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A SYSTEM OF NONLINEAR INTEGRAL EQUATIONS

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Abstract: The purpose of this paper is to establish some new variations on a system of nonlinear integral equations of one independent variable.

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1. Introduction

The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations,integral equations and inequalities of the various types (please, see Gronwall [7] and Guiliano [8]).

Closely related to the foregoing first-order ordinary differential operators is the following result of Bellman [5]:

Lemma1: If the functions $g(t)$ and $u(t)$ are nonnegative for t $\geq$ 0, and if c $\geq$ 0, then the inequality

$$u(t) \leq c + \int_0^t g(s)u(s)ds, \quad t \geq 0$$

Implies that

$$u(t) \leq c \exp\left(\int_0^t g(s)ds\right), \quad t \geq 0$$

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In analysing the dynamics of a physical system governed by certain differential and integral equations, one often needs some new kinds of inequalities[1-10].Green[6] proved the following interesting inequality, which can be used in the analysis of various problems in the theory of certain systems of simultaneous differential and integral equations.

Let k_1, k_2 and μ be nonnegative constants and let f, g and $h_i, (1,2,3,4)$ be nonnegative continuous functions defined for $t \in R_+$ with h_i bounded such that

$$f(t) \le k_1 + \int_0^t h_1(s) f(s) ds + \int_0^t e^{\mu s} h_2(s) g(s) ds$$

$$g(t) \le k_2 + \int_0^t e^{-\mu s} h_3(s) f(s) ds + \int_0^t h_4(s) g(s) ds$$

For all $t \in R_+$ .Then there exists constants c_1, c_2 and M_1, M_2 such that

$$f(t) \le M_1 e^{c_1 t}, \qquad g(t) \le M_2 e^{c_2 t}$$

For all $t \in R_+$.

2. Main Results:

Theorem 2.1: Let f, g, a, b, p and $h_i, (1,2,3,4)$ be nonnegative continuous functions defined for $t \in R_+$ with h_i bounded such that

$$f(t) \le a(t) + p(t)\left[\int_0^t h_1(s) f(s) ds + \int_0^t e^{p\mu s} h_2(s) g(s) ds\right] \tag{1}$$

$$g(t) \le b(t) + p(t)\left[\int_0^t e^{-p\mu s} h_3(s) f(s) ds + \int_0^t h_4(s) g(s) ds\right] \tag{2}$$

For all $t \in R_+$ .where μ be nonnegative constant and $p \ge 1$. Then

$$f(t) \le e^{p\mu t} Q(t), \qquad g(t) \le Q(t)$$

Where
$$Q(t) = m(t) + p(t)\int_0^t h(s) m(s) \exp\left[\int_s^t h(\partial) p(\partial) d\partial\right] ds \tag{3}$$

In which $m(t) = a(t) + b(t)$ and $h(t) = \max\{[h_1(t) + h_3(t)], [h_2(t) + h_4(t)]\}$ for all $t \in R_+$.

Proof: Multiplying both sides of (1) by $e^{-p\mu t}$,we get

$$e^{-p\mu t} f(t) \le a(t)e^{-p\mu t} + p(t)\left[\int_0^t e^{-p\mu s} h_1(s) f(s)ds + \int_0^t h_2(s)g(s)ds\right] \qquad (4)$$

Since t and μ are positive, we observe that $e^{-p\mu t} < 1$. For $0 \le s \le t$, $0 \ge -p\mu s \ge -p\mu t$ So $e^{-p\mu s} \ge e^{-p\mu t}$ .Therefore (4) can be rewritten as

$$e^{-p\mu t} f(t) \le a(t) + p(t)\left[\int_0^t e^{-p\mu s} h_1(s) f(s)ds + \int_0^t h_2(s)g(s)ds\right] \qquad (5)$$

Now define

$$V(t) = e^{-p\mu t} f(t) + g(t) \qquad (6)$$

By substituting from (2) and (5) in (6),we have

$$V(t) \le m(t) + p(t)\int_0^t h(s)V(s)ds \qquad (7)$$

Where $m(t) = a(t) + b(t)$ and $h(t) = \max\{[h_1(t) + h_3(t)], [h_2(t) + h_4(t)]\}$.

