

Available online at http://scik.org Adv. Inequal. Appl. 2014, 2014:29 ISSN: 2050-7461

A PROOF OF JENSEN'S INEQUALITY THROUGH A NEW STEFFENSEN'S INEQUALITY

MOHAMMED M. IDDRISU^{1,*}, CHRISTOPHER A. OKPOTI², KAZEEM A. GBOLAGADE³

¹Department of Mathematics, University for Development Studies, P. O. Box 24, Navrongo, Ghana ²Department of Mathematics, University of Education, Winneba, Ghana

³Department of Computer Science, University for Development Studies, P. O. Box 24, Navrongo, Ghana

Copyright © 2014 Iddrisu, Okpoti and Gbolagade. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. In this paper, we present more proofs of the new Steffensen's inequality for convex functions. First, we provide separate proofs for continuous functions followed by a general proof for all $L^1([0,1])$ functions. The last part is dedicated to the proof of the well known Jensen's inequality using the new inequality.

Keywords: Steffensen; Jensen; inequality; continuous function.

2010 AMS Subject Classification: 26D15.

1. Introduction

The following Steffensen's inequality (1) was unpopular in the research environment for some decades since its discovery in 1918:

(1)
$$\int_{b-\lambda}^{b} g(x)dx \le \int_{a}^{b} g(x)f(x)dx \le \int_{a}^{a+\lambda} g(x)dx,$$

where f and g are integrable functions defined on (a,b), g is decreasing and for each $x \in (a,b)$, $0 \le f(x) \le 1$ and $\lambda = \int_a^b f(x) dx$. This inequality (1) is known in literature as one of the useful inequalities in mathematical analysis (See [2]) and ([4], pp. 311-312). The second of its

^{*}Corresponding author

Received April 2, 2014

appearance was in [9]. Then in 1959, the inequality attracted the attention of R. Bellman [2] who gave the following generalization

(2)
$$\left(\int_{a}^{b} f(x)g(x)dx\right)^{p} \leq \int_{a}^{a+\lambda} g(x)^{p}dx$$

where $\lambda = (a + \int_a^b f(x)dx)^p$, $g \in L^p[a,b]$, $f \ge 0$ and $\int_a^b f^q \le 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ for p > 1. This inequality was noted to be invalid and was thus corrected by J.E. Pecaric [7] with the result

(3)
$$\left(\int_0^1 f(x)g(x)dx\right)^p \le \int_0^\lambda g(x)^p dx$$

where $g: [0,1] \longrightarrow \Re$ is a nonnegative and nonincreasing function, $f: [0,1] \longrightarrow \Re$ is an integrable function such that $0 \le f(x) \le 1$ ($\forall x \in [0,1]$), $p \ge 1$ and $\lambda = \left(\int_0^1 f(x) dx\right)^p$ (see also [5] and [8]).

Now the task in this paper is to give some proofs of (1) with involvement of convex functions and further deduce the Jensen's inequality from the results.

2. Preliminaries

Let us present some definitions of convex functions here. The study of convex functions were first defined and systematically studied by J. L. W. V. Jensen who gave the study of (algebraic) inequalities as principal object of his investigation of convex functions (see [1], [6]).

Definition 1.1. Let *I* be an interval in \Re . A function $\phi : I \longrightarrow \Re$ is said to be convex if for all $x_1, x_2 \in I$ and for all positive a_1 and a_2 satisfying $a_1 + a_2 = 1$,

(4)
$$\phi(a_1x_1 + a_2x_2) \le a_1\phi(x_1) + a_2\phi(x_2).$$

A convex function necessarily is continuous for $x_1, x_2 \in I$.

Definition 1.2. A function ϕ is said to be concave if $-\phi$ is convex (i.e. if the inequality (4) is reversed).

For example, |x|, x^k for k > 1, e^x etc. are convex functions whiles x^k for 0 < k < 1, $\log x$, \sqrt{x} for $x \ge 0$ etc. are concave functions.

3. Main results

We begin as follows:

Theorem 3.1. [3] Let the function $f : [0,1] \to \Re$ be continuous such that $0 \le f(x) \le 1$. If $\phi : [0,1] \to \Re$ is a convex, differentiable function with $\phi(0) = 0$. Then

(5)
$$\phi\left(\int_0^1 f(x)dx\right) \le \int_0^1 f(x)\phi'(x)dx$$

for all $x \in [0, 1]$ *.*

Next, we give two proofs.

