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FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR A NEW SUBCLASS OF MEROMORPHIC BI-UNIVALENT FUNCTIONS

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Abstract. In this article, we introduce a new subclass of meromorphic bi-univalent functions and obtain the general coefficient estimates for such functions in this class. For this purpose, we use the Faber polynomial expansions. In certain cases, our estimates improve some of existing coefficient bounds.

Keywords: Meromorphic univalent functions; Meromorphic bi-univalent functions; Faber polynomial expansions.

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1. Introduction

Let Σ denote the class of meromorphic univalent functions f of the form

$$f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{1.1}$$

defined on the domain $\triangle = \{z : z \in \mathbb{C} \;\; and \;\; 1 < |z| < \infty\}$. It is well known that every function $f \in \Sigma$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, (z \in \triangle),$$

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and

$$f(f^{-1}(w)) = w, (M < |w| < \infty,\ M > 0).$$

For $f \in \Sigma$ given by (1.1), the inverse map $g = f^{-1}$ has the following Faber polynomial expansion:

$$f^{-1}(w) = g(w) = w + \sum_{n=0}^{\infty} \frac{B_n}{w^n} = w - b_0 - \frac{b_1}{w} - \frac{b_1 b_0 + b_2}{w^2} - \frac{b_1^2 + b_1 b_0^2 + 2b_0 b_2 + b_3}{w^3} + \dots$$

$$= w - b_0 - \sum_{n \geq 1}^{\infty} \frac{1}{n} K_{n+1}^n \frac{1}{w^n}, \ \ (w \in \triangle), \tag{1.2}$$

where

$$\begin{aligned} K_{n+1}^n = {} & n b_0^{n-1} b_1 + n(n-1) b_0^{n-2} b_2 + \frac{1}{2} n(n-1)(n-2) b_0^{n-3} (b_3 + b_1^2) \\ & + \frac{n(n-1)(n-2)(n-3)}{3!} b_0^{n-4} (b_4 + 3b_1 b_2) + \sum_{k \geq 5} b_0^{n-k} V_k, \end{aligned} \tag{1.3}$$

and V_k with $5 \leq k \leq n$ is a homogeneous polynomial of degree k in the variables $b_1, b_2, \dots, b_n$. (See [1], [2], and [3]).

Analogous to the bi-univalent analytic functions, a function $f \in \Sigma$ is said to be meromorphic and bi-univalent if $f^{-1} \in \Sigma$. The class of all meromorphic and bi-univalent functions is denoted by $\Sigma_{\mathscr{M}}$.

The coefficient problem was investigated for various subclasses of the meromorphic univalent functions, for example, Schiffer [4] obtained the estimate $|b_2| \leq \frac{2}{3}$ for meromorphic univalent functions $f \in \Sigma$ with $b_0 = 0$. In 1983, Duren [5] obtained the inequality $|b_n| \leq \frac{2}{n+1}$ for $f \in \Sigma$ with $b_k = 0,\ 1 \leq k < \frac{n}{2}$. For the coefficient of the inverse of meromorphic univalent functions $g \in \Sigma_{\mathscr{M}}$, Springer [6] proved that

$$|B_3| \leq 1 \quad \textit{and} \quad |B_3 + \frac{1}{2} B_1^2| \leq \frac{1}{2},$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-1)!}{n!(n-1)!} \quad (n = 1, 2, \dots).$$

In 1977, Kubota [7] has proved that the Springer conjecture is true for $n = 3, 4, 5$, and subsequently Schober [8] obtained sharp bounds for $|B_{2n-1}|$ if $1 \leq n \leq 7$. In 2007, Kapoor and Mishra [9] found the coefficient estimates for the inverse of meromorphic starlike univalent functions of order α in $\triangle$. In 2011, Srivastava et al. [10] found sharp bounds for the coefficients of inverses of starlike univalent functions of order $\alpha(0 \leq \alpha < 1)$ having m-fold gap series representation. The interests in this direction is increasing. Recently, Hamidi et al. [11] introduced the following class of meromorphic bi-univalent functions $f \in \Sigma$ and used the Faber polynomial expansions to obtain bounds for the general coefficients $|b_n|$ of meromorphic bi-univalent functions in the class $B\Sigma(\alpha,\lambda)$.

2. Preliminaries

Definition 2.1. [11] For $0 \leq \alpha < 1$ and $\lambda \geq 1$, let $B\Sigma(\alpha,\lambda)$ be the class of meromorphic bi-univalent functions $f \in \Sigma$ so that:

$$\Re\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \alpha, \ \ z \in \triangle$$

and

$$\Re\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right) > \alpha, \ \ w \in \triangle,$$

where the function g is given by (1.2).

