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BOUNDS FOR THE GENERALIZATION OF TWO MAPPINGS RELATED TO THE HERMITE-HADAMARD INEQUALITY

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Abstract. In this paper, we give some results concerning the generalization of two mappings associated to the famous Hermite-Hadamard integral inequality for convex functions. As application, some new inequalities involving potential means are derived.

Keywords: convex functions; Hermite-Hadamard inequality; special means.

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1. Introduction

Let f be a convex function on $[a,b] \subset \mathbb{R}$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is known in the literature as the integral Hermite-Hadamard inequality [16].

It is well known that the Hermite-Hadamard inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there has been a large number of research papers written on this subject, see

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[9], [10], [11], and [12] and the references therein.

It has many applications for special means (see [8], [13], [14] and [17]) and also provides necessary and sufficient condition for a function f to be convex on (a,b) (see [19]).

Dragomir introduced in 1991. the following associated mapping $H\colon [0,1] \to \mathbb{R}$ defined by

$$H(t) = \frac{1}{b-a}\int_a^b f\Big(tx + (1-t)\frac{a+b}{2}\Big)\mathrm{d}x$$

for a given convex function $f\colon [a,b] \to \mathbb{R}$.

The corresponding double integral mapping $F\colon [0,1] \to \mathbb{R}$ in connection with the Hermite-Hadamard inequalities is defined as

$$F(t) = \frac{1}{(b-a)^2}\int_a^b \int_a^b f(tx + (1-t)y)\mathrm{d}x\mathrm{d}y.$$

For main properties of these mappings and some related results see [2], [5], [6], [7] and [18] and the references therein.

S.S.Dragomir [4] gave the following bounds for two mappings related to the Hermite-Hadamard inequality for convex functions:

Theorem 1.1. [4] *Let $f\colon [a,b] \to \mathbb{R}$ be a convex function on the interval $[a,b]$. Then we have*

$$\begin{aligned}&\frac{t}{b-a}\int_a^b f(x)\mathrm{d}x + (1-t)f\Big(\frac{a+b}{2}\Big) - H(t)\\ &\le t(1-t)\Big[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big]\end{aligned} \tag{2}$$

and

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - F(t)\\ &\le 2t(1-t)\Big[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big]\end{aligned} \tag{3}$$

for any $t \in [0,1]$.

In the present paper, we establish a weighted generalization of the above results involving a generalization of the two mappings associated to the Hermite-Hadamard inequality. Applications for potential means are also provided.

2. Preliminaries

Let $f\colon [a,b]\to\mathbb{R}$ be a convex function on the interval $[a,b]$. Let $p,g\colon [a,b]\to\mathbb{R}$ be integrable functions such that $p\geq 0$, $\int_a^b p(x)\mathrm{d}x = 1$ and $a\leq g(x)\leq b$ for any $x\in[a,b]$ and let $\bar{g}=\int_a^b p(x)g(x)\mathrm{d}x$.

In order to state our results, we first need to introduce the following associated mapping $H\colon [0,1]\to\mathbb{R}$ defined by

$$H(t;g)=\int_a^b p(x)f(tg(x)+(1-t)\bar{g})\mathrm{d}x.$$

Some of the main properties of the mapping H are:

1. H is convex on $[0,1]$;
2. H increases monotonically on $[0,1]$;
3. One has the bounds:

$$\inf_{t\in[0,1]} H(t;g)=H(0;g)=f(\bar{g})$$

$$\sup_{t\in[0,1]} H(t;g)=H(1;g)=\int_a^b p(x)f(g(x))\mathrm{d}x.$$

We also need to introduce the corresponding double integral mapping

$F\colon [0,1]\to\mathbb{R}$ defined by

$$F(t;g)=\int_a^b\int_a^b p(x)p(y)f(tg(x)+(1-t)g(y))\mathrm{d}x\mathrm{d}y.$$

Main results concerning this mapping are as follows:

1. $F(\tau+\frac{1}{2};g)=F(\frac{1}{2}-\tau;g)$ for every $\tau\in[0,\frac{1}{2}]$
2. $F(t;g)=F(1-t;g)$ for every $t\in[0,1]$
3. F is convex on $[0,1]$
4. F decreases monotonically on $[0,\frac{1}{2}]$ and increases monotonically on $[\frac{1}{2},1]$
5. We have the bounds:

$$\inf_{t\in[0,1]} F(t;g)=F(0;g)=F(1;g)=\int_a^b p(x)f(g(x))\mathrm{d}x$$

$$\sup_{t\in[0,1]} F(t;g)=F\Big(\frac{1}{2};g\Big)=\int_a^b\int_a^b p(x)p(y)f\Big(\frac{g(x)+g(y)}{2}\Big)\mathrm{d}x.$$

3. Main results

The following result gives us upper and lower bounds for the mappings F and H defined in the previous section.

