)\big) \subset M_{2t_1}(\varphi)$. Moreover, for $u, v \in M_{2t_1}(\varphi)$, we have

$$\begin{aligned} \left\|A^\alpha \big( (H(u))(t) - (H(v))(t) \big)\right\| \le\ & L_g M_\alpha t_1 \int_{t_1}^{t} \frac{e^{-w(t-s)}}{(t-s)^\alpha} \sup_{t_1 \le s \le 2t_1} \|u(s) - v(s)\|_\alpha ds \\ & + L_f M_\alpha \int_{t_1}^{t} \frac{e^{-w(t-s)}}{(t-s)^\alpha} \|\overline{u}_s - \overline{v}_s\|_{\mathscr{C}_\alpha} ds. \end{aligned}$$

Since $\overline{u} = \overline{v} = \varphi$ in $[-r, 0]$, we deduce that

$$\|(Hu) - (Hv)\|_{\mathscr{C}_\alpha} \le \left(t_1 L_g + L_f\right) M_\alpha \left(\int_0^{t_1} s^{-\alpha} e^{-ws} ds\right) \|u - v\|_{\mathscr{C}_\alpha}.$$

One can conclude that $H$ has an unique fixed point on $M_{2t_1}(\varphi)$ which extend the solution $u$ on $[t_1, 2t_1]$. Proceeding inductively, $u$ is uniquely and continuously extended to intervalles $[nt_1, (n+1)t_1]$, for $n \ge 1$ and this ends the proof. $\square$

Now we show the continuous dependence of the mild solutions with respect to the initial data.

**Theorem 3.2.** Assume that **(H1)-(H2)** hold. Then the mild solution $u(.,\varphi)$ of equation (1.4) defines a continuous Lipschitz operator $U(t), t \geq 0$ in $\mathscr{C}_\alpha$ by $U(t)\varphi = u_t(.,\varphi)$. Moreover there exists a real number $\delta$ and a scalar function $\beta$ such that for $t \geq 0$ and $\varphi_1, \varphi_2 \in \mathscr{C}_\alpha$ we have

$$\|U(t)\varphi_1 - U(t)\varphi_2\| \leq \beta(\delta)e^{\delta t}\|\varphi_1 - \varphi_2\|_\alpha. \tag{3.3}$$

**Proof.** The continuity is obvious on what the map $t \to u_t(.,\varphi)$ is continuous. Now, let $\varphi_1, \varphi_2 \in \mathscr{C}_\alpha$. If we pose $w(t) = u(t,\varphi_1) - u(t,\varphi_2)$, then we have

$$\begin{aligned} w(t) = T(t)\big(\varphi_1(0) - \varphi_2(0)\big) + \int_0^t T(t-s)\int_0^s \big[g\big(s-\tau, u(\tau,\varphi_1)\big) - g\big(s-\tau, u(\tau,\varphi_2)\big)\big]\, d\tau ds \\ + \int_0^t T(t-s)\big[F\big(s, u_s(\cdot,\varphi_1)\big) - F\big(s, u_s(\cdot,\varphi_2)\big)\big]\, ds. \end{aligned} \tag{3.4}$$

Taking the $\alpha$-norm, we obtain

$$\begin{aligned} \|w(t)\|_\alpha \leq & \big\|T(t)A^\alpha\big(\varphi_1(0) - \varphi_2(0)\big)\big\| \\ & + \left\|A^\alpha \int_0^t T(t-s)\int_0^s \big[g\big(s-\tau, u(\tau,\varphi_1)\big) - g\big(s-\tau, u(\tau,\varphi_2)\big)\big]\, d\tau ds\right\| \\ & + \left\|A^\alpha \int_0^t T(t-s)\big[F\big(s, u_s(\cdot,\varphi_1)\big) - F\big(s, u_s(\cdot,\varphi_2)\big)\big]\, ds\right\|. \end{aligned}$$

Which imply that

$$\begin{aligned} \|w(t)\|_\alpha \leq Me^{wt}\|\varphi_1(0) - \varphi_2(0)\|_\alpha + M_\alpha L_g t_1 \int_0^t \frac{e^{-w(t-s)}}{(t-s)^\alpha} \sup_{0\leq\tau\leq t_1} \|u(\tau,\varphi_1) - u(\tau,\varphi_2)\|_\alpha ds \\ + M_\alpha L_f \int_0^t \frac{e^{-w(t-s)}}{(t-s)^\alpha}\|u_s(\cdot,\varphi_1) - u_s(\cdot,\varphi_2)\|_{\mathscr{C}_\alpha} ds. \end{aligned} \tag{3.5}$$