Theorem 2.1. [11] *Let f be given by (1.1). For $0 \leq \alpha < 1$ and $\lambda \geq 1$ if $f \in B\Sigma(\alpha;\lambda)$ and $b_k = 0;\ 0 \leq k \leq n-1$,then:*

$$|b_n| \leq \frac{2(1-\alpha)}{\lambda(n+1)-1};\ \ n \geq 1,$$

$$|b_0| \leq \frac{2(1-\alpha)}{\lambda-1}, \quad |b_1| \leq \frac{2(1-\alpha)}{2\lambda-1},$$

$$|b_2| \leq \frac{2(1-\alpha)}{3\lambda-1}, \ \ \textit{and} \ \ |b_2 + b_0 b_1| \leq \frac{2(1-\alpha)}{3\lambda-1}.$$

For functions $f \in \Sigma$ in the form (1.1), we define the following new linear operator

$$D^0_\lambda f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z)$$
$$= z + \sum_{n=0}^{\infty}[-n]\frac{b_n}{z^n},$$

and

$$D^2 f(z) = D[Df(z)] = z + \sum_{n=0}^{\infty}[-n]^2\frac{b_n}{z^n},$$

hence, it can be easily seen that

$$D^k f(z) = D[D^{k-1} f(z)] = z + \sum_{n=0}^{\infty} [-n]^k \frac{b_n}{z^n}, \ \ (k \in \mathbb{N}_0). \tag{3.1}$$

For $0 \le \alpha < 1,\ \lambda \ge 0$ *and* $k \in \mathbb{N}_0$*, let* $B\Sigma(k,\alpha,\lambda)$ *be the class of meromorphic bi-univalent functions* $f \in \Sigma$ *so that*

$$\Re\left(\frac{(1-\lambda)D^k f(z) + \lambda D^{k+1} f(z)}{z}\right) > \alpha$$

and

$$\Re\left(\frac{(1-\lambda)D^k g(w) + \lambda D^{k+1} g(w)}{w}\right) > \alpha$$

where $z,w \in \triangle$ *and the function* $g = f^{-1}$ *is given by (1.2).*

In this paper, motivated by the previous works, we shall use the differential linear operator $D^k f(z)$ given above to obtain our results. Similar to the work done by [11], we use the Faber polynomial expansions to obtain bounds for the general coefficients $|b_n|$ of meromorphic bi-univalent functions in $B\Sigma(k,\alpha,\lambda)$.

3. Main results

Our first theorem introduces an upper bound for the coefficients $|b_n|$ of meromorphic bi-univalent functions in $B\Sigma(k,\alpha,\lambda)$.

Theorem 3.1. *For* $0 \le \alpha < 1,\ \lambda \ge 1,$ *and* $k \in \mathbb{N}_0$*, let* f *be given by (1.1). If* $f \in B\Sigma(\alpha;\lambda)$ *and* $b_k = 0; (n-1 \ge k \ge 0)$*, then*

$$|b_n| \le \frac{2(1-\alpha)}{[(n+1)\lambda - 1]^k}; \ \ n \ge 1. \tag{3.1}$$

Proof. For meromorphic functions $f \in B\Sigma(k,\alpha,\lambda)$ of the form (1.1), we have

$$\frac{(1-\lambda)D^k f(z) + \lambda D^{k+1} f(z)}{z} = 1 + \sum_{n=0}^{\infty} [1-\lambda(n+1)]^k \frac{b_n}{z^{n+1}}, \tag{3.2}$$

and for its inverse map, $g = f^{-1}$,we have

$$\frac{(1-\lambda)D^k g(w) + \lambda D^{k+1} g(w)}{w} = 1 + \sum_{n=0}^{\infty} [1-\lambda(n+1)]^k \frac{B_n}{w^{n+1}}$$

$$= 1-(1-\lambda)^k\frac{b_0}{w}-\sum_{n=1}^{\infty}[1-\lambda(n+1)]^k\frac{1}{n}K_{n+1}^{n}(b_0,b_2,...,b_n)\frac{1}{w^{n+1}}. \tag{3.3}$$

On the other hand, since $f\in B\Sigma(k,\alpha,\lambda)$ and $g=f^{-1}\in B\Sigma(k,\alpha,\lambda)$, by definition, there exist two positive real part functions

$$p(z)=1+\sum_{n=1}^{\infty}c_nz^{-n}\in\Sigma$$

and

$$q(w)=1+\sum_{n=1}^{\infty}d_nw^{-n}\in\Sigma,$$

where $\Re\{p(z)\}>0$ and $\Re\{q(w)\}>0$ in $\triangle$ so that

$$\frac{(1-\lambda)D^kf(z)+\lambda D^{k+1}f(z)}{z}=\alpha+(1-\alpha)p(z)=1+(1-\alpha)\sum_{n=1}^{\infty}K_n^1(c_1,c_2,...,c_n)z^{-n}, \tag{3.4}$$

and

$$\frac{(1-\lambda)D^kg(w)+\lambda D^{k+1}g(w)}{w}=\alpha+(1-\alpha)q(w)=1+(1-\alpha)\sum_{n=1}^{\infty}K_n^1(d_1,d_2,...,d_n)w^{-n}. \tag{3.5}$$

