Available online at http://scik.org
Advances in Fixed Point Theory, 3 (2013), No. 2, 418-427
ISSN: 1927-6303

FIXED POINTS OF EXPANSIVE TYPE MAPS FOR RATIONAL INEQUALITY IN 2-BANACH SPACES

PRABHA CHOUHAN[1,*], AND NEERAJ MALVIYA[2]

[1]Department of Mathematics, Scope College of Engineering, Bhopal (M.P.), India

[2]Department of Mathematics, NRI Institute of Information Science and Technology, Bhopal(M.P.),India

Abstract. In present paper, we prove some common fixed point theorems for expansive type mappings which is defined in [2] for rational inequality in 2-Banach space. Our theorems are extension of various results of literature ([9],[10],[15]) in 2-Banach spaces.

Keywords: 2-normed space, 2-Banach space, Expansive mapping, Rational inequality, Fixed point.

2000 AMS Subject Classification: 47H10; 54H25

1. Introduction

The research about fixed points of expansive mapping was initiated by Machuca (see [7]). Later Jungck discussed fixed points for other forms of expansive mapping (see [6]). In 1982, Wang et al. (see [16]) presented some interesting work on expansive mappings in metric spaces which correspond to some contractive mapping in [13]. Also, Zhang has done considerable work in this field. In order to generalize the results about fixed point theory, Zhang (See [18]) published his work Fixed Point Theory and Its Applications, in which the fixed point problem for expansive mapping is systematically presented in a chapter. As applications, he also investigated the existence of solutions of equations for

*Corresponding author

Received April 4, 2013

locally condensing mapping and locally accretive mapping. On the other hand Gahler ([3],[4]) investigated the idea of 2-metric and 2-Banach spaces and proved same results. Subsequently several authors including Iseki [5], Rhoades [11], White [17], Panja and Baisnab [8] and Saha et al [14] studied various aspects of the fixed point theory and proved fixed point theorems in 2-metric spaces and 2-Banach spaces. Recently, the study about fixed point theorem for expansive mapping is deeply explored and has extended too many others directions. Motivated and inspired by the above work, in this paper we investigate fixed point of expansive mappings for rational inequality in 2-Banach spaces. The presented theorems extend and improve many existing results of the literature ([9],[10],[15]) in 2-Banach space.

2. Preliminaries

Definition 2.1. *Let X be a real linear space and $\| .,. \|$ be a non-negative real valued function defined on $X \times X$ satisfying the following conditions:*

i) $\|x, y\| = 0$ *if and only if x and y are linearly dependent in X,*

ii) $\|x, y\| = \|y, x\|$ *for all $x, y \in X$*

iii) $\|x, ay\| = | a | \, \|x, y\|$ *a being real, $x, y \in X$*

iv) $\|x, y + z\| = \|x, y\| + \|x, z\|$ *for all $x, y, z \in X$*

Then $\| .,. \|$ is called a 2-norm and the pair $(X, \| .,. \|)$ is called a linear 2-normed space.

Some of the basic properties of 2-norms are that they are non negative satisfying $\|x, y + ax\| = \|x, y\|$, for all $x, y \in X$ and all real numbers a.

Definition 2.2. *A sequence $\{x_n\}$ in a linear 2-normed space $(X, \| .,. \|)$ is called Cauchy sequence if*

$$\lim_{m,n\to\infty} \| x_m - x_n, y \| = 0 \quad \text{for all } y \text{ in } X.$$

Definition 2.3. *A sequence $\{x_n\}$ in a linear 2-normed space $(X, \| .,. \|)$ is said to be convergent if there is a point x in X such that*

$$\lim_{n\to\infty} \| x_n - x, y \| = 0 \quad \textit{for all } y \textit{ in } X.$$

If $\{x_n\}$ converges to x, we write $\{x_n\} \to x$ as $n \to \infty$.

Definition 2.4. *A linear 2-normed space X is said to be complete if every Cauchy sequence is convergent to an element of X. We then call X to be a 2- Banach space.*

Definition 2.5. *Let X be a 2-Banach space and T be a self-mapping of X. T is said to be continuous at x if for every sequence $\{x_n\}$ in X, $\{x_n\} \to x$ as $n \to \infty$ implies $\{T(x_n)\} \to T(x)$ as $n \to \infty$.*

Example 2.6. *Let X is R^3 and consider the following 2-norm on X as*

$$\| \mathbf{x}, \mathbf{y} \| = \left| det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. Then $(X, \| .,. \|)$ is a 2-Banach space.