Let $\delta$ a real number such that $\omega - \delta < 0$. For $-r \leq \tau \leq 0$, we have

$$e^{-\delta\tau}\|w(\tau)\| \leq N_1 L \quad \text{where} \quad N_1 = \max\{e^{\delta r}, 1\} \quad \text{and} \quad L = \|\varphi_1 - \varphi_2\|_{\mathscr{C}_\alpha}. \tag{3.6}$$

Now, for $0 \leq \tau \leq t_1$, we have from (3.5)

$$e^{-\delta\tau}\|w_\tau\|_{\mathscr{C}_\alpha} \leq LMe^{(w-\delta)\tau} + M_\alpha L_g t_1 \int_0^\tau e^{-(w-\delta)(\tau-s)}\,(\tau-s)^{-\alpha}\,e^{-\tau s}\|w_s\|_{\mathscr{C}_\alpha} ds$$

$$+M_\alpha L_f \int_0^\tau e^{-(w-\delta)(\tau-s)}\,(\tau-s)^{-\alpha}\,e^{-\tau s}\|w_s\|_{\mathscr{C}_\alpha} ds. \tag{3.7}$$

Since $w-\delta \leq 0$, then we have

$$e^{-\delta\tau}\|w_\tau\|_{\mathscr{C}_\alpha} \leq LM + M_\alpha\left(L_f + L_g t_1\right)\int_0^\tau e^{-(w-\delta)(\tau-s)}\,(\tau-s)^{-\alpha}\,e^{-\tau s}\|w_s\|_{\mathscr{C}_\alpha} ds. \tag{3.8}$$

Then from (3.6) and (3.8) we deduce that

$$\sup_{-r\leq\tau\leq t_1} e^{-\delta\tau}\,\|w_\tau\|_{\mathscr{C}_\alpha} \leq LMN_1 + \left(t_1 L_g + L_f\right) M_\alpha W \Gamma(1-\alpha)(w-\delta)^{\alpha-1}, \tag{3.9}$$

where

$$W = \sup_{0\leq t\leq t_1} e^{-\delta t}\,\|w_t\|_{\mathscr{C}_\alpha} \quad \text{and} \quad \Gamma(1-\alpha)k^{\alpha-1} = \int_0^\infty e^{-ks}s^{-\alpha}ds,$$

with $k > 0$ (See [?], p.265). On the other hand, for $0 \leq t \leq t_1$, we have

$$\begin{aligned}
e^{-\delta t}\,\|w_t\|_{\mathscr{C}_\alpha} &= \underset{-r\leq\theta\leq 0}{sup}\, e^{\delta\theta}e^{-\delta(t+\theta)}\,\|w(t+\theta)\|_\alpha \\
&\leq N_2 \underset{-r\leq\theta\leq 0}{sup}\, e^{-\delta(t+\theta)}\,\|w(t+\theta)\|_\alpha \\
&\leq N_2 \underset{-r\leq\tau\leq t_1}{sup}\, e^{-\delta\tau}\,\|w_\tau\|_{\mathscr{C}_\alpha}.
\end{aligned}$$

where $N_2 = max\{e^{-\delta r}, 1\}$. Therefore, we have

$$W \leq LMN_1N_2 + N_2 M_\alpha\left(t_1 L_g + L_f\right)\Gamma(1-\alpha)(w-\delta)^{\alpha-1}W.$$

Then we deduce that

$$\|\mathrm{U}(\mathrm{t})\,\varphi_1 - \mathrm{U}(\mathrm{t})\,\varphi_2\|_{\mathscr{C}_\alpha} \leq \beta(\delta)\mathrm{e}^{\delta\mathrm{t}}\,\|\varphi_1-\varphi_2\|_{\mathscr{C}_\alpha},$$

where

$$\beta(\delta) = MN_1N_2\left(1 - N_2M_\alpha(t_1L_g+L_f)\Gamma(1-\alpha)(\omega-\delta)^{\alpha-1}\right)^{-1}.$$

This end the proof. $\square$

## 4. Regularity of the Mild Solution

In this section we prove, under certains conditions, that the mild solution obtained in Theorem 3.1 is a strict solution. To do this, we denote by $\mathscr{C}_\alpha^1 = \mathscr{C}^1\big([-r,0];\mathbb{X}_\alpha\big)$ as the espace of continuously differentiables functions from $[-r,0]$ to $\mathbb{X}_\alpha$ and we consider the following hypothesis:

(**H3**) The functions $F$ and $g$ are continuously differentiables, theire partials derivatives are loccaly Lipschitzian with respect to the second argument and $g(0,x)=g(0,y)$ for $x,y \in \mathbb{X}_\alpha$.