Proof 1. The function $\phi(x)$ is convex and differentiable on the interval [0, 1] and it's differential, $\phi'(x)$ is nondecreasing for all $x \in [0, 1]$. Then $-\phi'(x)$ is decreasing, and making the substitution of $g(x) = -\phi'(x)$, a = 0 and b = 1 into inequality (1) gives

$$\int_0^\lambda \phi'(x)dx \le \int_0^1 f(x)\phi'(x)dx \le \int_{1-\lambda}^1 \phi'(x)dx.$$

This simplifies to

$$\phi(\lambda) - \phi(0) \leq \int_0^1 f(x)\phi'(x)dx \leq \phi(1) - \phi(1-\lambda).$$

Since $\lambda = \int_0^1 f(x) dx$ and $\phi(0) = 0$, thus

$$\phi\left(\int_0^1 f(x)dx\right) \leq \int_0^1 f(x)\phi'(x)dx.$$

Proof 2. Let

(6)
$$F(x) = \int_0^x f(t)dt \le x$$

and

$$G(x) = \phi(F(x)) = \phi\left(\int_0^x f(t)dt\right).$$

Since *f* is continuous and the differential of F(x) denoted F'(x) = f(x), then by the chain rule of differentiation we obtain

$$G'(x) = F'(x)\phi'(F(x)) \le f(x)\phi'(x).$$

Thus

(7)
$$\int_{0}^{1} G'(x) dx \le \int_{0}^{1} f(x) \phi'(x) dx.$$

Since $\phi(0) = 0$ and

$$\int_0^1 G'(x)dx = G(1) - G(0)$$
$$= \phi(F(1)) - \phi(F(0))$$
$$= \phi\left(\int_0^1 f(x)dx\right) - \phi(0)$$
$$= \phi\left(\int_0^1 f(x)dx\right),$$

inequality (7) becomes

$$\phi\left(\int_0^1 f(x)dx\right) \le \int_0^1 f(x)\phi'(x)dx.$$

Example 3.1. Consider the convex function $\phi(u) = u^p$ for $1 \le p < \infty$, then

$$\left(\int_0^1 f(x)dx\right)^p \le p\int_0^1 f(x)x^{p-1}dx.$$

For a particular case of p = 2, we get

$$\left(\int_0^1 f(x)dx\right)^2 \le 2\int_0^1 xf(x)dx.$$

Let us also prove the new inequality for functions $f \in L^1([0,1])$. We denote by $C_c([0,1])$ the set of all continuous functions on [0,1] with compact support. Thus

$$C_c([0,1]) \subset L^1([0,1]).$$

Theorem 3.2. Let $f \in L^1([0,1])$ such that $0 \le f(x) \le 1$ for all $x \in [0,1]$. If $\phi : [0,1] \longrightarrow \Re$ is a convex, differentiable function with $\phi(0) = 0$ then

(8)
$$\phi\left(\int_0^1 f(x)dx\right) \le \int_0^1 f(x)\phi'(x)dx.$$

Proof. Let $f \in L^1([0,1])$ and $\varepsilon = \frac{1}{n} > 0$, there exists sequence $(f_n)_{n \in \mathbb{N}}$ of functions in $C_c([0,1])$ such that $||f - f_n||_1 < \frac{1}{n}$. Since f_n is continuous then by Theorem 3.1, we have

$$\phi\left(\int_0^1 f_n(x)dx\right) \le \int_0^1 f_n(x)\phi'(x)dx$$
$$\le \int_0^1 f(x)\phi'(x)dx + \int_0^1 [f_n(x) - f(x)]\phi'(x)dx.$$

Since

$$\left| \int_{0}^{1} f_{n}(x) dx - \int_{0}^{1} f(x) dx \right| \le \int_{0}^{1} |f_{n}(x) - f(x)| dx < \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty$$

Thus

$$\phi\left(\int_0^1 f(x)dx\right) \le \int_0^1 f(x)\phi'(x)dx.$$

Remark 3.1 Let us remark that $\phi'(x)$ is bounded in [0,1]. So

$$\int_0^1 |f_n(x) - f(x)|\phi'(x)dx \le ||\phi'||_{\infty} \int_0^1 |f_n(x) - f(x)|dx$$
$$\le \frac{1}{n} ||\phi'||_{\infty} \to 0 \quad \text{as} \quad n \to \infty.$$