Example 2.7. *Let P_n denotes the set of all real polynomials of degree $\leq n$, on the interval [0,1]. By considering usual addition and scalar multiplication, P_n is a linear vector space over the reals. Let $\{x_o, x_1, ..., x_{2n}\}$ be distinct fixed points in [0,1] and define the following 2-norm on P_n:$\| f, g \| = \sum_{k=0}^{2n} | f(x_k)\, g(x_k) |$, whenever f and g are linearly independent and $\| f, g \| = 0$, if f, g are linearly dependent. Then $(P_n, \| .,. \|)$ is a 2-Banach space.*

Example 2.8. *Let X is Q^3, the field of rational number and consider the following 2-norm on X as:*

$$\| \mathbf{x}, \mathbf{y} \| = \left| det \begin{pmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} \right|,$$

where $x = (x_1, x_2, x_3)$ *and* $y = (y_1, y_2, y_3)$. *Then* $(X, \| .,. \|)$ *is not a 2-Banach space but a 2-normed space.*

Definition 2.9. *Let* $(X, \| .,. \|)$ *be a 2-Banach space with 2-norm* $\| .,. \|$. *A mapping* T *of* X *into itself is said to be expansive if there exists a constant* $h > 1$ *such that*

$$\| Tx - Ty, a \| \geq h \| x - y, a \| \; for \;\; all \;\; x, y \in X$$

Example 2.10. *Let* $X = R^2$ *and consider the following 2-norm on* X *as* $\| x, y \| = | x_1y_2 - x_2y_1 |$ *where* $x = (x_1, x_2)$ *and* $y = (y_1, y_2)$. *Then* $(X, \| .,. \|)$ *is a 2-Banach space. Define a self map* T *on* X *as follows* $Tx = \beta x$ *where* $\beta > 1$ *for all* $x \in X$, *clearly* T *is an expansive mapping.*

3. Main results

The following theorem is an extension of theorem 3 of Popa. V [10]

Theorem 3.1. *Let* $(X, \| .,. \|)$ *be a 2-Banach space and* S,T *be surjective mappings of* X *into itself such that for every* $x, y, a \in X$,

$$\| Sx - Ty, a \| \geq \frac{a_1 \left[\|x-Sx,a\| \, \|x-y,a\| + \|y-Ty,a\| \, \|x-y,a\| \right] + b_1 \|x-Sx,a\| \, \|y-Ty,a\| + c_1 \|x-y,a\|^2}{\|x-Sx,a\| + \|y-Ty,a\| + \|x-y,a\|} \tag{3.1}$$

with $x \neq y$, *where* $a_1, b_1, c_1 \geq 0$ $2a_1 + b_1 + c_1 > 3$, *and* $c_1 > 1$. *Then* S *and* T *have a common unique fixed point in* X.

Proof: We define a sequence $\{x_n\}$ as follows for $n = 0, 1, 2, 3, \ldots$

$$x_{2n} = Sx_{2n+1}, \quad x_{2n+1} = Tx_{2n+2} \tag{3.2}$$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n then we see that x_{2n} is a fixed point of S and T. Therefore, we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal.

Now consider

$$\|x_{2n}-x_{2n+1},a\| = \|Sx_{2n+1}-Tx_{2n+2},a\|$$
$$\geq \frac{a_1\Big[\|x_{2n+1}-Sx_{2n+1},a\|\,\|x_{2n+1}-x_{2n+2},a\|+\|x_{2n+2}-Tx_{2n+2},a\|\,\|x_{2n+1}-x_{2n+2},a\|\Big]}{\|x_{2n+1}-Sx_{2n+1},a\|+\|x_{2n+2}-Tx_{2n+2},a\|+\|x_{2n+1}-x_{2n+2},a\|}$$
$$+\frac{b_1\|x_{2n+1}-Sx_{2n+1},a\|\,\|x_{2n+2}-Tx_{2n+2},a\|+c_1\|x_{2n+1}-x_{2n+2},a\|^2}{\|x_{2n+1}-Sx_{2n+1},a\|+\|x_{2n+2}-Tx_{2n+2},a\|+\|x_{2n+1}-x_{2n+2},a\|}$$

$$=\frac{a_1\Big[\parallel x_{2n+1}-x_{2n},a\parallel\parallel x_{2n+1}-x_{2n+2},a\parallel+\parallel x_{2n+2}-x_{2n+1},a\parallel\parallel x_{2n+1}-x_{2n+2},a\parallel\Big]}{\parallel x_{2n+1}-x_{2n},a\parallel+\parallel x_{2n+2}-x_{2n+1},a\parallel+\parallel x_{2n+1}-x_{2n+2},a\parallel}$$
$$+\frac{b_1\parallel x_{2n+1}-x_{2n},a\parallel\parallel x_{2n+2}-x_{2n+1},a\parallel+c_1\parallel x_{2n+1}-x_{2n+2},a\parallel^2}{\parallel x_{2n+1}-x_{2n},a\parallel+\parallel x_{2n+2}-x_{2n+1},a\parallel+\parallel x_{2n+1}-x_{2n+2},a\parallel}$$