**Theorem 4.1.** Assume that (**H1**)-(**H3**) hold. Let $\varphi \in \mathscr{C}_\alpha^1$ such that $\varphi^{'}(0) = -A\varphi(0) + F(0,\varphi)$ with $\varphi(0) \in D(A)$. Then the mild solution of equation (1.2) is a strict solution.

**Proof.** Let $a>0$ and $u = u(\cdot,\varphi)$ be the mild solution of equation (1.2) which is defined in the intervalle $[0,+\infty[$. Consider now the equation

$$(4.1)\quad \begin{cases} v(t) &= T(t)\varphi'(0) + \int_0^t T(t-s)\int_0^s \big[D_t g\big(s-\tau,u(\tau)\big) + D_\varphi g\big(s-\tau,u(\tau)\big)v(\tau)\big]\,d\tau ds \\ &\quad + \int_0^t T(t-s)\big[D_t F(s,u_s) + D_\varphi F(s,u_s)v_s\big]ds + \int_0^t T(t-s)g(0,u(s))ds \\ v_0 &= \varphi^{'}. \end{cases}$$

Using the strict contraction principle, we can show that there exists an unique continuous function $v$ solution in $[0,a]$ of equation (4.1). We introduce the function $w$ defined by:

$$w(t) = \varphi(0) + \int_0^t v(s)ds \quad \text{if } t \geq 0 \quad \text{and} \quad w(t) = \varphi(t) \quad \text{if } -r \leq t \leq 0.$$

It follows that

$$w_t = \varphi + \int_0^t v_s\,ds \qquad \text{for } t \in [0,a].$$

Then the maps $t \mapsto w_t$, $t \mapsto \int_0^t T(t-s)F(s,w_s)ds$ and $t \mapsto \int_0^t T(t-s)\int_0^s g\big(s-\tau,w(\tau)\big)d\tau ds$ are continuously differentiables and the following formula hold:

$$\frac{d}{dt}\int_0^t T(t-s)F(s,w_s)ds = T(t)F(0,\varphi) + \int_0^t T(t-s)\left[D_t F(s,w_s) + D_\varphi F(s,w_s)v_s\right]ds$$

and

$$\frac{d}{dt}\int_0^t T(t-s)\int_0^s g\big(s-\tau,w(\tau)\big)d\tau ds = \int_0^t T(t-s)g(0,w(s))ds$$
$$+\int_0^t T(t-s)\int_0^s \big[D_t g\big(s-\tau,w(\tau)\big)+D_\varphi g\big(s-\tau,w(\tau)\big)v(\tau)\big]\,d\tau ds.$$

Then

$$\begin{aligned}\int_0^t T(s)F(0,\varphi)ds &= \int_0^t T(t-s)F(s,w_s)ds\\ &\quad -\int_0^t\int_0^s T(s-\tau)\big[D_tF(\tau,w_\tau)+D_\varphi F(\tau,w_\tau)v_\tau\big]d\tau ds\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}\int_0^t\int_0^s T(s-\tau)g(0,w(\tau))d\tau ds &= \int_0^t T(t-s)\int_0^s g\big(s-\tau,w(\tau)\big)d\tau ds\\ &\quad -\int_0^t\int_0^s T(s-\tau)\int_0^\tau \big[D_t g\big(\tau-y,w(y)\big)+D_\varphi g\big(\tau-y,w(y)\big)v(y)\big]\,dyd\tau ds.\end{aligned}\tag{4.3}$$

