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COMMON FIXED POINT THEOREMS FOR FINITE FAMILY OF MAPPINGS INVOLVING CONTRACTIVE CONDITIONS OF RATIONAL TYPE IN DISLOCATED QUASI-METRIC SPACES

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Abstract. We proved the existence of common fixed point theorems for finite family of self- mappings involving contractive conditions of Rational type in dislocated quasi metric spaces by extending and generalizing some results in the literature. We also give some examples that support our results in this particular work.

Keywords: fixed point; dislocated quasi-metric space; rational type.

2020 Mathematics Subject Classification. 47H10, 54H25.

1. Introduction

The purpose of this paper is to prove the existence and the uniqueness of common fixed point of f inite family of compatible mappings in dislocated quasi metric spaces introduced by Wilson [18] as a generalization of metric spaces and also to provide some supporting examples to our main result.

Hitzler and Seda [2] introduced the concept of complete dislocated quasi metric space. They also generalized the Banach contraction principle [1] in dislocated metric space. Furthermore

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Zeyada et al. [4] introduced the notion of complete dislocated quasi metric space and established fixed point theorems by generalizing the results of Hitzler and Seda in the same space. Later on many papers of different authors have been published containing fixed point results for different type of contraction in the same space. Aage and Salunke([11] and [12]respectively) derived fixed point theorem in dislocated quasi-metric spaces. Sarwar et al.[21] established some fixed point results for single and a pair of continuous self-mappings in the context of dislocated quasi metric spaces which generalize, modify and unify the result of Aage and Salunke in 2014. Isufati [13] proved some fixed point theorem for continuous contractive condition with rational type expression in the context of dislocated quasi metric spaces. After these theorems in the literature, several authors have generalized and extended the fixed point results in various spaces for different types of contractive conditions and mappings in dislocated metric spaces. Rahman and Sarwar [3] obtained a unique fixed point result for a complete dislocated quasi-metric space in 2016. Yeshimabet Jira,Kidane Koyas and Aynalem Girma[5], proved fixed point result in the setting of dislocated quasi-metric spaces for a pair of self-mappings which generalize the result of Rahman and Sarwarin 2018.

In this paper, we generalize the following important results of Yeshimabet Jira,Kidane Koyas and Aynalem Girma[5] to the finite family of contractive mappings in dislocated quasi-metric spaces.

Theorem 1.1. [5] *Let (X,d) be a complete dislocated quasi metric spaces and $T,f : X \longrightarrow X$ be self-maps satisfying the following condition*
$i)$ $TX \subseteq X$
$ii)$ T and f are weakly compatible and fX is closed subset of X
$iii)$ $$d(Tx,Ty) \leq a\varphi\left(d(fx,fy)\right) + b\varphi\left(max\{d(fx,fy),d(fx,Tx)\}\right) + \frac{c\varphi\left(d(fx,fy)\left[1+\sqrt{d(fx,fy)d(fx,Tx)}\right]^2\right)}{(1+d(fx,fy))^2}$$
for all $x,y \in X$ and $a,b,c \geq 0$ with $a+b+c < 1$ and φ is a comparison. Then T and$f$ have a unique common fixed point if $T$and f commute at their coincidence points.

Remark 1.2. [5] For $f = I$ (I is identity on X) form of contractive condition of Theorem1.1, we get

$d(Tx,Ty) \leq a\varphi\left(d(x,y)\right) + b\varphi\left(max\{d(x,y),d(x,Tx)\}\right) + \frac{c\varphi\left(d(x,y)\left[1+\sqrt{d(x,y)d(x,Tx)}\right]^2\right)}{(1+d(x,y))^2}$,
whenever, $f = I$ contractive condition of Theorem 3.1

Theorem 1.3. [5] *Let (X,d) be a complete dq-metric space and let $T, f : X \longrightarrow X$ be continuous self mappings satisfying the contractive condition of Theorem 1.1. Then f and T have a unique common fixed point.*

Corollary 1.4. [5] [5] *Let (X,d) be a complete dislocated quasi-metric space. Let $T : X \longrightarrow X$ be a self mapping satisfying*
$d(Tx,Ty) \leq a\gamma(d(x,y)) + b\gamma(max\{d(x,y),d(x,Tx)\}) + c\gamma\left(d(x,y)\frac{\left(1+\sqrt{d(x,y)d(x,Tx)}\right)^2}{(1+d(x,y))^2}\right)$,
for all $x,y \in X, a,b,c \geq 0$ with $a+b+c < 1$ and γ is a comparison function. Then T has a unique fixed point.

In support of the following definitions we are motivated to generalize Theorem 1.1 for finite family of self-mappings.
Let the set of coincidence point $C(T_1T_2...T_{n-1},T_n)$ and the set of common fixed points $F(T_1T_2...T_{n-1},T_n)$ of finite family of self-maps $T_1,T_2,T_3,...,T_n$ respectively are denoted by $\{x \in X : T_1T_2...T_{n-1}x = T_nx\}$ and $\{x \in X : T_1T_2...T_{n-1}x = T_nx = x\}$. Then in the sequel we need to have the following definitions.

Definition 1.5. Finite family of self-maps $T_1,T_2,T_3,...,T_n$ on a nonempty set X are said to be commuting each other if $T_1T_2x = T_2T_1x, ..., T_1T_nx = T_nT_1x, T_2T_3x = T_3T_2x, ..., T_{n-1}T_n = T_nT_{n-1}x$ for all $x \in X$. That is self-maps $T_1,T_2,T_3,...,T_n$ on a non-empty set X are said to be commuting each other if

$$(T_1T_2...T_{n-1})T_nx = T_n(T_1T_2...T_{n-1}x) \text{ for all } x \in X.$$

Definition 1.6. Finite family of self-maps $T_1,T_2,T_3,...,T_n$ of a metric space (X,d) are called compatible if

$$\lim_{j\longrightarrow\infty} d(T_nT_1T_2...T_{n-1}x_j, T_1T_2...T_{n-1}T_nx_j) = 0,$$

whenever $\{x_j\}_{j=1}^{\infty}$ is a sequence in X such that

$$\lim_{j\longrightarrow\infty} T_nx_j = \lim_{j\longrightarrow\infty} T_1T_2...T_{n-1}x_j = t \text{ for some } t \in X.$$

Definition 1.7. Finite family of self-maps $T_1, T_2, T_3, ..., T_n$ of a metric space (X,d) are called weakly compatible if they commute each other at their coincidence points.
That is, $T_n u = T_1 T_2 ... T_{n-1} u$ for $u \in X$, then $T_1 T_2 ... T_{n-1} T_n u = T_n T_1 T_2 ... T_{n-1} u$ for u $\in X$.

Inspired with this, we establish common fixed point theorems for finite family of self-mappings and show the existence and uniqueness of common fixed point in dislocated quasi-metric spaces involving contractive contraction of rational type by extending Theorem 1.1

2. PRELIMINARIES

Now, we begin with some definitions and examples that support our definitions.

Definition 2.1. Let X be a nonempty and $d : X \times X \to [0, \infty)$ be a function, called a distance function, satisfies:
d_1: $d(x,x) = 0$,
$d_2 : d(x,y) = d(y,x) = 0$, then $x = y$,
$d_3 : d(x,y) = d(y,x)$
$d_4 : d(x,y) \leq d(x,z) + d(z,y)$, $\forall x, y, z \in X$.
Then, if d satisfies the condition $d_1 to d_4$, then d is called a metric on X. If it satisfies the conditions d_1, d_2 d_4, then it is called a quasi-metric space. If d satisfies conditionsd_2, d_3, d_4 then d is called a dislocated metric and if it satisfies only d_2 *and* d_4 then d is called a dislocated quasi-metric on X. Non empty set X together with metric d on X is called metric space and it is given by (X,d) and with $dq-metric\ d$, i.e. (X,d) is called a dislocated quasi-metric space

Definition 2.2. [4] Let (X,d) be a metric space and $T : X \longrightarrow X$ be a self - map. Then T is said to be a contraction mapping if there exists a constant $k \in [0,1)$, such that $d(Tx,Ty) \leq kd(x,y)$,$\forall x, y \in X$

Definition 2.3. [4] Let (X,d) be a metric space. Then, the mapping T: $X \longrightarrow X$ is said to be contractive mapping if $d(Tx,Ty) < d(x,y)$, $\forall x, y \in X$ with $x \neq y$.