$$\Rightarrow\parallel x_{2n}-x_{2n+1},a\parallel\Big[\parallel x_{2n+1}-x_{2n},a\parallel+2\parallel x_{2n+1}-x_{2n+2},a\parallel\Big]$$
$$\geq a_1\Big[\parallel x_{2n+1}-x_{2n},a\parallel\parallel x_{2n+1}-x_{2n+2},a\parallel+\parallel x_{2n+1}-x_{2n+2},a\parallel\parallel x_{2n+1}-x_{2n+2},a\parallel\Big]$$
$$+b_1\parallel x_{2n+1}-x_{2n},a\parallel\parallel x_{2n+1}-x_{2n+2},a\parallel+c_1\parallel x_{2n+1}-x_{2n+2},a\parallel^2$$
$$\Rightarrow\parallel x_{2n}-x_{2n+1},a\parallel^2+2\parallel x_{2n}-x_{2n+1},a\parallel,\parallel x_{2n+1}-x_{2n+2},a\parallel$$
$$\geq\parallel x_{2n+1}-x_{2n+2},a\parallel\ (2a_1+b_1+c_1)\min\{\parallel x_{2n}-x_{2n+1},a\parallel,\ \parallel x_{2n+1}-x_{2n+2},a\parallel\}$$
$$\Rightarrow\parallel x_{2n}-x_{2n+1},a\parallel^2\geq\parallel x_{2n+1}-x_{2n+2},a\parallel\ (2a_1+b_1+c_1-2)\min\{\parallel x_{2n}-x_{2n+1},a\parallel,$$
$$\parallel x_{2n+1}-x_{2n+2},a\parallel\}$$

Case I

$$\Rightarrow\parallel x_{2n}-x_{2n+1},a\parallel^2\geq(2a_1+b_1+c_1-2)\parallel x_{2n+1}-x_{2n+2},a\parallel^2$$
$$\Rightarrow\parallel x_{2n+1}-x_{2n+2},a\parallel\leq\left[\frac{1}{(2a_1+b_1+c_1-2)}\right]^{\frac{1}{2}}\parallel x_{2n}-x_{2n+1},a\parallel$$
$$\Rightarrow\parallel x_{2n+1}-x_{2n+2},a\parallel\leq k_1\parallel x_{2n}-x_{2n+1},a\parallel$$

where $k_1=\left[\frac{1}{(2a_1+b_1+c_1-2)}\right]^{\frac{1}{2}}<1(As\quad 2a_1+b_1+c_1>3)$

Similarly we can calculate

$$\Rightarrow \| x_{2n+2} - x_{2n+3}, a \| \leq k_1 \| x_{2n+1} - x_{2n+2}, a \|$$

where $k_1 = \left[\frac{1}{(2a_1+b_1+c_1-2)}\right]^{\frac{1}{2}} < 1$ and so on

Case II

$$\Rightarrow \| x_{2n} - x_{2n+1}, a \|^2 \geq (2a_1 + b_1 + c_1 - 2) \| x_{2n+1} - x_{2n+2}, a \| \| x_{2n} - x_{2n+1}, a \|$$

$$\Rightarrow \| x_{2n+1} - x_{2n+2}, a \| \leq \left[\frac{1}{(2a_1 + b_1 + c_1 - 2)}\right] \| x_{2n} - x_{2n+1}, a \|$$

$$\Rightarrow \| x_{2n+1} - x_{2n+2}, a \| \leq k_2 \| x_{2n} - x_{2n+1}, a \|$$

where $k_2 = \left[\frac{1}{(2a_1+b_1+c_1-2)}\right] < 1(As \quad 2a_1 + b_1 + c_1 > 3)$