It follows that

$$\begin{aligned}w(t) &= T(t)\varphi(0)+\int_0^t T(t-s)F(s,w_s)ds+\int_0^t T(t-s)\int_0^s g\big(s-\tau,w(\tau)\big)d\tau ds\\
&+\int_0^t\int_0^s T(s-\tau)\int_0^\tau \big[D_t g\big(\tau-y,u(y)\big)-D_t g\big(\tau-y,w(y)\big)\big]\,dyd\tau ds\\
&+\int_0^\tau \big[D_\varphi g\big(\tau-y,u(y)\big)v(y)-D_\varphi g\big(\tau-y,w(y)\big)v(y)\big]\,dyd\tau ds\\
&+\int_0^t\int_0^s T(s-\tau)\big[D_tF(\tau,u_\tau)-D_tF(\tau,w_\tau)\big]d\tau ds\\
&+\int_0^t\int_0^s T(s-\tau)\big[D_\varphi F(\tau,u_\tau)v_\tau-D_\varphi F(\tau,w_\tau)v_\tau\big]d\tau ds.\end{aligned}$$

We deduce, for $t\in[0,a]$, that

$$\begin{aligned}\|w(t)-u(t)\|_\alpha &\le \int_0^t \left\|A^\alpha T(t-s)\int_0^s \Big(g\big(s-\tau,u(\tau)\big)-g\big(s-\tau,w(\tau)\big)\Big)d\tau\right\|ds\\
&+\int_0^t \left\|A^\alpha T(t-s)\big(F(s,u_s)-F(s,w_s)\big)\right\|ds\\
&+\int_0^t\int_0^s \left\|A^\alpha T(s-\tau)\int_0^\tau D_t g\big(\tau-y,u(y)\big)-D_t g\big(\tau-y,w(y)\big)dy\right\|d\tau ds\\
&+\int_0^t\int_0^s \left\|A^\alpha T(s-\tau)\int_0^\tau \big[D_\varphi g\big(\tau-y,u(y)\big)-D_\varphi g\big(\tau-y,w(y)\big)\big]\,v(y)dy\right\|d\tau ds\end{aligned}$$

$$+\int_0^t\int_0^s \|A^{\alpha}T(s-\tau)\| \, \|D_t f(\tau,u_\tau)-D_t f(\tau,w_\tau)\| \, d\tau ds$$
$$+\int_0^t\int_0^s \|A^{\alpha}T(s-\tau)\| \, \left\| \left[D_\varphi f(\tau,u_\tau)-D_\varphi f(\tau,w_\tau)\right] v_\tau \right\| d\tau ds.$$

The set $K=\{u_s,w_s;s\in[0,a]\}$ is compact in $\mathscr{C}_\alpha$. Since the partial derivatives of $F$ and $g$ are locally Lipschitz with respect to the second argument, it is well-known that they are globally Lipschitz on $K$. Then we deduce that

$$\begin{aligned}
\|w(t)-u(t)\|_\alpha \le{}& M_\alpha a L_G \int_0^t \frac{e^{-w(t-s)}}{(t-s)^\alpha} \sup_{0\le\tau\le s}\|w(\tau)-u(\tau)\| \, ds \\
&+M_\alpha L_F \int_0^t \frac{e^{-w(t-s)}}{(t-s)^\alpha} \sup_{0\le\tau\le s}\|w(\tau)-u(\tau)\| \, ds \\
&+M_\alpha a Lip(D_t g)\int_0^t\int_0^s \frac{e^{-w(s-\tau)}}{(s-\tau)^\alpha} \sup_{0\le y\le\tau}\|u(y)-w(y)\| \, d\tau ds \\
&+M_\alpha a Lip(D_\varphi g)\int_0^t\int_0^s \frac{e^{-w(s-\tau)}}{(s-\tau)^\alpha} \sup_{0\le y\le\tau}\|u(y)-w(y)\| \, d\tau ds \\
&+M_\alpha Lip(D_t F)\int_0^t\int_0^s \frac{e^{-w(s-\tau)}}{(s-\tau)^\alpha} \sup_{0\le y\le\tau}\|u(y)-w(y)\| \, d\tau ds \\
&+M_\alpha Lip(D_\varphi F)\int_0^t\int_0^s \frac{e^{-w(s-\tau)}}{(s-\tau)^\alpha} \sup_{0\le y\le\tau}\|u(y)-w(y)\| \, d\tau ds.
\end{aligned}$$

