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REFINEMENTS OF THE CLASSICAL JENSEN'S INEQUALITY COMING FROM REFINEMENTS OF THE DISCRETE JENSEN'S INEQUALITY

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Abstract. In this paper new refinements of classical Jensen's inequality are obtained by using some refinements of discrete Jensen's inequality. To apply our refinements, new quasi-arithmetic means are introduced, the properties of these means are studied, and refinements of the left hand side of the Hermite-Hadamard inequality are given.

Keywords: Classical and discrete Jensen's inequality; Convex function, Quasi-arithmetic means, Hermite-Hadamard inequality.

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1. Introduction

Jensen's inequality is one of the most important inequality in mathematics. The key to this inequality is convexity: the real function f defined on a convex subset C of a real vector space V is called convex if

$$f(\alpha x+(1-\alpha)y) \leq \alpha f(x)+(1-\alpha)f(y), \quad x,y \in C, \quad 0 \leq \alpha \leq 1.$$

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The measure theoretical setting of Jensens inequality:

Theorem A. (classical Jensen's inequality, see [6]) *Let g be an integrable function on a probability space $(X,\mathscr{A},\mu)$ taking values in an interval $I\subset\mathbb{R}$. Then $\int_X g d\mu$ lies in I. If f is a convex function on I such that $f\circ g$ is integrable, then*

$$f\left(\int_X g d\mu\right)\leq\int_X f\circ g d\mu. \tag{1}$$

The discrete version of Jensen's inequality:

Theorem B. (discrete Jensen's inequality, see [6]) *Let C be a convex subset of a real vector space V, and let $f:C\to\mathbb{R}$ be a convex function. If $p_1,\ldots,p_n$ are nonnegative numbers with $\sum_{i=1}^{n}p_i=1$, and $v_1,\ldots,v_n\in C$, then*

$$f\left(\sum_{i=1}^{n}p_iv_i\right)\leq\sum_{i=1}^{n}p_if\left(v_i\right). \tag{2}$$

In recent years, many papers dealing with refinements of discrete Jensen's inequality have appeared in the literature, see the recent monograph by Horváth, Khuram Ali Khan and Pečarić [6], where further references are given. However, refinements of classical Jensen's inequality have been much less extensively studied, results are quite rare, see Horváth [3] and [5]. A natural and interesting problem is whether we can construct refinements of classical Jensen's inequality by using some refinements of discrete Jensen's inequality. The next result in this direction is obtained by Brnetić, Pearce and Pečarić [1] (the existence of all integrals are supposed):

Theorem 1. *Suppose $I\subset\mathbb{R}$ is an interval. Let $f:I\to\mathbb{R}$ be a convex function, $g:[a,b]\to I$, and $w:[a,b]\to\mathbb{R}$ a positive function. Let $p_1,\ldots,p_n$ be positive numbers with $\sum_{i=1}^{n}p_i=1$, and $\overline{w}:=\int_a^b w\left(x\right)dx$. Further, let*

$$\mathscr{F}_{k,n}:=\frac{1}{\binom{n-1}{k-1}\overline{w}^k}$$

$$\cdot \sum_{1\leq 1<i_1<\ldots<i_k\leq n} \sum_{r=1}^{k} p_{i_r} \int_a^b \ldots \int_a^b \left(\prod_{s=1}^{k} w\left(x_{i_s}\right)\right) f\left(\frac{\sum_{j=1}^{k} p_{i_j} g\left(x_{i_j}\right)}{\sum_{j=1}^{k} p_{i_j}}\right) dx_{i_1}\ldots dx_{i_k}.$$

Then

$$f\left(\frac{1}{\overline{w}}\int_a^b w(x)\,g(x)\,dx\right) \leq \mathscr{F}_{n,n} \leq \ldots \leq \mathscr{F}_{k+1,n} \leq \mathscr{F}_{k,n} \leq \ldots \leq \mathscr{F}_{1,n}$$

$$= \frac{1}{\overline{w}}\int_a^b w(x)\,f(g(x))\,dx.$$

The previous result is based on the following well-known refinements of discrete Jensen's inequality (see Pečarić [8] and Horváth and Pečarić [4]):