Similarly we can calculate

$$\Rightarrow \| x_{2n+2} - x_{2n+3}, a \| \leq k_2 \| x_{2n+1} - x_{2n+2}, a \|$$

where $k_2 = \left[\frac{1}{(2a_1+b_1+c_1-2)}\right] < 1$ and so on

So, in general

$$\| x_n - x_{n+1}, a \| \leq k \| x_{n-1} - x_n, a \| \quad for\ n = 1, 2, 3...$$

where $k = \max\{k_1, k_2\}$ then $k < 1$

$$\Rightarrow \| x_n - x_{n+1}, a \| \leq k^n \| x_0 - x_1, a \|$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. For this, for every positive integer p we have,

$$\begin{aligned}
\| x_n - x_{n+p}, a \| &\leq \| x_n - x_{n+1}, a \| + \| x_{n+1} - x_{n+2}, a \| + ... + \| x_{n+p-1} - x_{n+p}, a \| \\
&\leq (k^n + k^{n+1} + k^{n+2} + ... + k^{n+p-1}) \| x_0 - x_1, a \| \\
&= k^n(1 + k + k^2 + ... + k^{p-1}) \| x_0 - x_1, a \| \\
&< \frac{k^n}{1-k} \| x_0 - x_1, a \|
\end{aligned}$$

as $n \to \infty$, $\| x_n - x_{n+p}, a \| \to 0$, it follows that $\{x_n\}$ is a Cauchy sequence in X. So there exists a point x in X such that

$$\{x_n\} \to x \text{ as } n \to \infty \tag{3.3}$$

Existence of fixed point: Since S and T are surjective maps and hence there exist two points y and y' in C such that

$$x = Sy \; and \; x = Ty' \tag{3.4}$$

Consider,

$$\| x_{2n} - x, a \| = \| Sx_{2n+1} - Ty', a \|$$

$$\geq \frac{a_1 \Big[\| x_{2n+1} - Sx_{2n+1}, a \| + \| y' - Ty', a \| \Big] \| x_{2n+1} - y', a \|}{\| x_{2n+1} - Sx_{2n+1}, a \| + \| y' - Ty', a \| + \| x_{2n+1} - y', a \|}$$

$$+ \frac{b_1 \| x_{2n+1} - Sx_{2n+1}, a \| \| y' - Ty', a \| + c_1 \| x_{2n+1} - y', a \|^2}{\| x_{2n+1} - Sx_{2n+1}, a \| + \| y' - Ty', a \| + \| x_{2n+1} - y', a \|}$$

$$= \frac{a_1 \Big[\|x_{2n+1} - x_{2n}, a\| + \|y' - Ty', a\| \Big] \|x_{2n+1} - y', a\| + b_1 \|x_{2n+1} - x_{2n}, a\| \, \|y' - Ty', a\| + c_1 \|x_{2n+1} - y', a\|^2}{\|x_{2n+1} - x_{2n}, a\| + \|y' - Ty', a\| + \|x_{2n+1} - y', a\|}$$

As $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{x_n\}$ as $n \to \infty$, $\{x_{2n}\} \to x$, $\{x_{2n+1}\} \to x$ (Using 3)

Therefore

$$\| x - x, a \| \geq \frac{a_1 \Big[\|x - x, a\| + \|y' - x, a\| \Big] \|x - y', a\| + b_1 \|x - x, a\| \, \|y' - x, a\| + c_1 \|x - y', a\|^2}{\|x - x, a\| + \|y' - x, a\| + \|x - y', a\|} \quad [Using \; 3.4]$$

$$0 \geq \frac{1}{2}(a_1 + c_1) \| y' - x, a \|$$

$$\Rightarrow \| x - y', a \| = 0 (as \; a_1 + c_1 > 1)$$

$$\Rightarrow x = y' \tag{3.5}$$

In an exactly similar way we can prove that,

$$x = y \tag{3.6}$$

The fact (3.4) along with (3.5) and (3.6) shows that x is a common fixed point of S and T.

Uniqueness: Let z be another common fixed point of S and T, that is

$$Sz = z \; and \; Tz = z \tag{3.7}$$

$$\begin{aligned}
\| x - z, a \| &= \| Sx - Tz, a \| \\
&\geq \frac{a_1\left[\|x-Sx,a\| \, \|x-z,a\| + \|z-Tz,a\| \|x-z,a\|\right] + b_1 \|x-Sx,a\| \, \|z-Tz,a\| + c_1 \|x-z,a\|^2}{\|x-Sx,a\| + \|z-Tz,a\| + \|x-z,a\|} \\
&\Rightarrow \| x - z, a \|^2 \geq c_1 \; \| x - z, a \|^2 \\
&\Rightarrow (1 - c_1) \| x - z, a \|^2 \geq 0 \\
&\Rightarrow \| x - z, a \|^2 = 0 \quad (as \; c_1 > 1) \\
&\Rightarrow x = z
\end{aligned}$$

This completes the proof of the theorem 3.1.

The following theorem is an example in which fixed point is not unique and it extend the theorem 3 of [15] in 2-Banach spaces.