Definition 2.4. [4] Let (X,d) be a dislocated quasi-metric space. A mapping $T : X \longrightarrow X$ is called contraction if there exists a constant $k \in [0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x, y \in X$.

Lemma 2.5. [4] *Limit of a convergent sequence in a dq-metric space is unique.*

Theorem 2.6. [4] *Let (X,d) be a complete dislocated quasi-metric space and $T\colon X \longrightarrow X$ be a contraction. Then T has a unique fixed point in X.*

Definition 2.7. Two self-maps f and g of a non empty set X are said to be commuting if $fgx = gfx$ for all x in X .If $fx = gx$ for some x in X, then x is called coincidence point of f and g.

Definition 2.8. [9] Let (X,d) be a metric space. Then two self-mappings $f,g : X \longrightarrow X$ if $fx = gx = x$ are called weakly compatible if they commute at their coincidence points.

Definition 2.9. Let X be non empty set and $T : X \longrightarrow X$ be a self-map. For a given $x \in X$ we define $T_n(x)$ inductively by T_0 and we call $T_n(x)$ is the n^{th} iterate of x under T.

3. Main Results

In this section we study the existence and uniqueness of common fixed point for finite family of self-mappings and show it in dislocated quasi- metric spaces involving contractive contraction of rational type.

Theorem 3.1. *Let (X,d) be a complete dislocated quasi metric spaces and $T_1,T_2,...,T_n : X \longrightarrow X$ be finite family of self-mappings satisfying the following conditions.*
$i)$ $T_1X \subseteq T_2X \subseteq ... \subseteq T_nX$;
$ii)$ $T_1,T_2,...,T_{n-1}$ and T_n are weakly compatible and T_nX is closed subset of X;
$iii)$

$$d(T_1T_2...T_{n-1}x, T_1T_2...T_{n-1}y) \leq a\gamma(d(T_nx,T_ny)) + b\gamma(max\{d(T_nx,T_ny), d(T_nx,T_1T_2...T_{n-1}x)\}) + c\gamma\left(\frac{d(T_nx,T_ny)\left[1+\sqrt{d(T_nx,T_ny)d(T_nx,T_1T_2...T_{n-1}x)}\right]^2}{(1+d(T_nx,T_ny))^2}\right)$$

for all $x,y \in X$ and $a,b,c \geq 0$ with $a+b+c<1$, and γ is a comparison function. Then $T_1,T_2,...,T_{n-1}$ and T_n have a unique common fixed point if $T_1,T_2,...,T_{n-1}$ and T_n commute each other at their coincidence points.

Proof. Let $x_0 \in X$, so that $y_0 = T_1x_0 = T_2x_1 = ... = T_nx_{n-1}$.
By condition (i) we have that

$$T_1x_1 \in T_2X, T_2x_2 \in T_3X, ..., T_{n-1}x_{n-1} \in T_nX.$$

Then there exist $x_n \in X$ such that $y_1 = T_1x_1 = T_2x_2 = ... = T_nx_n$.

Continuing this process we construct a sequence $\{x_j\}$ and $\{y_j\}$ such that $y_j = T_1x_j = T_2x_{j+1} = ... = T_nx_{j+(n-1)}$ for $j \in \{0,1,2,...\}$.

Now considering two cases we have the following proof.

Case - i :

Suppose $y_j = y_{j+1} = ... = y_{j+(n-1)}$ for some $j \in \{0,1,2,...\}$. Then we have $y_j = T_1x_j = T_1x_{j+1} = ... = T_1x_{j+(n-1)} = y_{j+1} = T_2x_{j+1} = ... = T_2x_{j+(n-1)} = ... = y_{j+(n-1)} = T_nx_{j+(n-1)}$ of which $x_{j+(n-1)}$ is coincidence point of $T_1, T_2, ...T_n$.

Let $T_1x_j = T_1x_{j+1} = ... = T_1x_{j+(n-1)} = T_2x_{j+1} = ... = T_2x_{j+(n-1)} = ... = T_nx_{j+(n-1)} = w$, for some $w \in T_nX$. Then by the weakly compatibility of $T_1, T_2, ...T_n$ we get

$$\begin{aligned}
T_1w &= T_1T_2x_{j+(n-1)} = T_2T_1x_{j+(n-1)} \\
&= T_2w = T_2T_3x_{j+(n-1)} = T_3T_2x_{j+(n-1)} \\
&= T_3w = ... = T_{n-1}w = T_{n-1}T_nx_{j+(n-1)} \\
&= T_nT_{n-1}x_{j+(n-1)} = T_nw \qquad (3.1)
\end{aligned}$$

Therefore, $T_1T_2w = T_2T_1w = ... = T_{n-1}T_nw = T_nT_{n-1}w$ for $w \in X$ which gives $T_1, T_2, ..., T_n$ commute each other at their coincidence point w and by composition it gives that $T_1T_2...T_{n-1}w = T_nw$.

Therefore $(T_1T_2...T_{n-1})T_nw = T_n(T_1T_2...T_{n-1}w)$ for $w \in X$.

Claim -1: $d(T_1T_2...T_{n-1}w, T_1T_2...T_{n-1}w) = 0$

By using condition (iii) of Theorem 4.1 given above we have that

$$\begin{aligned}
d(T_1T_2...T_{n-1}w, T_1T_2...T_{n-1}w) &\le a\gamma(d(T_nw, T_nw)) \\
&+ b\gamma(max\{d(T_nw, T_nw), d(T_nw, T_1T_2...T_{n-1}w)\}) \\
&+ c\gamma\left(\frac{d(T_nw, T_nw)\left[1+\sqrt{d(T_nw, T_nw)d(T_nw, T_1T_2...T_{n-1}w)}\right]^2}{(1+d(T_nw, T_nw))^2}\right)
\end{aligned}$$

$$d(T_1w, T_1w) \le a\gamma d(T_nw, T_nw)) + b\gamma(max\{d(T_nw, T_nw), d(T_nw, T_1w)\})$$

$$
\begin{aligned}
&+c\gamma\left(\frac{d(T_nw,T_nw)\left[1+\sqrt{d(T_nw,T_nw)d(T_nw,T_1w)}\right]^2}{(1+d(T_nw,T_nw))^2}\right)\\
&=a\gamma(d(T_1w,T_1w))+b\gamma(max\{d(T_1w,T_1w),d(T_1w,T_1w)\})\\
&+c\gamma\left(\frac{d(T_1w,T_1w)\left(1+\sqrt{d(T_1w,T_1w)d(T_1w,T_1w)}\right)^2}{(1+(T_1w,T_1w))^2}\right).
\end{aligned}
$$

Since $\gamma(t)\le t$ for all $t\ge 0$, then we obtain

$d(T_1w,T_1w)\le ad(T_1w,T_1w)+bd(T_1w,T_1w)+cd(T_1w,T_1w)$

$d(T_1w,T_1w)\le (a+b+c)d(T_1w,T_1w)$.