Here $Lip\left(D_\varphi g\right)$, $Lip(D_t g)$, $Lip\left(D_t F\right))$ and $Lip\left(D_\varphi F\right)$ are respectively the Lipschitz constant of $D_\varphi g$, $D_t g$, $D_t g$ and $D_\varphi F$. Since the map

$$s \longrightarrow \int_0^s \frac{e^{-w(s-\tau)}}{(s-\tau)^\alpha} \sup_{0\le y\le\tau}\|u(y)-w(y)\| \, d\tau$$

is a nondecresing function, then we deduce that

$$\|w(t)-u(t)\|_\alpha \le M_\alpha \beta(a)\int_0^a \frac{e^{-w(t-s)}}{(t-s)^\alpha} \mathop{sup}_{0\le\tau\le a}\|u(\tau)-w(\tau)\|_\alpha ds,$$

where

$$\beta(a)=aL_g+L_F+a^2\left(Lip(D_t g)+Lip\left(D_\varphi g\right)\right)+Lip(D_t F)+Lip\left(D_\varphi F\right).$$

Then it follows that

$$\|u-w\|_{\mathscr{C}_\alpha} \le \left(M_\alpha\beta(a)\int_0^a \frac{e^{-ws}}{s^\alpha}ds\right)\|u-w\|_{\mathscr{C}_\alpha}.$$

Now we choose $a$ such that

$$M_\alpha \beta(a) \int_0^a \frac{e^{-ws}}{s^\alpha} ds < 1,$$

then $u = w$ in $[0, a]$. We claim that $u = w$ in $[0, +\infty[$. Indeed, suppose that there exists $t_1 > 0$ such that $u(t_1) \neq w(t_1)$. Let $t_2 = \inf\{t > 0 : \|u(t) - w(t)\|_\alpha > 0\}$. By continuity, we has $u(t) = w(t)$ for $t \leq t_2$ and there exists $\varepsilon > 0$ such that $\|u(t) - w(t)\|_\alpha > 0$ for $t \in ]t_2, t_2 + \varepsilon[$. On the other hand

$$\|u(t) - w(t)\|_\alpha \leq M_\alpha \beta(\varepsilon) \int_0^\varepsilon \frac{e^{-ws}}{s^\alpha} \sup_{\varepsilon \leq \tau \leq t_1 + \varepsilon} \|u(\tau) - w(\tau)\|_\alpha ds.$$

Now we choose $\varepsilon$ such that

$$M_\alpha \beta(\varepsilon) \int_0^\varepsilon \frac{e^{-ws}}{s^\alpha} ds < 1.$$

Then $u = w$ in $[t_2, t_2 + \varepsilon]$ which gives a contradiction. Therefore, $u(t) = w(t)$ for $t \geq 0$. We conclude that $t \mapsto u_t$ from $[0, +\infty[$ to $\mathscr{D}(A^\alpha)$, $(t, s) \mapsto g\big(t - s, u(s)\big)$ from $\mathbb{R}^+ \times \mathbb{X}_\alpha$ to $\mathbb{X}_\alpha$ and $t \mapsto F(t, u_t)$ from $\mathbb{R}^+ \times \mathscr{C}_\alpha$ to $\mathbb{X}$ are continiusly differentiables. Then $u$ is a stricte solution $\square$

## 5. Application

We consider the following system for illustration

$$(5.1) \qquad \begin{cases} \dfrac{\partial}{\partial t} w(t, x) = \dfrac{\partial^2}{\partial x^2} w(t, x) + \displaystyle\int_0^t g(t - s, w(s, x)) ds \\ \qquad\qquad + \displaystyle\int_{-r}^0 k(t, \frac{\partial}{\partial x} w(t + \theta, x)) d\theta \quad \text{for } t \geq 0 \text{ and } x \in [0, \pi], \\ w(t, 0) = w(t, \pi) = 0 \quad \text{for } t \geq 0, \\ w(\theta, x) = w_0(\theta, x) \quad \text{for } \theta \in [-r, 0] \text{ and } x \in [0, \pi], \end{cases}$$

where $w_0 : [-r, 0] \times [0, \pi] \longrightarrow \mathbb{R}$, $k : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ and $h : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are appropriates functions. To study this equation, we choose $\mathbb{X} = L^2([0, \pi])$, with with its usual norm $\|.\|$. We define the operator $A : \mathbb{Y} = D(A) \subset \mathbb{X} \to \mathbb{X}$ by