Theorem 2. *Let C be a convex subset of a real vector space V, and let $f : C \to \mathbb{R}$ be a convex function. If $x_1,\ldots,x_n \in C$, $p_1,\ldots,p_n$ are positive numbers with $\sum_{i=1}^{n} p_i = 1$, and*

$$F_{k,k} = \frac{1}{\binom{n-1}{k-1}} \sum_{1\leq i_1<\ldots<i_k\leq n} \left(\sum_{s=1}^{k} p_{i_s}\right) f\left(\frac{\sum_{s=1}^{k} p_{i_s} x_{i_s}}{\sum_{s=1}^{k} p_{i_s}}\right), \quad k = 1,\ldots,n.$$

then

$$f\left(\sum_{r=1}^{n} p_r x_r\right) = F_{n,n} \leq \ldots \leq F_{k,k} \leq F_{k-1,k-1} \leq \ldots \leq F_{1,1} = \sum_{r=1}^{n} p_r f(x_r).$$

In this paper, among others, we obtain new refinements of classical Jensen's inequality by using an essential extension of Theorem 2 by Horváth and Pečarić [4]. The methods and results are somewhat similar to those of Brnetić, Pearce and Pečarić [1], but there are key differences. To apply our refinements, new quasi-arithmetic means are introduced, the properties of these means are studied, and refinements of the left hand side of the Hermite-Hadamard inequality are given.

2. New refinements of the classical Jensen's inequality

First, we recall some of the essential basic tools from [4] needed in the rest of the paper.

Let X be a set. The power set of X is denoted by $P(X)$. $|X|$ means the number of elements in X. The usual symbol $\mathbb{N}$ is used for the set of natural numbers (including 0). Let $u \geq 1$ and $v \geq 2$ be fixed integers. Define the functions

$$S_{v,w} : \{1,\ldots,u\}^v \to \{1,\ldots,u\}^{v-1}, \quad 1 \leq w \leq v,$$

$$S_v : \{1,\ldots,u\}^v \to P\left(\{1,\ldots,u\}^{v-1}\right),$$

and

$$T_v : P(\{1,\ldots,u\}^v) \to P\left(\{1,\ldots,u\}^{v-1}\right)$$

by

$$S_{v,w}(i_1,\ldots,i_v) := (i_1,i_2,\ldots,i_{w-1},i_{w+1},\ldots,i_v), \quad 1 \leq w \leq v,$$

$$S_v(i_1,\ldots,i_v) := \bigcup_{w=1}^{v} \{S_{v,w}(i_1,\ldots,i_v)\},$$

and

$$T_v(I) := \begin{cases} \emptyset, & \text{if} \quad I = \emptyset, \\ \bigcup\limits_{(i_1,\ldots,i_v)\in I} S_v(i_1,\ldots,i_v), & \text{if} \quad I \neq \emptyset. \end{cases}$$

Next, let the function

$$\alpha_{v,i} : \{1,\ldots,u\}^v \to \mathbb{N}, \quad 1 \leq i \leq u, \tag{3}$$

be given by: $\alpha_{v,i}(i_1,\ldots,i_v)$ means the number of occurrences of i in the sequence $(i_1,\ldots,i_v)$. For each $I \in P(\{1,\ldots,u\}^v)$ let

$$\alpha_{I,i} := \sum_{(i_1,\ldots,i_v)\in I} \alpha_{v,i}(i_1,\ldots,i_v), \quad 1 \leq i \leq u.$$

The following hypothesis plays an important rule in the generalization of Theorem 2.

(H_1) Let $n \geq 1$ and $k \geq 2$ be fixed integers, and let I_k be a subset of $\{1,\ldots,n\}^k$ such that

$$\alpha_{I_k,i} \geq 1, \quad 1 \leq i \leq n.$$

Starting from I_k, we introduce the sets $I_l \subset \{1,\ldots,n\}^l$ $(k-1 \geq l \geq 1)$ inductively by

$$I_{l-1} := T_l(I_l), \quad k \geq l \geq 2.$$

Obviously, $I_1 = \{1, \ldots, n\}$, by (3), and this insures that $\alpha_{I_1,i} = 1$ $(1 \leq i \leq n)$. From (3) again, we have that $\alpha_{I_l,i} \geq 1$ $(k-1 \geq l \geq 1, 1 \leq i \leq n)$. For any $k \geq l \geq 2$ and for any $(j_1, \ldots, j_{l-1}) \in I_{l-1}$ let

$$H_{I_l}(j_1, \ldots, j_{l-1})$$
$$:= \left\{((i_1, \ldots, i_l), m) \in I_l \times \{1, \ldots, l\} \mid S_{l,m}(i_1, \ldots, i_l) = (j_1, \ldots, j_{l-1})\right\}.$$