Since $0\le a+b+c<1$, we have

$$
\begin{aligned}
d(T_1T_2...T_{n-1}w,T_1T_2...T_{n-1}w)\le\ & a\gamma(d(T_nw,T_nw))\\
&+b\gamma(max\{d(T_nw,T_nw),d(T_nw,T_1T_2...T_{n-1}w)\})\\
&+c\gamma\left(\frac{d(T_nw,T_nw)\left[1+\sqrt{d(T_nw,T_nw)d(T_nw,T_1T_2...T_{n-1}w)}\right]^2}{(1+d(T_nw,T_nw))^2}\right)
\end{aligned}
$$

is satisfied if

$$d(T_1T_2...T_{n-1}w,T_1T_2...T_{n-1}w)=0. \tag{3.2}$$

<u>Claim-2</u>: $T_1T_2...T_{n-1}w=w$

$$
\begin{aligned}
d(T_1T_2...T_{n-1}w,w)&=d(T_1T_2...T_{n-1}w,T_1T_2...T_{n-1}x_{j+(n-1)})=d(T_1w,T_1x_{j+(n-1)})\\
&\le a\gamma\left(d(T_nw,T_nx_{j+(n-1)})\right)\\
&\quad+b\gamma(max\left\{d(T_nw,T_1T_2...T_{n-1}w),d(T_nw,T_nx_{j+(n-1)})\right\})+\\
&\quad c\gamma\left(\frac{d\left(T_nw,T_nx_{j+(n-1)}\right)\left(1+\sqrt{d(T_nw,T_nx_{j+(n-1)})d(T_nw,T_1T_2...T_{n-1}w)}\right)^2}{\left(1+(T_nw,T_nx_{j+(n-1)})\right)^2}\right)\\
&=a\gamma\left(d(T_nw,T_nx_{j+(n-1)})\right)\\
&\quad+b\gamma(max\left\{d(T_nw,T_1w),d(T_nw,T_nx_{j+(n-1)})\right\})\\
&\quad c\gamma\left(\frac{d\left(T_nw,T_nx_{j+(n-1)}\right)\left(1+\sqrt{(d(T_nw,T_nx_{j+(n-1)})d(T_nw,T_1w))}\right)^2}{\left(1+(T_1w,T_nx_{j+(n-1)})\right)^2}\right)\\
&=a\gamma(d(T_1w,w))+b\gamma(max\{d(T_1w,T_1w),d(T_1w,w)\})\\
&\quad+c\gamma\left(\frac{d(T_1w,w)\left(1+\sqrt{(d(T_1w,w)d(T_1w,T_1w))}\right)^2}{(1+(T_1w,w))^2}\right);
\end{aligned}
$$

Since $d(T_1w,T_1w)=0$, then $(max\{d(T_1w,T_1w),d(T_1w,w)\})$ is $d(T_1w,w)$ and $\left(\frac{d(T_1w,w)\left(1+\sqrt{d(T_1w,w)d(T_1w,T_1w)}\right)^2}{(1+(T_1w,w))^2}\right)\le d(T_1w,w)$. Thus,

$$d(T_1w,w)\le a\gamma(d(T_1w,w))+b\gamma(d(T_1w,w))+c\gamma(d(T_1w,w)).$$

Since $\gamma(t) \leq t$ for all $t \geq 0$, then we obtain

$$d(T_1w,w) \leq ad(T_1w,w) + bd(T_1w,w) + cd(T_1w,w)$$
$$\leq (a+b+c)d(T_1w,w).$$

Since $0 \leq a+b+c < 1$, we get the following.

$$d(T_1T_2...T_{n-1}w,w) \leq a\gamma\left(d(T_nw,T_nx_{j+(n-1)})\right)$$
$$+ b\gamma(max\left\{d(T_nw,T_1T_2...T_{n-1}w), d(T_nw,T_nx_{j+(n-1)})\right\})$$
$$+ c\gamma\left(\frac{d\left(T_nw,T_nx_{j+(n-1)}\right)\left(1+\sqrt{d\left(T_nw,T_nx_{j+(n-1)}\right)d\left(T_nw,T_1T_2...T_{n-1}w\right)}\right)^2}{\left(1+\left(T_nw,T_nx_{j+(n-1)}\right)\right)^2}\right)$$

is possible if

$$d\left(T_1T_2...T_{n-1}w,w\right) = d(T_1w,w) = 0. \tag{3.3}$$

Similarly,

$$d(w,T_1T_2...T_{n-1}w) = d(T_1x_{j+(n-1)},T_1w) \leq a\gamma(d(T_nx_{j+(n-1)},T_nw))$$
$$+ b\gamma(max\left\{d(T_nx_{j+(n-1)},T_nw), d(T_nx_{j+(n-1)},T_1T_2...T_{n-1}x_{j+(n-1)})\right\})$$
$$+ c\gamma\left(d(T_nx_{j+(n-1)},T_nw)\frac{\left(1+\sqrt{d\left(T_nx_{j+(n-1)},T_nw\right)d\left(T_nx_{j+(n-1)},T_1T_2...T_{n-1}x_{j+(n-1)}\right)}\right)^2}{\left(1+d\left(T_nx_{j+(n-1)},T_nw\right)\right)^2}\right)$$
$$d(w,T_1T_2...T_{n-1}w) \leq a\gamma(d(w,T_1w)) + b\gamma(max\left\{d(w,T_1w), d(w,w)\right\})$$
$$+ c\gamma\left(\frac{d(w,T_1w)\left(1+\sqrt{d(w,T_1w)d(w,w)}\right)^2}{(1+(w,T_1w))^2}\right).$$

Since $d(w,w) = 0$, then $max\left\{d(w,T_1w), d(w,w)\right\}$ is $d(w,T_1w)$ and

$$d(w,T_1w)\frac{\left(1+\sqrt{d(w,T_1w)d(w,w)}\right)^2}{(1+(w,T_1w))^2} \leq d(w,T_1w).$$

Thus,

$$d(w,T_1w) \leq a\gamma(d(w,T_1w) + b\gamma(d(w,T_1w)) + c\gamma(d(w,T_1w))$$

Since $\gamma(t) \leq t$ for all $t \geq 0$, then we obtain

$$d(w,T_1w) \leq ad(w,T_1w) + bd(w,T_1w) + cd(w,T_1w)$$
$$\leq (a+b+c)d(w,T_1w)$$

Since $0 \leq a+b+c < 1$, so

$$d(w,T_1T_2...T_{n-1}w) \leq a\gamma(d(T_nx_{j+(n-1)},T_nw))$$
$$+ b\gamma(max\left\{d(T_nx_{j+(n-1)},T_nw), d(T_nx_{j+(n-1)},T_1T_2...T_{n-1}x_{j+(n-1)})\right\})$$
$$+ c\gamma\left(d(T_nx_{j+(n-1)},T_nw)\frac{\left(1+\sqrt{d\left(T_nx_{j+(n-1)},T_nw\right)d\left(T_nx_{j+(n-1)},T_1T_2...T_{n-1}x_{j+(n-1)}\right)}\right)^2}{\left(1+d\left(T_nx_{j+(n-1)},T_nw\right)\right)^2}\right).$$

is possible if

$$d(w, T_1T_2...T_{n-1}w) = 0. \tag{3.4}$$

From (3.3) and (3.4) we obtain $T_1T_2...T_{n-1}w = w$.

By (3.1), we obtain $T_1w = T_2w = ... = T_nw = w$. Therefore, w is a common fixed point of $T_1, T_2, ..., T_n$.