Now an application of Lemma1 in (7) with suitable modifications, yields

$$Z(t) \le \int_0^t h(s)m(s)\exp\left[\int_s^t h(\partial)p(\partial)d\partial\right]ds \qquad (8)$$

Since $V(t) \le Z(t)$, therefore from (7) and (8), we get

$$V(t) \le m(t) + p(t)\int_0^t h(s)m(s)\exp\left[\int_s^t h(\partial)p(\partial)d\partial\right]ds \qquad (9)$$

Substituting the value of $V(t)$ in the right side of (9) ,takes the form

$$e^{-p\mu t} f(t) + g(t) \le m(t) + p(t)\int_0^t h(s)m(s)\exp\left[\int_s^t h(\partial)p(\partial)d\partial\right]ds \qquad (10)$$

By comparing both sides of (10), we have

$$f(t) \le e^{p\mu t} Q(t) \qquad \text{and} \qquad g(t) \le Q(t)$$

Where $Q(t)$ is defined as in (3).

Theorem 2.2: Let k_1, k_2 and μ be nonnegative constants and let f, g and $h_i, (1,2,3,4)$ be nonnegative continuous functions defined for $t \in R_+$ with h_i bounded such that

$$f(t) \le k_1 + \int_0^t h_1(s) f(s) ds + \int_0^t e^{p\mu s} h_2(s) g^p(s) ds \qquad (11)$$

$$g(t) \le k_2 + \int_0^t e^{-p\mu s} h_3(s) f(s) ds + \int_0^t h_4(s) g^p(s) ds \qquad (12)$$

For all $t \in R_+$ .and $p \ge 1$, then there exists constants a_1, a_2 and N_1, N_2 such that

$$f(t) \le N_1 e^{a_1 t}, \qquad g(t) \le N_2 e^{a_2 t}$$

where $N_1 = k_1 + k_2$ and $N_2 = (k_1 + k_2)^{\frac{1}{p}}$. Also $a_1 = p\mu + R, a_2 = \frac{R}{p}, \ R = h(s)$.

Proof: Multiplying both sides of (11) by $e^{-p\mu t}$,we get

$$e^{-p\mu t} f(t) \le k_1 e^{-p\mu t} + \int_0^t e^{-p\mu s} h_1(s) f(s) ds + \int_0^t h_2(s) g^p(s) \qquad (13)$$

Since t and μ are positive, we observe that $e^{-p\mu t} < 1$. For $0 \le s \le t$, $0 \ge -p\mu s \ge -p\mu t$ So $e^{-p\mu s} \ge e^{-p\mu t}$ .Therefore (13) can be rewritten as

$$e^{-p\mu t} f(t) \le k_1 + \int_0^t e^{-p\mu s} h_1(s) f(s) ds + \int_0^t h_2(s) g^p(s) \qquad (14)$$

Now define

$$V(t) = e^{-p\mu t} f(t) + g^p(t) \qquad (15)$$

By substituting from (12) and (14) in (15),we have

$$V(t) \le N_1 + \int_0^t h(s) V(s) ds \qquad (16)$$

Where $N_1 = k_1 + k_2$ and $h(t) = \max\{[h_1(t) + h_3(t)], [h_2(t) + h_4(t)]\}$.

Now an application of Lemma1 in (16) with suitable modifications, yields

$$Z(t) \le N_1 \exp\left[\int_0^t h(s) ds\right] \qquad (17)$$

Since $V(t) \le Z(t)$, therefore from(15), (16) and (17),we get

$$e^{-p\mu t} f(t) + g^{p}(t) \le N_1 \exp\left[\int_0^t h(s)ds\right] \tag{18}$$

By comparing both sides of (18), we have

$$f(t) \le N_1 e^{a_1 t}, \qquad g(t) \le N_2 e^{a_2 t}$$

where $N_1 = k_1 + k_2$ and $N_2 = (k_1 + k_2)^{\frac{1}{p}}$. Also $a_1 = p\mu + R, a_2 = \frac{R}{p}$, $R = h(s)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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