Theorem 3.1. *Let the conditions stated above hold. Then we have*

$$\begin{aligned} 0 &\leq t\int_a^b p(x)f(g(x))\mathrm{d}x+(1-t)f(\bar{g})-H(t;g) \\ &\leq t(1-t)\int_a^b p(x)f'(g(x))\mathrm{d}x\left[\frac{\int_a^b p(x)g(x)f'(g(x))\mathrm{d}x}{\int_a^b p(x)f'(g(x))\mathrm{d}x}-\bar{g}\right] \end{aligned} \tag{4}$$

and

$$\begin{aligned} 0 &\leq \int_a^b p(x)f(g(x))\mathrm{d}x-F(t;g) \\ &\leq 2t(1-t)\int_a^b p(x)f'(g(x))\mathrm{d}x\left[\frac{\int_a^b p(x)g(x)f'(g(x))\mathrm{d}x}{\int_a^b p(x)f'(g(x))\mathrm{d}x}-\bar{g}\right] \end{aligned} \tag{5}$$

for any $t \in [0,1]$.

Proof. Function f is convex, so the following inequality holds

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y) \tag{6}$$

for all $x,y \in [a,b]$ and for all $t \in [0,1]$. We can first replace x with $g(x)$ and y with $\bar{g}$, and then x with $g(x)$ and y with $g(y)$ in (6) because $\bar{g}, g(x), g(y) \in [a,b]$ for all $x,y \in [a,b]$, and get respectively

$$f(tg(x)+(1-t)\bar{g}) \leq tf(g(x))+(1-t)f(\bar{g}) \tag{7}$$

and

$$f(tg(x)+(1-t)g(y)) \leq tf(g(x))+(1-t)f(g(y)). \tag{8}$$

Multiplying the inequality (7) by $p(x) \geq 0$ and then integrating it over x on $[a,b]$ we get the first inequality in (4) and multiplying the inequality (8) by $p(x) \geq 0$ and $p(y) \geq 0$ and then integrating it over x and y on $[a,b]$ we get the first inequality in (5).

Since the class of convex differentiable functions is dense in the uniform topology in the class of all convex functions defined on the interval $[a,b]$, we can assume that f is differentiable on (a,b).

If we use the convexity of the function f, we get the gradient inequality

$$f(u)-f(v)\geq f'(v)(u-v) \tag{9}$$

for any $u,v\in(a,b)$.

Because $tx+(1-t)y\in(a,b)$ holds for any $x,y\in(a,b)$ and $t\in[0,1]$, from (9) we get

$$f(tx+(1-t)y)-f(x)\geq(1-t)f'(x)(y-x) \tag{10}$$

and

$$f(tx+(1-t)y)-f(y)\geq -tf'(y)(y-x). \tag{11}$$

Now, if we multiply (10) by t and (11) by $(1-t)$, and add together the obtained inequalities, we get

$$\begin{aligned} &tf(x)+(1-t)f(y)-f(tx+(1-t)y)\\ &\leq t(1-t)[f'(y)-f'(x)](y-x) \end{aligned} \tag{12}$$

for any $x,y\in(a,b)$ and $t\in[0,1]$.

Since $a\leq g(x),\bar{g}\leq b$, we can replace x with $g(x)$ and y with $\bar{g}$ in (12), multiply the obtained inequality by $p(x)\geq 0$ and then integrate it over x on $[a,b]$ and get

$$\begin{aligned} &t\int_a^b p(x)f(g(x))\mathrm{d}x+(1-t)\int_a^b p(x)f(\bar{g})\mathrm{d}x-\int_a^b p(x)f(tg(x)+(1-t)\bar{g})\mathrm{d}x\\ &\leq t(1-t)\int_a^b p(x)[f'(\bar{g})-f'(g(x))](\bar{g}-g(x))\mathrm{d}x, \end{aligned} \tag{13}$$

which is equivalent to (4).

Further more, if we replace x with $g(x)$ and y with $g(y)$ in (12), and then multiply that inequality by $p(x)\geq 0$ and $p(y)\geq 0$ and integrate it over x and y on $[a,b]$ we can obtain the following inequality

$$
\begin{aligned}
& t\int_a^b\int_a^b p(x)p(y)f(g(x))\mathrm{d}x\mathrm{d}y+(1-t)\int_a^b\int_a^b p(x)p(y)f(g(y))\mathrm{d}x\mathrm{d}y \\
& -\int_a^b\int_a^b p(x)p(y)f(tg(x)+(1-t)g(y))\mathrm{d}x\mathrm{d}y \\
& \leq t(1-t)\int_a^b\int_a^b p(x)p(y)[f'(g(y))-f'(g(x))](g(y)-g(x))\mathrm{d}x\mathrm{d}y. \qquad (14)
\end{aligned}
$$

After some calculations, from (14) we easily get (5), and this completes the proof.