Example 3.2. Let

$$f(x) = \begin{cases} a_1 & \text{if } 0 \le x < x_1 \\ a_2 & \text{if } x_1 \le x < x_2 \\ \vdots & & \\ a_n & \text{if } x_{n-1} \le x \le 1 \end{cases}$$

where $0 < a_1 < a_2 < \cdots < a_n < 1$, $x_0 = 0$ and $x_n = 1$. Putting *f* into (8) and evaluating the result, yields

(9)
$$\phi\left(\sum_{i=1}^{n}a_{i}(x_{i}-x_{i-1})\right) \leq \sum_{i=1}^{n}a_{i}\{\phi(x_{i})-\phi(x_{i-1})\}.$$

Theorem 3.3. (Jensen's inequality) Let $0 \le b_1, b_2, \dots, b_n \le 1$ with $\sum_{j=1}^n b_j = 1$ and let $0 < x_1 \le x_2 \le \dots \le x_n \le 1$. If $\phi : [0,1] \longrightarrow \Re$ is convex and differentiable with $\phi(0) = 0$, then

(10)
$$\phi\left(\sum_{j=1}^{n}b_{j}x_{j}\right) \leq \sum_{j=1}^{n}b_{j}\phi(x_{j}).$$

Proof. Expansion of inequality (9) gives

$$\phi \{a_1(x_1 - x_0) + a_2(x_2 - x_1) + \dots + a_n(x_n - x_{n-1})\}$$

$$\leq a_1[\phi(x_1) - \phi(x_0)] + a_2[\phi(x_2) - \phi(x_1)] + \dots + a_n[\phi(x_n) - \phi(x_{n-1})].$$

Since $x_0 = 0$, we have

(11)
$$\phi\left(a_n x_n + \sum_{j=1}^{n-1} (a_j - a_{j+1}) x_j\right) \le a_n \phi(x_n) + \sum_{j=1}^{n-1} (a_j - a_{j+1}) \phi(x_j).$$

Now, we consider $0 < b_1, b_2, \dots, b_n < 1$ with $\sum_{j=1}^n b_j = 1$. Let

$$a_{1} = b_{1} + b_{2} + \dots + b_{n-1} + b_{n} = 1$$

$$a_{2} = b_{2} + b_{3} + \dots + b_{n-1} + b_{n}$$

$$a_{3} = b_{3} + b_{4} + \dots + b_{n-1} + b_{n}$$

$$\vdots$$

$$a_{n-1} = b_{n-1} + b_{n}$$

$$a_{n} = b_{n}.$$

This shows that

(12)
$$a_j - a_{j+1} = b_j$$
, for $j = 1, \dots, n-1$.

Putting (12) into (11), yields

(13)
$$\phi\left(a_nx_n+\sum_{j=1}^{n-1}b_jx_j\right) \le a_n\phi(x_n)+\sum_{j=1}^{n-1}b_j\phi(x_j).$$

Since $a_n = b_n$, thus (13) becomes $\phi\left(\sum_{j=1}^n b_j x_j\right) \le \sum_{j=1}^n b_j \phi(x_j)$ as required.

Conclusion

The Steffensen's inequality was discussed for convex and differentiable functions. Simple analytical proofs were provided for the new Steffensen's inequality. The well known Jensen's inequality was also proved from the new inequality.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

The first author wish to extend his appreciation to Frederic SYMESAK for his care and attention during his research period at the University of Angers, France.

REFERENCES

- [1] E. F. Beckenbach, Convex Functions, Bull. Amer. Math. Soc. 54 (1948), 439-460
- [2] R. Bellman, On inequalities with alternating signs, proc. Amer. Math. Soc. 10 (1959), 807-809
- [3] M.M. Iddrisu, C.A. Okpoti, K.A. Gbolagade, Geometrical proof of new Steffensen's inequality and Applications, Adv. Inequal. Appl. 2014, Accepted for publication
- [4] D.S. Mitrinovic, J.E. Pecaric, A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, 1993
- [5] D.S. Mitrinovic and J.E. Pecaric, On the Bellman Generalization of Steffensen's Inequality III, J. Math. Anal. and Appl. 135 (1988), 342-345
- [6] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications: A contemporary approach, Springer, Berlin Heidelberg, 2004
- [7] J.E. Pecaric, On the Bellman Generalization of Steffensen's Inequality, J. Math. Anal. and Appl., 88 (1982), 505–507
- [8] J.E. Pecaric, On the Bellman Generalization of Steffensen's Inequality II, J. Math. Anal. Appl. 104 (1984), 432-434
- [9] J. Steffensen, "Bounds of certain trigonometric integrals," Tenth Scandinavian Mathematical congress (1946), Copenhagan, Hellerup (1947), 181-186