Using these sets we define the functions $t_{I_k,l} : I_l \to \mathbb{N}$ $(k \geq l \geq 1)$ inductively by

$$t_{I_k,k}(i_1, \ldots, i_k) := 1, \quad (i_1, \ldots, i_k) \in I_k;$$
$$t_{I_k,l-1}(j_1, \ldots, j_{l-1}) := \sum_{((i_1,\ldots,i_l),m) \in H_{I_l}(j_1,\ldots,j_{l-1})} t_{I_k,l}(i_1, \ldots, i_l).$$

We use some special expressions, which we now describe. Associate to each $k \geq l \geq 1$ the number

$$A_{k,l} = A_{k,l}(I_k, x_1, \ldots, x_n, p_1, \ldots, p_n)$$
$$:= \frac{(k-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l) \in I_l} t_{I_k,l}(i_1, \ldots, i_l) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s}}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right).$$

Now, we are in a position to formulate the main result in [4] which is an essential generalization of Theorem 2 (see the examples of this paper).

Theorem 3. *Assume* (H_1). *Let* C *be a convex subset of a real vector space* V, *and let* $f : C \to \mathbb{R}$ *be a convex function. If* $x_1, \ldots, x_n \in C$, $p_1, \ldots, p_n$ *are positive numbers with* $\sum_{i=1}^{n} p_i = 1$, *then*

(a)

$$f\left(\sum_{r=1}^{n} p_r x_r\right) \leq A_{k,k} \leq A_{k,k-1} \leq \ldots \leq A_{k,2} \leq A_{k,1} = \sum_{r=1}^{n} p_r f(x_r). \tag{4}$$

(b) Suppose $|H_{I_l}(j_1, \ldots, j_{l-1})| = \beta_{l-1}$ *for any* $(j_1, \ldots, j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$. *Then*

$$A_{k,l} = \frac{n}{l\,|I_l|} \sum_{(i_1,\ldots,i_l) \in I_l} \left(\sum_{s=1}^{l} p_{i_s}\right) f\left(\frac{\sum_{s=1}^{l} p_{i_s} x_{i_s}}{\sum_{s=1}^{l} p_{i_s}}\right), \quad (k \geq l \geq 1).$$

We list further hypotheses that will be used to derive our main results.

(H_2) Let $(X,\mathscr{B},\mu)$ be a probability space, and let $p_1,\ldots,p_n$ be positive numbers with $\sum_{i=1}^{n} p_i = 1$.

(H_3) Let g be an integrable function on X taking values in an interval $I \subset \mathbb{R}$.

(H_4) Let f be a convex function on I such that $f \circ g$ is integrable.

Let $l \geq 2$ be a fixed integer. The σ-algebra in X^l generated by the projection mappings $pr_m :$

$X^l \to X \ (m = 1,\ldots,l)$

$$pr_m(x_1,\ldots,x_l) := x_m$$

is denoted by $\mathscr{B}^l$. μ^l means the product measure on $\mathscr{B}^l$: this measure is uniquely (μ is σ-finite) specified by

$$\mu^l(B_1 \times \ldots \times B_l) := \mu(B_1)\ldots\mu(B_l), \quad B_m \in \mathscr{B}, \quad m = 1,\ldots,l.$$

To establish a new refinement of the classical Jensen's inequality, we begin with the introduction of some expressions and a function: under the hypotheses (H_1)-(H_4), assign to each $k \geq l \geq 1$ the number

$$\mathscr{A}_{k,l} = \mathscr{A}_{k,l}(I_k,f,g,\mu,p_1,\ldots,p_n) := \frac{(k-1)!}{(l-1)!} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(i_1,\ldots,i_l)$$

$$\cdot \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) \int_{X^n} f\left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} g(x_{i_s})}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right) d\mu^n(x_1,\ldots,x_n), \tag{5}$$

and define the function H_k on $[0,1]$ by

$$H_k(t) = H_k(t,I_k,f,g,\mu,p_1,\ldots,p_n) := \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right)$$

$$\cdot \int_{X^n} f\left(t\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} g(x_{i_s})}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} + (1-t)\int_X g\,d\mu \right) d\mu^n(x_1,\ldots,x_n). \tag{6}$$