$$Aw = -w'' \quad \text{with domain } D(A) = H^2(0, \pi) \cap H_0^1(0, \pi).$$

For $\alpha = 1/2$, we define $\mathbb{X}_{1/2} = \big(D(A^{1/2}), |\cdot|_{1/2}\big)$ where $|x|_{1/2} = \|A^{1/2}x\|$ for each $x \in \mathbb{X}_{1/2}$. We define $\mathscr{C}_{1/2}$= $C([-r,0],\mathbb{X}_{1/2})$ equipped with norm $|\cdot|_\infty$ and the functions, $u$ and $\varphi$ and $F$ by $u(t) = w(t,x)$, $\varphi(\theta)(x) = w_0(\theta,x)$ for a.e $x \in [0,\pi]$ and $\theta \in [-r,0]$, $t \geq 0$ and finaly

$$F(t,\varphi)(x) = \int_{-r}^{0} k(t,\frac{\partial}{\partial x}\varphi(\theta)(x))d\theta \quad \text{for a.e } x \in [0,\pi] \quad \text{and } \varphi \in \mathscr{C}_{1/2}.$$

Then the equation (5.1) takes the abstract form

$$(5.2) \qquad \begin{cases} u'(t) = -Au(t) + \displaystyle\int_0^t g(t-s,u(s))ds + F(t,u_t) & \text{for} \quad t \geq 0, \\ u_0 = \varphi \in \mathscr{C}_{1/2} = C\big([-r,0],X_{1/2}]\big). \end{cases}$$

The operator $-A$ is closed operator and generates an analytic compact semigroup $(T(t))_{t\geq 0}$ on $\mathbb{X}$. Thus, there exists $\delta$ in $(0,\pi/2)$ and $M \geq 0$ such that

$$\Lambda = \Big\{\lambda \in \mathbb{C} : |arg\lambda| < \frac{\pi}{2} + \delta\Big\} \cup \{0\}$$

is contained in $\rho(-A)$, the resolvent set of $-A$ and $\|R(\lambda,-A)\| < M/|\lambda|$ for $\lambda \in \Lambda$. The operator $A$ has a discrete spectrum, the eigenvalues are $n^2$ and the corresponding normalized eigenvectors are

$$e_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx), n = 1,2,\cdots.$$

Moreover the following formula hold.

**(i)** $Au = \displaystyle\sum_{n=1}^{\infty} n^2\langle u,e_n\rangle e_n \quad u \in D(A)$,

**(ii)** $A^{-1/2}u = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\langle u,e_n\rangle e_n \quad$ for $u \in \mathbb{X}$,

**(iii)** $A^{1/2}u = \displaystyle\sum_{n=1}^{\infty} n\langle u,e_n\rangle e_n \quad$ for $u \in D(A^{1/2}) = \{u \in \mathbb{X} : \sum_{n=1}^{\infty}\frac{1}{n}\langle u,e_n\rangle e_n \in \mathbb{X}\}$.

One also have the following result.

**Lemma 5.1.** [20] Let $\varphi \in \mathbb{X}_{1/2}$. Then $\varphi$ is absolutely continuous, $\varphi' \in \mathbb{X}$ and

$$\|\varphi'\| = \|A^{\frac{1}{2}}\varphi\|.$$

We assume the following assumptions.

**(H4)** The functions $k : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous and Lipschitz with respect to the second variable.

The functions $F$ is continuous in the first variable from the fact that $k$ is continuous in the first variable. Moreover from Lemma 5.1 and the continuity of $k$, we deduce that $F$ is continuous with respect to the second argument. This yields the continuity of $F$ in $\mathbb{R}_+ \times \mathscr{C}_{1/2}$. In addition, by assumption **(H4)** we deduce that

$$\|F(t,\varphi_1) - F(t,\varphi_2)\| \leq rL_f \|\varphi_1 - \varphi_2\|_{\mathscr{C}_{1/2}}.$$

Where $L_f$ is the Lipchitz constante of $F$. Then $F$ is a continuous globally Lipschitz function with respect to the second argument. We obtain the following important result.

**Proposition 5.1.** Suppose that the assumptions **(H4)** hold. Then the equation (5.2) has a mild solution wich is defined for $t \geq 0$.

## Conflict of Interests

The author(s) declare that there is no conflict of interests.

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