Next we show the uniqueness of w. Suppose w and z are two distinct fixed points of $T_1, T_2, ..., T_n$. That means,

$$T_1w = T_2w = ... = T_nw = w \text{ and } T_1z = T_2z = ... = T_nz = z.$$

Then by condition (iii) of the above Theorem 3.1, we have the following.

$$\begin{aligned}
d(w,z) &= d(T_1T_2...T_{n-1}w, T_1T_2...T_{n-1}z) \\
&\leq a\gamma(d(T_nw, T_nz)) \\
&+ b\gamma(max\{d(T_nw, T_nz), d(T_nw, T_1T_2...T_{n-1}w)\}) \\
&+ c\gamma\left(d(T_nw, T_nz)\frac{(1+\sqrt{d(T_nw, T_nz)d(T_nw, T_1T_2...T_{n-1}w)})^2}{(1+d(T_nw, T_nz))^2}\right) \\
&\leq a\gamma(d(w,z)) + b\gamma(max\{d(w,z), d(w,w)\}) \\
&+ c\gamma\left(d(w,z)\frac{\left(1+\sqrt{(d(w,z)d(w,w))}\right)^2}{(1+d(w,z))^2}\right).
\end{aligned}$$

Since $d(w,w) = 0$ then $max\{d(w,z), d(w,w)\}$ is $d(w,z)$ and
$d(w,z)\frac{\left(1+\sqrt{d(w,z)d(w,w)}\right)^2}{(1+d(w,z))^2} \leq d(w,z)$.

Thus,

$$d(w,z) \leq a\gamma(d(w,z)) + b\gamma(d(w,z)) + c\gamma d((w,z)).$$

Since $\gamma(t) \leq t$ for all $t \geq 0$, then we obtain

$$\begin{aligned}
d(w,z) &\leq ad(w,z) + bd(w,z) + cd(w,z) \\
&\leq (a+b+c)d(w,z).
\end{aligned}$$

Since $0 \leq a+b+c<1$,

$$d(w,z) \leq a\gamma(d(T_nw,T_nz)) + b\gamma(max\{d(T_nw,T_nz), d(T_nw,T_1T_2...T_{n-1}w)\}) + c\gamma\left(d(T_nw,T_nz)\frac{\left(1+\sqrt{d(T_nw,T_nz)d(T_nw,T_1T_2...T_n-1w)}\right)^2}{(1+d(T_nw,T_nz))^2}\right)$$

is possible if

$$d(w,z)=0. \tag{3.5}$$

Similarly,

$$\begin{aligned} d(z,w) = d(T_1T_2...T_{n-1}z, T_1T_2...T_{n-1}w) &\leq a\gamma(d(T_nz,T_nw)) \\ &+ b\gamma(max\{d(T_nz,T_nw), d(T_nz,T_1T_2...T_{n-1}z)\}) \\ &+ c\gamma\left(d(T_nz,T_nw)\frac{\left(1+\sqrt{d(T_nz,T_nw)d(T_nz,T_1T_2...T_{n-1}z)}\right)^2}{(1+d(T_nz,T_nw))^2}\right) \\ &= a\gamma(d(z,w)) + b\gamma(maxd(z,w), d(z,z)) \\ &+ c\gamma\left(d(z,w)\frac{\left(1+\sqrt{d(z,w)d(z,z)}\right)^2}{(1+d(z,w))^2}\right) \end{aligned}$$

Since $d(z,z)=0$, then $max\{d(z,w),d(z,z)\}$ is $d(z,w)$ and $d(z,w)\frac{\left(1+\sqrt{(d(z,w)d(z,z))}\right)^2}{(1+(z,w))^2} \leq d(z,w)$.

Thus,

$$d(z,w) \leq ad(z,w) + bd(z,w) + cd(z,w) \leq (a+b+c)d(z,w).$$

Since $0 \leq a+b+c<1$, so we obtain the following.

$$\begin{aligned} d(z,w)) \leq a\gamma(d(T_nz,T_nw)) & \\ &+ b\gamma(max\{d(T_nz,T_nw), d(T_nz,T_1T_2...T_{n-1}z)\}) \\ &+ c\gamma\left(d(T_nz,T_nw)\frac{\left(1+\sqrt{d(T_nz,T_nw)d(T_nz,T_1T_2...T_{n-1}z)}\right)^2}{(1+d(T_nz,T_nw))^2}\right) \end{aligned}$$

is possible if

$$d(z,w)=0. \tag{3.6}$$

From (3.5) and (3.6), we have that $w=z$. Hence, w is a unique common fixed point of $T_1,T_2,...,T_n$.

Case- ii : Suppose $y_j \neq y_{j+1} \neq y_{j+2} \neq ... \neq y_{j+(n-1)}$ for each $j \in \{0,1,2,...\}$. Then,

$$\begin{aligned} d\left(y_j,y_{j+1}\right) &= d\left(T_1x_j,T_1x_{j+1}\right) \\ &= d\left(T_1x_{j+1},T_1x_{j+2}\right) = ... = d\left(T_1x_{j+(n-2)},T_1x_{j+(n-1)}\right) \end{aligned}$$

$$
\begin{aligned}
&= d\left(y_{j+1}, y_{j+2}\right) = d\left(T_2 x_{j+1}, T_2 x_{j+2}\right) \\
&= \ldots = d\left(T_2 x_{j+(n-2)}, T_2 x_{j+(n-1)}\right) = \ldots = d\left(y_{j+(n-2)}, y_{j+(n-1)}\right) \\
&= d\left(T_{n-1} x_{j+(n-2)}, T_{n-1} x_{j+(n-1)}\right). \qquad (3.7)
\end{aligned}
$$

$d\left(T_1T_2...T_{n-1}x_{j+(n-2)}, T_1T_2...T_{n-1}x_{j+(n-1)}\right) \leq a\gamma\left(d\left(T_n x_{j+(n-2)}, T_n x_{j+(n-1)}\right)\right)$
$+ b\gamma\left(max\left\{d\left(\left(T_n x_{j+(n-2)}, T_n x_{j+(n-1)}\right)\right), d\left(T_n x_{j+(n-2)}, T_1T_2...T_{n-1}x_{j+(n-2)}\right)\right\}\right)$
$+ c\gamma\left(d\left(T_n x_{j+(n-2)}, T_n x_{j+(n-1)}\right) \frac{\left(1+\sqrt{d\left(T_n x_{j+(n-2)}, T_n x_{j+(n-1)}\right) d\left(T_n x_{j+(n-2)}, T_1T_2...T_{n-1}x_{j+(n-2)}\right)}\right)^2}{\left(1+d\left(T_n x_{j+(n-2)}, T_n x_{j+(n-1)}\right)\right)^2}\right)$
From (3.7) we have,
$d\left(T_1T_2...T_{n-1}x_{j+(n-2)}, T_1T_2...T_{n-1}x_{j+(n-1)}\right) = d\left(T_1 x_{j+(n-2)}, T_1 x_{j+(n-1)}\right)$
Then,
$d\left(y_{j+(n-2)}, y_{j+(n-1)}\right) = d\left(T_1 x_{j+(n-2)}, T_1 x_{j+(n-1)}\right)$
$d\left(T_1 x_{j+(n-2)}, T_1 x_{j+(n-1)}\right) \leq a\gamma\left(d\left(T_n x_{j+(n-2)}, T_n x_{j+(n-1)}\right)\right) +$
$b\gamma\left(max\left\{d\left(T_n x_{j+(n-2)}, T_n x_{j+(n-1)}\right), d\left(T_n x_{j+(n-2)}, T_1T_2...T_{n-1}x_{j+(n-2)}\right)\right\}\right)$
$+ c\gamma\left(d\left(T_n x_{j+(n-2)}, T_n x_{j+(n-1)}\right) \frac{\left(1+\sqrt{d\left(T_n x_{j+(n-2)}, T_n x_{j+(n-1)}\right) d\left(T_n x_{j+(n-2)}, T_1T_2...T_{n-1}x_{j+(n-2)}\right)}\right)^2}{\left(1+d\left(T_n x_{j+(n-2)}, T_n x_{j+(n-1)}\right)\right)^2}\right)$