Remark 4. *(a) It follows from Lemma 2.1 (b) in* [3] *that the integrals in (5) exist and finite (there are at most n different between the variables* $x_{i_1},\ldots,x_{i_l}$ *for every* $k \geq l \geq 1$*). Let* $l \in \{1,\ldots,k\}$*, and let* $(i_1,\ldots,i_l) \in I_l$ *be fixed. If* $i_1,\ldots,i_l$ *are all different (in this case* $n \geq l \geq 1$*), then*

$$\int_{X^n} f\left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} g\left(x_{i_s}\right)}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right) d\mu^n\left(x_1,\ldots,x_n\right)$$

$$= \int_{X^l} f\left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} g\left(x_{i_s}\right)}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right) d\mu^l\left(x_{i_1},\ldots,x_{i_l}\right),$$

while if there are identical members, then similar integral can be obtained after contraction. For the sake of lucidity, we use integrals over X^n*.*

(b) If μ *is a finite measure on* $\mathscr{B}$ *such that* $\mu(X) \neq 0$*, then* $\frac{\mu}{\mu(X)}$ *is a probability measure on* $\mathscr{B}$*, and we can therefore extend our results to finite measures on* $\mathscr{B}$*.*

(c) Slight modification of the proof of Lemma 2.1 (b) in [3] *shows that the integrals in (6) exist and finite for every* $t \in [0,1]$*.*

First of all, some important properties of the function H_k are derived in the next result.

Theorem 5. *Assume (*H_1*)-(*H_4*). Then*

(a) H_k *is convex.*

(b)

$$\min_{t\in[0,1]} H_k(t) = H_k(0) = f\left(\int_X g\,d\mu\right),$$

$$\max_{t\in[0,1]} H_k(t) = H_k(1) = \mathscr{A}_{k,k}.$$

(c) H_k *is increasing.*

After these preparations we can present the following new refinements of the classical Jensen's inequality.

Theorem 6. *Assume (H_1)-(H_4).*

(a) Then for every $t \in [0,1]$

$$f\left(\int_X g d\mu\right) \leq H_k(t) \leq \mathscr{A}_{k,k} \leq \mathscr{A}_{k,k-1} \leq \ldots \leq \mathscr{A}_{k,2} \leq \mathscr{A}_{k,1} = \int_X f \circ g d\mu.$$

(b) Suppose $|H_{I_l}(j_1,\ldots,j_{l-1})| = \beta_{l-1}$ for any $(j_1,\ldots,j_{l-1}) \in I_{l-1}$ $(k \geq l \geq 2)$.

Then for any $k \geq l \geq 1$

$$\mathscr{A}_{l,l} := \mathscr{A}_{k,l} = \frac{n}{l\,|I_l|} \sum_{(i_1,\ldots,i_l)\in I_l} \left(\sum_{s=1}^{l} p_{i_s}\right) \int_{X^n} f\left(\frac{\sum_{s=1}^{l} p_{i_s} g(x_{i_s})}{\sum_{s=1}^{l} p_{i_s}}\right) d\mu^n(x_1,\ldots,x_n),$$

and

$$H_k(t) = \frac{n}{k\,|I_k|} \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} p_{i_s}\right)$$

$$\cdot \int_{X^n} f\left(t\frac{\sum_{s=1}^{k} p_{i_s} g(x_{i_s})}{\sum_{s=1}^{k} p_{i_s}} + (1-t)\int_X g d\mu\right) d\mu^n(x_1,\ldots,x_n), \quad t \in [0,1].$$

It will be seen in the next section that the previous result is a significant sharpening of Theorem 1.

3. Examples in special cases

First, we give some special cases of Theorem 6.

Proposition 7. *Let*

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 < \ldots < i_k\right\}, \quad 1 \leq k \leq n,$$

and assume (H_2)-(H_4). Then

$$f\left(\int_X g d\mu\right) \leq H_n(t) \leq \mathscr{A}_{n,n}$$

$$\leq \mathscr{A}_{n-1,n-1} \leq \ldots \leq \mathscr{A}_{2,2} \leq \mathscr{A}_{1,1} = \int_X f \circ g d\mu, \quad t \in [0,1],$$

where for every $n \geq l \geq 1$

$$\mathscr{A}_{l,l} = \frac{1}{\binom{n-1}{l-1}} \sum_{1 \leq i_1 < \ldots < i_l \leq n} \left(\sum_{s=1}^{l} p_{i_s} \right) \int_{X^l} f \left(\frac{\sum_{s=1}^{l} p_{i_s} g(x_{i_s})}{\sum_{s=1}^{l} p_{i_s}} \right) d\mu^l (x_{i_1}, \ldots, x_{i_l}),$$

and for each $t \in [0,1]$

$$H_n(t) = \int_{X^n} f \left(t \sum_{s=1}^{n} p_s g(x_s) + (1-t) \int_X g d\mu \right) d\mu^n (x_1, \ldots, x_n).$$

Proof We deduce this from Theorem 6 by using Example 2 in [4].