Note that, by the Caratheodory Lemma (e.g., [5]),

$$|c_n|\le 2 \quad \textit{and} \quad |d_n|\le 2 \quad (n\in\mathbb{N}).$$

Comparing the corresponding coefficients of (3.2) and (3.4), for any $n\ge 2$, yields

$$(1-\lambda(n+1))^kb_n=(1-\alpha)K_{n+1}^1(c_1,c_2,...,c_{n+1}), \tag{3.6}$$

and similarly, from (3.3) and (3.5) we find

$$(1-\lambda(n+1))^kB_n=(1-\alpha)K_{n+1}^1(d_1,d_2,...,d_{n+1}). \tag{3.7}$$

Note that for $b_k=0;\ (0\le k\le n-1)$, we have $B_n=-b_n$ and so

$$(1-\lambda(n+1))^kb_n=(1-\alpha)c_{n+1}, \tag{3.8}$$

and

$$-(1-\lambda(n+1))^kb_n=(1-\alpha)d_{n+1}. \tag{3.9}$$

Now taking the absolute values of the above equalities and applying the Caratheodory lemma, we obtain

$$|b_n| = \frac{(1-\alpha)|c_{n+1}|}{|[1-\lambda(n+1)]^k|} = \frac{(1-\alpha)|d_{n+1}|}{|[1-\lambda(n+1)]^k|} \leq \frac{2(1-\alpha)}{[\lambda(n+1)-1]^k}.$$

which completes the proof of Theorem 3.1.

Remark 3.1. *Let $k=1$, we have $f \in B\Sigma(1,\alpha,\lambda) = B\Sigma(\alpha,\lambda)$, Theorem 3.1 reduced to Theorem 1.1 in* [11].

By imposing coefficient restrictions on Theorem 3.1, we capture the initial Taylor-Maclaurin coefficients of functions f in $B\Sigma(k,\alpha,\lambda)$ as well as a bound for the coefficient combination of $(b_2+b_0b_1)$ in the following theorem:

Theorem 3.2. *For $0 \leq \alpha < 1,\ \lambda \geq 1$, and $k \in \mathbb{N}_0$, let $f \in B\Sigma(k,\lambda,\alpha)$ be given by (1.1), then*

$$(i)\ |b_0| \leq \frac{2(1-\alpha)}{(\lambda-1)^k}. \tag{3.10}$$

$$(ii)\ |b_1| \leq \frac{2(1-\alpha)}{(2\lambda-1)^k}. \tag{3.11}$$

$$(iii)\ |b_2| \leq \frac{2(1-\alpha)}{(3\lambda-1)^k}. \tag{3.12}$$

$$(iv)\ |b_2+b_0b_1| \leq \frac{2(1-\alpha)}{(3\lambda-1)^k}. \tag{3.13}$$

Proof. Comparing Eqs. (3.2) and (3.4) for $n=0,1,2$, we obtain:

$$(1-\lambda)^k b_0 = (1-\alpha)c_1, \tag{3.14}$$

$$(1-2\lambda)^k b_1 = (1-\alpha)c_2, \tag{3.15}$$

and

$$(1-3\lambda)^k b_2 = (1-\alpha)c_3, \tag{3.16}$$

Also, from (3.3) and (3.5), for $n=2$ we have:

$$-(1-3\lambda)^k(b_2+b_0b_1) = (1-\alpha)d_3. \tag{3.17}$$

Solving Eqs. (3.14), (3.15), (3.16) and (3.17) for b_0, b_1, b_2 and $(b_2+b_0b_1)$, respectively, taking their absolute values and then applying the Caratheodory Lemma, we obtain:

$$|b_0| \leq \frac{(1-\alpha)|c_1|}{|(1-\lambda)^k|} \leq \frac{2(1-\alpha)}{(\lambda-1)^k},$$

$$|b_1| \leq \frac{(1-\alpha)|c_2|}{|(1-2\lambda)^k|} \leq \frac{2(1-\alpha)}{(2\lambda-1)^k},$$

$$|b_2| \leq \frac{(1-\alpha)|c_3|}{|(1-3\lambda)^k|} \leq \frac{2(1-\alpha)}{(3\lambda-1)^k},$$

and

$$|b_2+b_0b_1| \leq \frac{(1-\alpha)|d_3|}{|(1-3\lambda)^k|} \leq \frac{2(1-\alpha)}{(3\lambda-1)^k}.$$

Remark 3.2 *From the above discussion it is understood that the estimates of b_0, b_1, b_2 and $(b_2+b_0b_1)$ in Theorem 3.2 is the same as the corresponding estimates of Theorem 1.2 in* [11]*, when $k=1$.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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