Also we have
$d\left(y_{j+(n-2)}, y_{j+(n-1)}\right) = d\left(T_1 x_{j+(n-2)}, T_1 x_{j+(n-1)}\right) = d(y_j, y_{j+1}) = d(T_1 x_j, T_1 x_{j+1})$ and

$$
\begin{aligned}
d\left(y_{j+(n-2)}, y_{j+(n-1)}\right) = d\left(y_j, y_{j+1}\right) &\leq a\gamma\left(d(y_{j-1}, y_j)\right) \\
&+ b\gamma\left(max\left\{d(y_{j-1}, y_j), d(y_{j-1}, y_j)\right\}\right) \\
&+ c\gamma\left(d(y_{j-1}, y_j)\frac{\left(1+\sqrt{d(y_{j-1}, y_j) d(y_{j-1}, y_j)}\right)^2}{\left(1+d(y_{j-1}, y_j)\right)^2}\right)
\end{aligned}
$$

Since $\gamma(t) \leq t$ for all $t \geq 0$, then we obtain

$$
\begin{aligned}
d(y_j, y_{j+1}) &\leq ad(y_{j-1}, y_j) + bd(y_{j-1}, y_j) + cd(y_{j-1}, y_j) \\
&\leq (a+b+c)d(y_{j-1}, y_j);
\end{aligned}
$$

Let $p = a+b+c$. Then,

$$
d(y_j, y_{j+1}) \leq pd(y_{j-1}, y_j). \qquad (3.8)
$$

Since $0 \leq p < 1$, we obtain $d(y_{j-1}, y_j) \leq pd(y_{j-2}, y_{j-1})$.

Then,

$$
\begin{aligned}
d(y_j, y_{j+1}) &\leq p d(y_{j-1}, y_j) \\
&\leq p(p d(y_{j-2}, y_{j-1})) = p^2 d(y_{j-2}, y_{j-1}) \\
&\leq p^2 d(y_{j-2}, y_{j-1}).
\end{aligned}
$$

If we continue this process, we get that $d(y_j, y_{j+1}) \leq p^j d(y_0, y_1)$.

Since $0 \leq p < 1$, we have $\lim\limits_{j \longrightarrow \infty} p^j d(y_0, y_1) = 0$.

Thus,

$$\lim_{j \longrightarrow \infty} d(y_j, y_{j+1}) = 0. \tag{3.9}$$

Similarly, we can easily show that,

$$\lim_{j \longrightarrow \infty} d(y_{j+1}, y_j) = 0. \tag{3.10}$$

Now we show that $\{y_j\}$is a Cauchy sequence in X. Let $m, j \in \mathbb{N}$ with $m > j$, applying triangular inequality

$$
\begin{aligned}
d(y_j, y_m) &\leq d(y_j, y_{j+1}) + d(y_{j+1}, y_m) \\
&\leq d(y_j, y_{j+1}) + d(y_{j+1}, y_{j+2}) + \ldots + d(y_{m-1}, y_m) \\
&\leq p^j d(y_0, y_1) + p^{j+1} d(y_0, y_1) + \ldots + p^{m-1} d(y_0, y_1) \\
&\leq p^j \left(1 + p + \ldots + p^{m-j-1}\right) d(y_0, y_1) = p^j \left(\textstyle\sum_{i=0}^{m-j-1} p^i\right) d(y_0, y_1) \\
&\leq p^j \left(\textstyle\sum_{i=0}^{\infty} p^i\right) d(y_0, y_1) = \tfrac{p^j}{1-p} d(y_0, y_1) \\
&\leq \tfrac{p^j}{1-p} d(y_0, y_1).
\end{aligned}
$$

Since $0 \leq p < 1$, then $\lim\limits_{p \longrightarrow 0} \frac{p^j}{1-p} d(y_0, y_1) = 0$

This implies,

$$\lim_{j,m \longrightarrow \infty} d(y_j, y_m) = 0. \tag{3.11}$$

Let $m, j \in \mathbb{N}$ with $m < j$

Applying triangular inequality

$$
\begin{aligned}
d(y_m, y_j) &\leq d(y_m, y_{m+1}) + d(y_{m+1}, y_j) \\
&\leq d(y_m, y_{m+1}) + d(y_{m+1}, y_{m+2}) + \ldots + d(y_{j-1}, y_j) \\
&\leq p^m d(y_0, y_1) + p^{m+1} d(y_0, y_1) + \ldots + p^{j-1} d(y_0, y_1) \\
&\leq p^m (1 + p + \ldots + p^{j-m-1}) d(y_0, y_1) = p^m \left(\textstyle\sum_{i=0}^{j-m-1} p^i\right) d(y_0, y_1) \\
&\leq p^m \left(\textstyle\sum_{i=0}^{\infty} p^i\right) d(y_0, y_1) = \tfrac{p^m}{1-p} d(y_0, y_1)
\end{aligned}
$$

$\leq \frac{p^m}{1-p} d(y_0, y_1)$.

Since $0 \leq p < 1$, then $\lim\limits_{p \longrightarrow 0} \frac{p^m}{1-p} d(y_0, y_1) = 0$.

This implies,

$$\lim_{j,m \longrightarrow \infty} d(y_m, y_j) = 0. \tag{3.12}$$

From (3.11) and (3.12) we get

$$\lim_{m,j \longrightarrow \infty} d(y_j, y_m) = \lim_{m,j \longrightarrow \infty} d(y_m, y_j) = 0.$$

Thus, $\{y_j\}$ is a Cauchy sequence in X for $j \in \{0, 1, 2, ...\}$.

Since, X is complete there exists $q \in X$ such that $\lim\limits_{j \longrightarrow \infty} y_j = q$.

Thus,

$$\lim_{j \longrightarrow \infty} T_1 x_j = \lim_{j \longrightarrow \infty} T_2 x_{j+1} = ... = \lim_{j \longrightarrow \infty} T_n x_{j+(n-1)} = q$$

from which we have

$$\lim_{j \longrightarrow \infty} T_1 T_2 ... T_{n-1} x_{j+(n-2)} = \lim_{j \longrightarrow \infty} T_n x_{j+(n-1)} = q.$$

Since $T_n X$ are a closed subset of X, there is $w \in T_n X$ such that

$$q = T_n w.$$

Now we show that $T_1 T_2 ... T_{n-1} w = q$.

$$\begin{aligned} d\left(T_1 T_2 ... T_{n-1} w, T_1 T_2 ... T_{n-1} x_{j+(n-2)}\right) &\leq a\gamma\left(d(T_n w, T_n x_{j+(n-2)})\right) \\ &+ b\gamma\left(max\left\{d(T_n w, T_n x_{j+(n-2)}), d\left(T_n w, T_1 T_2 ... T_{n-1} w\right)\right\}\right) \\ &+ c\gamma\left(d(T_n w, T_n x_{j+(n-2)}) \frac{\left(1+\sqrt{d(T_n w, T_n x_{j+(n-2)}) d(T_n w, T_1 T_2 ... T_{n-1} w)}\right)^2}{(1+d(T_n w, T_n x_{j+(n-2)}))^2}\right). \end{aligned}$$

By the equation (3.7), we have

$$\begin{aligned} d\left(T_1 T_2 ... T_{n-1} w, T_1 T_2 ... T_{n-1} x_{j+(n-2)}\right) &= d\left(T_1 T_2 ... T_{n-1} w, T_1 x_{j+(n-2)}\right) \\ &= d\left(T_1 T_2 ... T_{n-1} w, q\right). \end{aligned}$$

Then,

$$\begin{aligned} d(T_1 T_2 ... T_{n-1} w, q) = d\left(T_1 T_2 ... T_{n-1} w, T_1 x_{j+(n-2)}\right) &\leq a\gamma\left(d(T_n w, T_n x_{j+(n-2)})\right) \\ &+ b\gamma\left(max\left\{d(T_n w, T_n x_{j+(n-2)}), d\left(T_n w, T_1 T_2 ... T_{n-1} w\right)\right\}\right) \\ &+ c\gamma\left(d(T_n w, T_n x_{j+(n-2)}) \frac{\left(1+\sqrt{d(T_n w, T_n x_{j+(n-2)}) d(T_n w, T_1 T_2 ... T_{n-1} w)}\right)^2}{(1+d(T_n w, T_n x_{j+(n-2)}))^2}\right) \end{aligned}$$