Theorem 3.2. *Let $(X, \| .,. \|)$ be a 2-Banach space and S, T be surjective mappings of X into itself satisfying.*

$$\| Sx - Ty, a \| \geq \frac{a_1 \|x-Sx,a\| \, \|x-y,a\| + b_1 \|y-Ty,a\| \, \|x-y,a\| + c_1 \|x-Sx,a\| \, \|y-Ty,a\|}{\|x-Sx,a\| + \|y-Ty,a\|} \tag{3.8}$$

for every $x, y \in X$ and $\| x - Sx \| + \| y - Ty \| \neq 0$ Where $a_1, c_1 \geq 0$, $b_1 > 0$ and $a_1 + b_1 + c_1 > 2$. Then S and T have a common fixed point in X.

Proof: The proof of this theorem is similar to the proof of the theorem 3.1.

Theorem 3.3. *Let $(X, \| .,. \|)$ be a 2-Banach space and S, T be surjective mappings of X into itself such that for every $x, y, a \in X$ satisfying*

$$\begin{aligned}
\| Sx - Ty, a \| \geq \frac{a_1 \| x - Sx, a \| \| y - Ty, a \|}{\| x - y, a \|} + b_1 \Big[\| x - Sx, a \| + \| y - Ty, a \| \Big] \\
+ c_1 \| x - y, a \|
\end{aligned} \tag{3.9}$$

with $x \neq y$ *Where* $a_1, b_1 \geq 0$, $c_1 > 1$. *Then* S *and* T *have a common unique fixed point in* X.

Proof: The proof of this theorem is similar to the proof of the theorem 3.1 and it extend our own theorem 3.3 of [9] in 2-Banach space.

References

[1] Y.J. Cho, M.S. Khan and S.L. Sing, Common fixed points of weakly commuting mappings, Univ.u. Novom Sadu, Zb.Rad. Period.-Mat.Fak.Ser.Mat, 18 1(1988)129-142. MR1034710.

[2] P. Chouhan and N. Malviya, Fixed points of expansive type mappings in 2-Banach spaces communicated in Acta Mathematica Academiae Paedagogiace Nyregyhziensis.

[3] S. Ghler, 2-metric Raume and ihre topologische strucktur, Math. Nachr., 26(1963), 115-148. MR0162224.

[4] S. Ghler, Uber die unifromisieberkeit 2-metrischer Raume, Math. Nachr. 28(1965), 235 - 244. MR0178452.

[5] K. Iseki, Fixed point theorems in 2-metric space, Math.Seminar.Notes, Kobe Univ.,3(1975), 133 - 136. MR0405395.

[6] G. Jungck, Commuting mappings and fixed points The American Mathematical Monthly, Vol.83, no.4,PP.261-263, 1976.

[7] R. Machuca, A coincidence theorem The American Mathematical Monthly, Vol.74, no.5,P. 569, 1967.

[8] C. Panja and A.P. Baisnab, Asymptotic regularity and fixed point theorems, The Mathematics Student, 46 1(1978), 54-59 . MR0698179.

[9] S.S. Pagey and N. Malviya, Some common fixed point theorems for pair of non commuting expansive type mappings in Hilbert spaces Accepted for publication in the international journal of Mathematical analysis Ruse 7005, Bulgaria

[10] V. Popa, Some fixed point theorems of expansion mappings, Demonstratio Math. 19 (1986), 699-702.

[11] B.E. Rhoades, Contractive type mappings on a 2-metric space, Math. Nachr., 91(1979), 151-155. MR0563606.

[12] B.E. Rhoades, Some fixed point theorems for pairs of mappings, Jnanabha 15 (1985), 151-156.

[13] B.E. Rhoades, A comparison of various definitions of contractive mappings. Tran. Am. Math. Soc.226, 257-290 (1977)

[14] M. Saha, D. Dey, A. Ganguly, and L. Debnath Asymptotic regularity and fixed point theorems on a 2-Banach space Surveys in Mathematics and its Applications Vol.7(2012), 31-38.

[15] M. Telci and K. Tas some fixed point theorems for pairs of expansive type mappings, Demonstratio Math. 27 (1994), 401-405.

[16] S.Z. Wang, B.Y. Li, Z.M. Gao, and K. Iseki, Some fixed point theorems on expansion mappings, Math. Japonica 29 (1984), 631-636.

[17] A. White, 2-Banach spaces, Math. Nachr., 42(1969), 43 - 60. MR0257716 (41 2365) Zbl 0185.20003.

[18] S. Zhang, Fixed point theory and its applications, Chongqing press, Chongqing China, 1984.