By considering Remark 4 (b), it follows that Theorem 1 is a special case of the previous result.

Proposition 8. *Let*

$$I_k := \left\{ (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \mid i_1 \leq \ldots \leq i_k \right\}, \quad k \geq 1,$$

and assume (H_2)-(H_4). Then

(a)

$$f\left(\int_X g d\mu \right) \leq \ldots \leq \mathscr{A}_{l,l} \leq \mathscr{A}_{l-1,l-1} \leq \ldots \leq \mathscr{A}_{2,2} \leq \mathscr{A}_{1,1} = \int_X f \circ g d\mu,$$

where for every $l \geq 1$

$$\mathscr{A}_{l,l} = \frac{1}{\binom{n+l-1}{l-1}} \sum_{1 \leq i_1 \leq \ldots \leq i_l \leq n} \left(\sum_{s=1}^{l} p_{i_s} \right) \int_{X^n} f \left(\frac{\sum_{s=1}^{l} p_{i_s} g(x_{i_s})}{\sum_{s=1}^{l} p_{i_s}} \right) d\mu^n (x_1, \ldots, x_n).$$

(b) For every fixed $k \geq 1$

$$f\left(\int_X g d\mu \right) \leq \frac{1}{\binom{n+k-1}{k-1}} \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \left(\sum_{s=1}^{k} p_{i_s} \right)$$

$$\cdot \int_{X^n} f \left(t \frac{\sum_{s=1}^{k} p_{i_s} g(x_{i_s})}{\sum_{s=1}^{k} p_{i_s}} + (1-t) \int_X g d\mu \right) d\mu^n (x_1, \ldots, x_n) \leq \mathscr{A}_{k,k}, \quad t \in [0,1].$$

Proof. It can be obtained from Theorem 6 by using Example 3 in [4].

Proposition 9. *Let*

$$I_k := \{1, \ldots, n\}^k, \quad k \geq 1,$$

and assume (H_2)*-*(H_4)*. Then*

(a)

$$f\left(\int_X g d\mu\right) \leq \ldots \leq \mathscr{A}_{l,l} \leq \mathscr{A}_{l-1,l-1} \leq \ldots \leq \mathscr{A}_{2,2} \leq \mathscr{A}_{1,1} = \int_X f \circ g d\mu,$$

where for every $l \geq 1$

$$\mathscr{A}_{l,l} = \frac{1}{n^{l-1} l} \sum_{(i_1,\ldots,i_l) \in \{1,\ldots,n\}^l} \left(\sum_{s=1}^{l} p_{i_s}\right) \int_{X^n} f\left(\frac{\sum_{s=1}^{l} p_{i_s} g\left(x_{i_s}\right)}{\sum_{s=1}^{l} p_{i_s}}\right) d\mu^n\left(x_1, \ldots, x_n\right).$$

(b) For every fixed $k \geq 1$

$$f\left(\int_X g d\mu\right) \leq \frac{1}{n^{k-1} k} \sum_{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k} \left(\sum_{s=1}^{k} p_{i_s}\right)$$

$$\cdot \int_{X^n} f\left(t \frac{\sum_{s=1}^{k} p_{i_s} g\left(x_{i_s}\right)}{\sum_{s=1}^{k} p_{i_s}} + (1-t) \int_X g d\mu\right) d\mu^n\left(x_1, \ldots, x_n\right) \leq \mathscr{A}_{k,k}, \quad t \in [0,1].$$

Proof. It follows from Theorem 6 by using Example 4 in [4].

Proposition 10. *Let* $c_i \geq 1$ *be an integer* $(i = 1, \ldots, n)$*, let* $k := \sum_{i=1}^{n} c_i$*, and let* $I_k = P^{c_1,\ldots,c_n}$ *consist of all sequences* $(i_1, \ldots, i_k)$ *in which the number of occurrences of* $i \in \{1, \ldots, n\}$ *is* c_i $(i = 1, \ldots, n)$*. Assume* (H_2)*-*(H_4)*. Then*

$$f\left(\int_X g d\mu\right) \leq \mathscr{A}_{k,k-1} \leq \int_X f \circ g d\mu,$$

where

$$\mathscr{A}_{k,k-1} = \frac{1}{k-1} \sum_{i=1}^{n} \left(c_i - p_i\right) \int_{X^n} f\left(\frac{\sum_{r=1}^{n} p_r g\left(x_r\right) - \frac{p_i}{c_i} g\left(x_i\right)}{1 - \frac{p_i}{c_i}}\right) d\mu^n\left(x_1, \ldots, x_n\right).$$

Proof. This is a consequence of Theorem 6 and Example 5 in [4].