Since $T_1 w = T_2 w = ... = T_n w$, then we obtain that

$$d(T_1w,q) \le a\gamma\left(d(T_1w,T_nx_{j+(n-2)})\right)$$
$$+b\gamma\left(max\left\{d(T_1w,T_nx_{j+(n-2)}),d(T_1w,T_1w)\right\}\right)$$
$$+c\gamma\left(d(T_1w,T_nx_{j+(n-2)})\frac{\left(1+\sqrt{d(T_1w,T_nx_{j+(n-2)})d(T_1w,T_1w)}\right)^2}{\left(1+d(T_1w,T_nx_{j+(n-2)})\right)^2}\right)$$

Letting $j \longrightarrow \infty$ we get $\lim\limits_{j\longrightarrow\infty} T_nx_{j+(n-2)} = q$.

$$d\left(T_1T_2...T_{n-1}w,q\right) \le a\gamma(d(T_1w,q)+b\gamma(max\{d(T_1w,q),d(T_1w,T_1w)\})$$
$$+c\gamma\left(d(T_1w,q)\frac{\left(1+\sqrt{d(T_1w,q)d(T_1w,T_1w)}\right)^2}{(1+(T_1w,q))^2}\right)$$

Since $d(T_1w,T_1w)=0$, then $max\{d(T_1w,q),d(T_1w,T_1w)\} = d(T_1w,q)$ and $d(T_1w,q)\frac{\left(1+\sqrt{d(T_1w,q)d(T_1w,T_1w)}\right)^2}{(1+(T_1w,q))^2} \le d(T_1w,q)$.

Thus,

$$d(T_1T_2...T_{n-1}w,q) \le a\gamma(d(T_1w,q))+b\gamma(d(T_1w,q))+c\gamma d(T_1w,q).$$

Since $\gamma(t) \le t$ for all $t \ge 0$, then we get

$$d(T_1T_2...T_{n-1}w,q) \le ad(T_1w,q)+bd(T_1w,q)+cd(T_1w,q)$$
$$\le (a+b+c)d(T_1w,q).$$

Since $0 \le a+b+c < 1$, so the given inequality is satisfied if

$$d(T_1T_2...T_{n-1}w,q) = 0. \tag{3.13}$$

Similarly,

$$d(q,T_1T_2...T_{n-1}w) \le a\gamma(d(T_nx_{j+(n-2)},T_nw))$$
$$+b\gamma\left(max\left\{d(T_nx_{j+(n-2)},T_nw),d(T_nx_{j+(n-2)},T_1T_2...T_{n-1}x_{j+(n-2)})\right\}\right)$$
$$+c\gamma\left(d(T_nx_{j+(n-2)},T_nw)\frac{\left(1+\sqrt{d(T_nx_{j+(n-2)},T_nw)d\left(T_nx_{j+(n-2)},T_1T_2...T_{n-1}x_{j+(n-2)}\right)}\right)^2}{\left(1+d(T_nx_{j+(n-2)},T_nw)\right)^2}\right).$$

Since $T_1T_2...T_{n-1}w = T_nw$ and $T_1T_2...T_{n-1}x_{j+(n-2)} = T_1x_j+(n-2)$, then

$$d\left(q,T_1T_2...T_{n-1}w\right) \le a\gamma\left(d(T_nx_{j+(n-2)},T_1w)\right)$$
$$+b\gamma\left(max\left\{d(T_nx_{j+(n-2)},T_1w),d(T_nx_{j+(n-2)},T_1x_j+(n-2))\right\}\right)$$
$$+c\gamma\left(d(T_nx_{j+(n-2)},T_1w)\frac{\left(1+\sqrt{d(T_nx_{j+(n-2)},T_1w)d(T_nx_{j+(n-2)},T_1x_j+(n-2))}\right)^2}{\left(1+d(T_nx_{j+(n-2)},T_1w)\right)^2}\right).$$

Letting $j \longrightarrow \infty$ we get $\lim\limits_{j\longrightarrow\infty} T_1x_j+(n-2) = \lim\limits_{j\longrightarrow\infty} T_nx_{j+(n-2)} = q$.

By (3.7) we have

$$d\left(q,T_1T_2...T_{n-1}w\right) \le a\gamma(d(q,T_1w))+b\gamma(max\{d((q,T_1w)),d(q,q)\})$$

$$+c\gamma\left(d(q,T_1w)\frac{\left(1+\sqrt{d(q,T_1w)d(q,q)}\right)^2}{(1+d(q,T_1w))^2}\right)$$

Since $d(q,q)=0$, then $max\{d((q,T_1w)),d(q,q)\}$ is $d(q,T_1w)$ and

$c\gamma\left(d(q,T_1w)\frac{\left(1+\sqrt{d(q,T_1w)d(q,q)}\right)^2}{(1+d(q,T_1w))^2}\right)\leq d(q,T_1w)$

Thus,

$d(q,T_1T_2...T_{n-1}w)\leq a\gamma(d(q,T_1w))+b\gamma(d(q,T_1w))+c\gamma(d(q,T_1w)$

Since $\gamma(t)\leq t$ for all $t\geq 0$, then we get that

$$\begin{aligned}d(q,T_1T_2...T_{n-1}w)&\leq ad(q,T_1w)+bd(q,T_1w)+cd(q,T_1w)\\&\leq (a+b+c)d(q,T_1w)\end{aligned}$$

Since $0\leq a+b+c<1$, so the given inequality is satisfied if

$$d(q,T_1T_2...T_{n-1}w)=0. \tag{3.14}$$

Using (3.13) and (3.14), we have $q=T_1T_2...T_{n-1}w$.

Then, $q=T_1w=T_2w=...=T_nw$ from which we have

$q=T_1T_2...T_{n-1}w=T_nw$.

By the weakly compatibility of $T_1,T_2,...,T_n$, we have

$(T_1T_2...T_{n-1})\,T_nw=T_n\,(T_1T_2...T_{n-1}w)\,.$

Then,

$(T_1T_2...T_{n-1})\,T_nw=(T_1T_2...T_{n-1})\,q=T_n\,(T_1T_2...T_{n-1}w)=T_nq.$

Thus q is a coincidence point of $T_1,T_2,...,T_n$.

Consider

$$\begin{aligned}d\,(T_1T_2...T_{n-1}q,q)=d\,(T_1T_2...T_{n-1}q,T_1T_2...T_{n-1}w)&=d\,(T_1T_2...T_{n-1}q,T_1w)\\&\leq a\gamma(d(T_nq,T_nw))\\&\quad+b\gamma(max\{d\,((T_nq,T_nw))\,d\,(T_nq,T_1T_2...T_{n-1}q)\})\\&\quad+c\gamma\left(d(T_nq,T_nw)\frac{\left(1+\sqrt{d(T_nq,T_nw)d(T_nq,T_1T_2...T_{n-1}q)}\right)^2}{(1+d(T_nq,T_nw))^2}\right)\end{aligned}$$

Since $T_1T_2...T_{n-1}q=T_nq$ and $T_nw=q$, then

$$\begin{aligned}d\,(T_1T_2...T_{n-1}q,q)&\leq a\gamma(d(T_1q,q))+b\gamma(max\{d(T_1q,q),d(T_1q,T_1q)\})\\&\quad+c\gamma\left(d(T_1q,q)\frac{\left(1+\sqrt{d(T_1q,q)d(T_1q,T_1q)}\right)^2}{(1+d(T_1q,q))^2}\right)\end{aligned}$$

Since $\gamma(t)\leq t,(max\{d((T_1q,q)),d(T_1q,T_1q)\})$ is $d(T_1q,q)$ and

$d(T_1q,q)\frac{\left(1+\sqrt{d(T_1q,q)d(T_1q,T_1q)}\right)^2}{(1+d(T_1q,q))^2} \leq d(T_1q,q)$.