4. New quasi-arithmetic means

In this section some new quasi-arithmetic means are introduced. The conditions (H_3) and (H_4) will be changed:

($\hat{H}_3$) Let g be a measurable function on X taking values in an interval $I \subset \mathbb{R}$.

($\hat{H}_4$) Let $\varphi,\ \psi : I \to \mathbb{R}$ be continuous and strictly monotone functions.

Definition 11. *Assume (H_1)-(H_2) and ($\hat{H}_3$)-($\hat{H}_4$).*

(a) For $k \geq l \geq 1$, we define quasi-arithmetic means with respect to (5) by

$$M_{\psi,\varphi}(I_k, g, \mu, l) := \psi^{-1}\left(\frac{(k-1)!}{(l-1)!}\sum_{(i_1,\dots,i_l)\in I_l} t_{I_k,l}(i_1,\dots,i_l)\right.$$

$$\left.\cdot\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)\int_{X^n}(\psi\circ\varphi^{-1})\left(\frac{\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\varphi(g(x_{i_s}))}{\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)d\mu^n(x_1,\dots,x_n)\right), \tag{7}$$

if the integrals exist and finite.

(b) For $t \in [0,1]$, different kind of quasi-arithmetic means can be defined with respect to (6) by

$$M_{\psi,\varphi}(t, I_k, g, \mu) := \psi^{-1}\left(\sum_{(i_1,\dots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)\right. \tag{8}$$

$$\left.\cdot\int_{X^n}(\psi\circ\varphi^{-1})\left(t\frac{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\varphi(g(x_{i_s}))}{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}+(1-t)\int_X\varphi\circ g\,d\mu\right)d\mu^n(x_1,\dots,x_n)\right)$$

if the integrals exist and finite.

$M_{\psi,\varphi}(I_k, g, \mu, l)$ and $M_{\psi,\varphi}(t, I_k, g, \mu)$ really define means, since it follows from the proof of Lemma 9 in [4] that

$$\frac{(k-1)!}{(l-1)!}\sum_{(i_1,\dots,i_l)\in I_l} t_{I_k,l}(i_1,\dots,i_l)\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) = 1, \quad k \geq l \geq 1. \tag{9}$$

If $\varphi \circ g$ and $\psi \circ g$ are integrable on X, and $\psi \circ \varphi^{-1}$ is either convex or concave, then the integrals in (7) exist and finite (see Remark 4).

Based on the results of [3], quasi-arithmetic means like (7) have been obtained in the work Khuram Ali Khan and Pečarić [7]. Their construction involves not just one but finitely many probability spaces. However, they use only the set (see Proposition 7)

$$I_n = \{(1, \ldots, n)\},$$

and just consider the mean $M_{\psi,\varphi}(I_n, g, \mu, n)$.

To investigate the monotonicity of the introduced means, some other means are also needed. Assume ($\hat{\mathrm{H}}_3$), and let $h : I \to \mathbb{R}$ be a continuous and strictly monotone function such that $h \circ g$ is integrable on X. Define the mean

$$M_h(g, \mu) := h^{-1}\left(\int_X h \circ g d\mu\right).$$

Theorem 12. *Assume* (H_1)-(H_2), ($\hat{H}_3$)-($\hat{H}_4$), *and assume that* $\varphi \circ g$ *and* $\psi \circ g$ *are integrable on* X.*Then*

(a)

$$M_\varphi(g, \mu) \le M_{\psi,\varphi}(t, I_k, g, \mu) \le M_{\psi,\varphi}(I_k, g, \mu, k)$$
$$\le \ldots \le M_{\psi,\varphi}(I_k, g, \mu, 1) = M_\psi(g, \mu), \quad t \in [0,1],$$

if either $\psi \circ \varphi^{-1}$ *is convex and* ψ *is increasing or* $\psi \circ \varphi^{-1}$ *is concave and* ψ *is decreasing.*