Then we get the following

$$\begin{aligned} d(T_1T_2...T_{n-1}q,q) &\leq ad(T_1q,q) \leq bd(T_1q,q) \leq cd(T_1q,q) \\ &\leq (a+b+c)d(T_1q,q). \end{aligned}$$

Since $0 \leq (a+b+c) < 1$, so the above inequality is satisfied if

$$d(T_1T_2...T_{n-1}q,q) = 0. \tag{3.15}$$

Similarly,

$$\begin{aligned} d\left(q,T_1T_2...T_{n-1}q\right) = d\left(T_1T_2...T_{n-1}w,T_1T_2...T_{n-1}q\right) &= d\left(T_1w,T_1T_2...T_{n-1}q\right) \\ &\leq a\gamma(d(T_nw,T_nq)) \\ &\quad + b\gamma(max\{d(T_nw,T_nq),d\left(T_nw,T_1T_2...T_{n-1}w\right)\}) \\ &\quad + c\gamma\left(d(T_nw,T_nq)\frac{\left(1+\sqrt{d(T_nw,T_nq)d(w,T_1T_2...T_{n-1}w)}\right)^2}{(1+d(T_nw,T_nq))^2}\right) \end{aligned}$$

Since $T_1T_2...T_{n-1}w = T_nw = q$ and $T_nq = T_1q$, then we have

$$\begin{aligned} d(q,T_1T_2...T_{n-1}q) &\leq a\gamma(d(T_1w,T_1q)) \\ &\quad +b\gamma(max\{d(T_1w,T_1q),d(T_1w,T_1w)\}) \\ &\quad +c\gamma\left(d(T_1w,T_1q)\frac{\left(1+\sqrt{d(T_1w,T_1q)d(T_1w,T_1w)}\right)^2}{(1+d(T_1w,T_1q))^2}\right) \\ d(q,T_1T_2...T_{n-1}q) &\leq a\gamma(d(q,T_1q)) + b\gamma(max\{d(q,T_1q),d(q,q)\}) \\ &\quad +c\gamma\left(d(q,T_1q)\frac{\left(1+\sqrt{d(q,T_1q)d(q,q)}\right)^2}{(1+d(q,T_1q))^2}\right) \end{aligned}$$

Since, $max\{d(q,T_1q),d(q,q)\}$ is $d(q,T_1q)$ and $\gamma(t) \leq t$, then we have

$$\begin{aligned} d(q,T_1T_2...T_{n-1}q) &\leq ad(q,T_1q) + bd(q,T_1q) + cd(q,T_1q) \\ &\leq (a+b+c)d(q,T_1q). \end{aligned}$$

Since $0 \leq a+b+c < 1$, so the given inequality is possible if

$$d\left(q,T_1T_2...T_{n-1}q\right) = 0. \tag{3.16}$$

So, from (3.15) and (3.16), we have $T_1T_2...T_{n-1}q = q$.

Thus, $T_1T_2...T_{n-1}q = T_nq = q$. Therefore, q is a common fixed point of $T_1T_2...T_{n-1}$ and T_n in X.

i.e., q is a common fixed point of $T_1,T_2,...,T_n$ in X.

Next we show that q is unique in X.

Let r be another common fixed point of $T_1,T_2,...,T_{n-1}$ and T_n in X.

$T_1r = T_2r = ... = T_{n-1}r = T_nr = r$, That is, $T_1T_2...T_{n-1}r = T_nr = r$.

Consider,

$$\begin{aligned} d(T_1T_2...T_{n-1}q, T_1T_2...T_{n-1}r) = d(q,r) \le\ & a\gamma(d(T_nq,T_nr)) \\ & + b\gamma(max\{d((T_nq,T_nr)), d(T_nq, T_1T_2...T_{n-1}q)\}) \\ & + c\gamma\left(d(T_nq,T_nr)\frac{\left(1+\sqrt{d(T_nq,T_nr)d(T_nq,T_1T_2...T_{n-1}q)}\right)^2}{(1+d(T_nq,T_nr))^2}\right) \end{aligned}$$

$$d(q,r) \le a\gamma(d(q,r)) + b\gamma(max\{d(d(q,r)), d(q,q)\}) + c\gamma\left(d(q,r)\frac{\left(1+\sqrt{d(T_1q,q)d(q,q)}\right)^2}{(1+d(q,r))^2}\right)$$

Since $\gamma(t) \le t, max\{d(d(q,r)), d(q,q)\}$ is $d(q,r)$ and

$d(q,r)\left(\frac{\left(1+\sqrt{d(T_1q,q)d(q,q)}\right)^2}{(1+d(q,r))^2}\right)$ then we obtain

$d(q,r) \le ad(q,r) + bd(q,r) + cd(q,r) \le (a+b+c)d(q,r)$

Since $0 \le a+b+c < 1$, so the given inequality is satisfied if

$$d(q,r) = 0. \tag{3.17}$$

Similarily,

$$\begin{aligned} d(T_1T_2...T_{n-1}r, T_1T_2...T_{n-1}q) = d(r,q) \le\ & a\gamma(d(T_nr,T_nq)) \\ & + b\gamma(max\{d((T_nr,T_nq)), d(T_nr, T_1T_2...T_{n-1}r)\}) \\ & + c\gamma\left(d(T_nr,T_nq)\frac{\left(1+\sqrt{d(T_nr,T_nq)d(T_nr,T_1T_2...T_{n-1}r)}\right)^2}{(1+d(T_nr,T_nq))^2}\right) d(r,q) \\ & \le a\gamma(d(r,q)) + b\gamma(max\{d(r,q), d(r,r)\}) \\ & + c\gamma\left(d(r,q)\frac{\left(1+\sqrt{d(r,q)d(r,r)}\right)^2}{(1+d(r,q))^2}\right) \end{aligned}$$

Since $\gamma(t) \le t, max\{d(r,q), d(r,r)\}$ is $d(r,q)$ and

$d(r,q)\frac{\left(1+\sqrt{d(r,q)d(r,r)}\right)^2}{(1+d(r,q))^2} \le d(r,q)$, then we obtain

$d(r,q) \le ad(r,q) + bd(r,q) + cd(r,q) \le (a+b+c)d(r,q)$.

Since $0 \le a+b+c < 1$, so the given inequality is satisfied if

$$d(r,q) = 0. \tag{3.18}$$

From (3.17) and (3.18), we conclude that $r = q$.

So,q is a unique common fixed point of $T_1, T_2, ..., T_n$ in X. □

Theorem 3.2. *Let (X,d) be a complete dq-metric space and*

$T_1,T_2,...,T_n : X \longrightarrow X$ be continuous self-mappings satisfying the contractive condition of Theorem 4.1. Then $T_1,T_2,...,T_n$ have a unique common fixed point.

Proof. We follow the proof of Theorem 3.1 and construct a sequence $\{y_j\}$. Let sub-sequences of $\{y_j\}$ be $\{x_{2j}\},\{x_{2j+1}\},\{x_{2(j+1)}\},...,\{x_{2j+(n-1)}\}$.

We define $x_{2j+(n-1)} = T_1T_2...T_{n-1}x_{2j+(n-2)}$ and $x_{2j+(n-2)} = T_nx_{2j+(n-3)}$.

Similarly we can show that the sequence $\{y_j\}$ is a Cauchy sequence.

By the completeness of X one can show that $\lim_{j\longrightarrow\infty} y_j = w$ for $w \in X$.

Since $\{x_{2j}\},\{x_{2j+1}\},\{x_{2(j+1)}\},...,\{x_{2j+(n-1)}\}$ are subsequences of $\{y_j\}$, then

$\lim_{j\longrightarrow\infty} x_{2j} = \lim_{j\longrightarrow\infty} x_{2j+1} = ... = \lim_{j\longrightarrow\infty} x_{2j+(n-1)} = w.$

Since $T_1,T_2,...,T_n$ are continuous then we arrive at

$$T_1w = T_1\lim_{j\longrightarrow\infty} x_{2j} = \lim_{j\longrightarrow\infty} T_1x_{2j} = \lim_{j\longrightarrow\infty} x_{2j+1} = ... = \lim_{j\longrightarrow\infty} T_1x_{2j+(n-2)} = \lim_{j\longrightarrow\infty} x_{2j+(n-1)} = w$$

$$\begin{aligned} T_2w &= T_2\lim_{j\longrightarrow\infty} x_{2j+1} = \lim_{j\longrightarrow\infty} T_2x_{2j+1} = \lim_{j\longrightarrow\infty} x_{2(j+1)}... = \lim_{j\longrightarrow\infty} T_2x_{2j+(n-2)} \\ &= \lim_{j\longrightarrow\infty} x_{2j+(n-1)} = w \end{aligned}$$

$\vdots$

$$\begin{aligned} T_{n-1}w &= T_{n-1}\lim_{j\longrightarrow\infty} x_{2j+(n-2)} = \lim_{j\longrightarrow\infty} T_{n-1}x_{2j+(n-2)} \\ &= \lim_{j\longrightarrow\infty} x_{2j+(n-1)} = w. \end{aligned}$$

Then, $T_1w = ... = T_{n-1}w = w$ which gives that

$$\begin{aligned} T_1...T_{n-1}\lim_{j\longrightarrow\infty} x_{2j+(n-2)} &= \lim_{j\longrightarrow\infty} T_1...T_{n-1}x_{2j+(n-2)} = \lim_{j\longrightarrow\infty} x_{2j+(n-1)} = w \\ &= T_1...T_{n-1}w = w. \end{aligned} \tag{3.19}$$

Similarly,

$T_nw = T_n\lim_{j\longrightarrow\infty} x_{2j+(n-3)} = \lim_{j\longrightarrow\infty} T_nx_{2j+(n-3)} = \lim_{j\longrightarrow\infty} x_{2j+(n-2)} = w.$ Then,

$$T_nw = w. \tag{3.20}$$

So, from (3.19) and (3.20) we obtain that $T_1T_2...T_{n-1}w = w = T_nw$.

Therefore w is common fixed point of $T_1,T_2,...,T_n$.

To show the uniqueness of w, let z be another common fixed point of $T_1,T_2,...,T_n$.

That is $T_1T_2...T_{n-1}z = z = T_nz$. Then by (iii) of Theorem 3.1, we have the following.

$$d(w,z) = d\left(T_1T_2...T_{n-1}w, T_1T_2...T_{n-1}z\right)$$
$$\leq a\gamma(d(T_nz,T_nz)) + b\gamma(max\{d(T_nw,T_nz), d\left(T_nw, T_1T_2...T_{n-1}w\right)\})$$
$$+c\gamma\left(d(T_nw,T_nz)\frac{\left(1+\sqrt{d(T_nw,T_nz)d(T_nw,T_1T_2...T_{n-1}w)}\right)^2}{(1+d(T_nw,T_nz))^2}\right)$$

$$d(w,z) \leq a\gamma(d(w,z)) + b\gamma(max\{d(w,z), d(w,w)\})$$
$$c\gamma\left(d(w,z)\frac{\left(1+\sqrt{d(w,z)d(w,w)}\right)^2}{(1+d(w,z))^2}\right)$$

Since $d(w,w) = 0$ then $max\{d(w,z), d(w,w)\}$ is d(w,z) and
$d(w,z)\frac{\left(1+\sqrt{d(w,z)d(w,w)}\right)^2}{(1+(w,z))^2} \leq d(w,z)$.
Thus, $d(w,z) \leq a\gamma(d(w,z)) + b\gamma(d(w,z)) + c\gamma(d(w,z))$

Since $\gamma(t) \leq t$ for all $t \geq 0$, then we obtain

$d(w,z) \leq ad(w,z) + bd(w,z) + cd(w,z) \leq (a+b+c)d(w,z)$

Since $0 \leq a+b+c < 1$, so

$$d(w,z) \leq a\gamma(d(T_nz,T_nz)) + b\gamma(max\{d(T_nw,T_nz), d(T_nw, T_1T_2...T_{n-1}w)\})$$
$$+c\gamma\left(d(T_nw,T_nz)\frac{\left(1+\sqrt{d(T_nw,T_nz)d(T_nw,T_1T_2...T_{n-1}w)}\right)^2}{(1+d(T_nw,T_nz))^2}\right)$$

is possible if,

$$d(w,z) = 0. \tag{3.21}$$

Similarly,

$$d(z,w) = d\left(T_1T_2...T_{n-1}z, T_1T_2...T_{n-1}w\right) \leq a\gamma(d(T_nz,T_nw))$$
$$+b\gamma(max\{d(T_nz,T_nw), d\left(T_nz, T_1T_2...T_{n-1}z\right)\})$$
$$+c\gamma\left(d(T_nz,T_nw)\frac{\left(1+\sqrt{d(T_nz,T_nw)d(T_nz,T_1T_2...T_{n-1}z)}\right)^2}{(1+d(T_nz,T_nw))^2}\right)$$

$$d(z,w) \leq a\gamma(d(z,w)) + b\gamma(max\{d(z,w), d(z,z)\}) + c\gamma\left(d(z,w)\frac{\left(1+\sqrt{d(z,w)d(z,z)}\right)^2}{(1+d(z,w))^2}\right).$$

Since $d(z,z) = 0$, then $max\{d(z,w), d(z,z)\}$ is $d(z,w)$ and
$d(z,w)\frac{\left(1+\sqrt{d(z,w)d(z,z)}\right)^2}{(1+(z,w))^2} \leq d(z,w)$.
Thus,

$d(z,w) \leq ad(z,w) + bd(z,w) + cd(z,w) \leq (a+b+c)d(z,w)$.

Since $0 \leq a+b+c < 1$, so

$$d(z,w) \leq a\gamma(d(T_nz,T_nw)) + b\gamma(max\{d(T_nz,T_nw), d\left(T_nz, T_1T_2...T_n - 1z\right)\})$$

$$+c\gamma\left(d(T_nz,T_nw)\frac{\left(1+\sqrt{d(T_nz,T_nw)d\left(T_nz,T_1T_2?T_(n-1)z\right)}\right)^2}{(1+d(T_nz,T_nw))^2}\right)$$

is possible if,

$$d(z,w)=0. \tag{3.22}$$

From (3.21) and (3.22) , we have that $w=z$. Thus, w is a unique common fixed point of $T_1,T_2,...,T_n$. □

Remark 3.3. In contractive condition, from our Theorem 3.1 if $T_n=I$, where I is identity map on X and $n\geq 2$, we obtain the following corollary which is the simplified form of contractive condition of Theorem 2.3 of Rahman M. U., Sarwar M.[3].

Corollary 3.4. *Let (X,d) be a complete dislocated quasi-metric space. Let $T_1:X\longrightarrow X$ be a self mapping satisfying*

$$d(T_1x,T_1y)\leq a\gamma(d(x,y))+b\gamma(max\{d(x,y),d(x,T_1x)\})+c\gamma\left(d(x,y)\frac{\left(1+\sqrt{d(x,y)d(x,T_1x)}\right)^2}{(1+d(x,y))^2}\right),$$

for all $x,y\in X,a,b,c\geq 0$ with $a+b+c<1$ and γ is a comparison function. Then T_1 has a unique fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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