(b)

$$M_\varphi(g, \mu) \ge M_{\psi,\varphi}(t, I_k, g, \mu) \ge M_{\psi,\varphi}(I_k, g, \mu, k)$$
$$\ge \ldots \ge M_{\psi,\varphi}(I_k, g, \mu, 1) = M_\psi(g, \mu), \quad t \in [0,1],$$

if either $\psi \circ \varphi^{-1}$ *is convex and* ψ *is decreasing or* $\psi \circ \varphi^{-1}$ *is concave and* ψ *is increasing.*

Proof. Theorem 6 (a) can be applied to the pair of functions $\psi \circ \varphi^{-1}$ and $\varphi \circ g$ ($\varphi(I)$ is an interval), if $\psi \circ \varphi^{-1}$ is convex, and to the pair of functions $-\psi \circ \varphi^{-1}$ and $\varphi \circ g$, if $\psi \circ \varphi^{-1}$ is concave, then upon taking ψ^{-1}, we get (a) and (b).

5. Refinements of the left hand side of the Hermite-Hadamard inequality

The set of Borel subsets of $[a,b] \subset \mathbb{R}$ is denoted by $\mathscr{B}$. λ means the Lebesgue measure on $\mathscr{B}$. The classical Hermite-Hadamard inequality (see [2]) says: if $f : [a,b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f d\lambda \leq \frac{f(a)+f(b)}{2}.$$

We can obtain the following refinements of the left hand side of the Hermite-Hadamard inequality:

Theorem 13. *Assume (H_1), let $[a,b] \subset \mathbb{R}$ $(a < b)$, and let $p_1, \ldots, p_n$ be positive numbers with $\sum_{i=1}^{n} p_i = 1$. If $f : [a,b] \to \mathbb{R}$ is a convex function, then*

$$f\left(\frac{a+b}{2}\right) \leq H_k^s(t)$$

$$\leq \mathscr{A}_{k,k}^s \leq \mathscr{A}_{k,k-1}^s \leq \ldots \leq \mathscr{A}_{k,2}^s \leq \mathscr{A}_{k,1}^s = \frac{1}{b-a}\int_a^b f d\lambda, \quad t \in [0,1],$$

where for every $k \geq l \geq 1$

$$\mathscr{A}_{k,l}^s := \frac{1}{(b-a)^n}\frac{(k-1)!}{(l-1)!}\sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(i_1,\ldots,i_l)$$

$$\cdot\left(\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)\int_{[a,b]^n} f\left(\frac{\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}x_{i_s}}{\sum_{s=1}^{l}\frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right) d\lambda^n(x_1,\ldots,x_n),$$

and for each $t \in [0,1]$

$$H_k^s(t) := \frac{1}{(b-a)^n}\sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)$$

$$\cdot\int_{[a,b]^n} f\left(t\frac{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}x_{i_s}}{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}} + (1-t)\frac{a+b}{2}\right) d\lambda^n(x_1,\ldots,x_n).$$

Proof. Theorem 6 (a) can be applied with $X = [a,b]$, $\mu = \frac{1}{(b-a)}\lambda$ and $g(x) = x$ $(x \in [a,b])$.

If (see Proposition 7)

$$I_k := \left\{(i_1,\ldots,i_k) \in \{1,\ldots,n\}^k \mid i_1 < \ldots < i_k\right\}, \quad 1 \leq k \leq n,$$

then the main results in the paper Yang and Wang [9] contain

$$f\left(\frac{a+b}{2}\right) \le H_k^s(t) \le \mathscr{A}_{k,k}^s \le \mathscr{A}_{k,k-1}^s \le \frac{1}{b-a}\int_a^b f d\lambda, \quad t \in [0,1],$$

which is a very special case of the previous theorem.

6. Proofs of Theorem 5 and 6

Proof. [Proof of Theorem 5](a) Since the function

$$t \to t\frac{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}g(x_{i_s})}{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}} + (1-t)\int_X g d\mu, \quad t \in [0,1]$$

is linear for each $(i_1,\ldots,i_k) \in I_k$ and for all $x_{i_1},\ldots,x_{i_k} \in X$, the convexity of f and the monotonicity of the integral yield that the function

$$t \to \int_{X^n} f\left(t\frac{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}g(x_{i_s})}{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}} + (1-t)\int_X g d\mu\right) d\mu^n(x_1,\ldots,x_n), \quad t \in [0,1]$$

is convex for every $(i_1,\ldots,i_k) \in I_k$. The result follows from this, because the sum of convex functions is also convex.

(b) By the classical Jensen's inequality

$$H_k(t) \ge \sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)$$

$$\cdot f\left(\int_{X^n}\left(t\frac{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}g(x_{i_s})}{\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}} + (1-t)\int_X g d\mu\right) d\mu^n(x_1,\ldots,x_n)\right)$$

$$= \sum_{(i_1,\ldots,i_k)\in I_k}\left(\sum_{s=1}^{k}\frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(t\int_X g d\mu + (1-t)\int_X g d\mu\right)$$

$$= \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) f\left(\int_X g d\mu \right) = f\left(\int_X g d\mu \right) = H_k(0), \quad t\in[0,1]. \tag{10}$$

(9) for $l=k$ was used in (10).

It follows from this and from (a) that

$$H_k(t) \le tH_k(0) + (1-t)H_k(1) \le tH_k(1) + (1-t)H_k(1) = H_k(1), \quad t\in[0,1].$$

(c) Suppose $0\le t_1 < t_2 \le 1$. The convexity of H_k, and $H_k(t)\ge H_k(0)$ $(t\in[0,1])$ imply that

$$\frac{H_k(t_2)-H_k(t_1)}{t_2-t_1} \ge \frac{H_k(t_2)-H_k(0)}{t_2} \ge 0,$$

and thus

$$H_k(t_2) \ge H_k(t_1).$$

The proof is complete.

Proof. [Proof of Theorem 6](a) The proof is divided into four parts.

First, we prove that

$$f\left(\int_X g d\mu \right) \le \mathscr{A}_{k,k}.$$

By Lemma 2.1 (a) in [3]

$$f\left(\int_X g d\mu \right) = f\left(\int_{X^n} \sum_{s=1}^{n} p_s g(x_s)\, d\mu^n(x_1,\ldots,x_n) \right),$$

and therefore the classical Jensen's inequality and the first inequality in (4) yield

$$f\left(\int_X g d\mu \right) \le \int_{X^n} f\left(\sum_{s=1}^{n} p_s g(x_s) \right) d\mu^n(x_1,\ldots,x_n)$$

$$\le \sum_{(i_1,\ldots,i_k)\in I_k} \left(\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} \right) \int_{X^n} f\left(\frac{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}} x_{i_s}}{\sum_{s=1}^{k} \frac{p_{i_s}}{\alpha_{I_k,i_s}}} \right) d\mu^n(x_1,\ldots,x_n) = \mathscr{A}_{k,k}.$$

Now, we show that

$$\mathscr{A}_{k,l} \le \mathscr{A}_{k,l-1}, \quad k\ge l\ge 2. \tag{11}$$

Let $l \in \{2, \ldots k\}$. It comes from Theorem 3 (a) that for every fixed $x_1, \ldots, x_n \in X$

$$\frac{1}{(k-1)\ldots l} \sum_{(i_1,\ldots,i_l)\in I_l} t_{I_k,l}(i_1,\ldots,i_l) \left(\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right) f\left(\frac{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}} g(x_{i_s})}{\sum_{s=1}^{l} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right)$$

$$\leq \frac{1}{(k-1)\ldots(l-1)} \sum_{(i_1,\ldots,i_{l-1})\in I_{l-1}} t_{I_k,l-1}(i_1,\ldots,i_{l-1}) \left(\sum_{s=1}^{l-1} \frac{p_{i_s}}{\alpha_{I_k,i_s}}\right)$$

$$\cdot f\left(\frac{\sum_{s=1}^{l-1} \frac{p_{i_s}}{\alpha_{I_k,i_s}} g(x_{i_s})}{\sum_{s=1}^{l-1} \frac{p_{i_s}}{\alpha_{I_k,i_s}}}\right),$$

thus (11) can be obtained by integrating over X^n both sides of the inequality.

Next, we prove that

$$\mathscr{A}_{k,1} = \int_X f \circ g d\mu.$$

By applying Lemma 9 in [4], it follows that

$$\mathscr{A}_{k,1} = \frac{1}{(k-1)!} \sum_{i=1}^{n} t_{I_k,1}(i) \frac{p_i}{\alpha_{I_k,i}} \int_{X^n} f(g(x_i))\, d\mu^n(x_1,\ldots,x_n) = \int_X f \circ g d\mu.$$

Finally, the task of confirming

$$f\left(\int_X g d\mu\right) \leq H_k(t) \leq \mathscr{A}_{k,k}, \quad t \in [0,1]$$

remains. We can derive it from Theorem 5 (b) and (c).

(b) Theorem 3 (b) insures it.

The proof is complete.

Conflict of Interests

The authors declare that there is no conflict of interests.

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