Remark 3.2. If we replace t with $1-t$ in (4), add together the obtained results, and then divide it by 2, we get the symmetric inequality

$$
\begin{aligned}
& \frac{1}{2}\Big[\int_a^b p(x)f(g(x))\mathrm{d}x+f(\bar{g})\Big]-\frac{H(t;g)+H(1-t;g)}{2} \\
& \leq t(1-t)\int_a^b p(x)f'(g(x))\mathrm{d}x\left[\frac{\int_a^b p(x)g(x)f'(g(x))\mathrm{d}x}{\int_a^b p(x)f'(g(x))\mathrm{d}x}-\bar{g}\right] \qquad (15)
\end{aligned}
$$

for any $t\in[0,1]$.

Remark 3.3.

(i) Let the conditions of Theorem 2.1 hold. Then the integral version of the Slater inequality for convex functions found in [1] is valid:

$$
0\leq\int_a^b p(x)f(g(x))\mathrm{d}x-f(\bar{g})\leq\int_a^b p(x)f'(g(x))(g(x)-\bar{g})\mathrm{d}x. \qquad (16)
$$

If we multiply the inequalities in (16) with $1-t$ and add it to (4), we get the following inequalities:

$$
\begin{aligned}
& 0\leq\int_a^b p(x)f(g(x))\mathrm{d}x-H(t;g) \\
& \leq(1-t^2)\int_a^b p(x)f'(g(x))(g(x)-\bar{g})\mathrm{d}x. \qquad (17)
\end{aligned}
$$

(ii) Now, if we subtract the inequalities in (5) from the inequalities in (17) we get

$$
\begin{aligned}
& 0\leq F(t;g)-H(t;g) \\
& \leq(1-t)^2\int_a^b p(x)f'(g(x))(g(x)-\bar{g})\mathrm{d}x. \qquad (18)
\end{aligned}
$$

4. Application for potential means

Let $f, w\colon [a,b] \to \mathbb{R}$ be positive integrable functions. The potential mean of order r of a function f with weight function w is given by

$$M_r(f,w) = \left[\frac{\int_a^b w(x) f(x)^r \mathrm{d}x}{\int_a^b w(x)\mathrm{d}x}\right]^{1/r}, \qquad r \neq 0$$

$$M_0(f,w) = \exp\left[\frac{\int_a^b w(x)\ln f(x)\mathrm{d}x}{\int_a^b w(x)\mathrm{d}x}\right], \quad r = 0$$

(19)

Let us consider the convex mapping $f\colon (0,\infty) \to \mathbb{R}$, $f(x) = x^p$, $p \in (-\infty,0) \cup (1,\infty)$ and $0 < a < b$. We define the mapping

$$H_p(t;g) = \frac{1}{W}\int_a^b w(x)(tg(x) + (1-t)\bar{g})^p \mathrm{d}x,\ t \in [0,1], \tag{20}$$

where $W = \int_a^b w(x)\mathrm{d}x$ and $\bar{g} = \frac{1}{W}\int_a^b w(x)g(x)\mathrm{d}x$.

It is obvious that $H_p(0;g) = \frac{1}{W}\int_a^b w(x)\bar{g}^p\mathrm{d}x = \bar{g}^p$ and $H_p(1;g) = \frac{1}{W}\int_a^b w(x)g(x)^p\mathrm{d}x = M_p^p(g,w)$, and for $t \in (0,1)$ and $p \in \mathbb{N}$ we have

$$H_p(t;g) = \frac{1}{W}\int_a^b w(x)(tg(x)+(1-t)\bar{g})^p\mathrm{d}x = \sum_{k=0}^{p}\binom{p}{i}(tM_i(g,w))^i((1-t)\bar{g})^{p-i}.$$

Now, consider the function

$$F_p(t;g) = \frac{1}{W^2}\int_a^b\int_a^b w(x)w(y)(tg(x)+(1-t)g(y))^p\mathrm{d}x\mathrm{d}y,\ t \in [0,1].$$

We observe that $F_p(0;g) = F_p(1;g) = \frac{1}{W}\int_a^b w(x)g(x)^p\mathrm{d}x = M_p^p(g,w)$ and we can calculate that for $p \in \mathbb{N}$

$$\begin{aligned} F_p\Big(\frac{1}{2};g\Big) &= \frac{1}{W^2}\int_a^b\int_a^b w(x)w(y)\Big(\frac{g(x)+g(y)}{2}\Big)^p\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2^p}\sum_{k=0}^{p}\binom{p}{i}M_{p-i}^{p-i}(g,w)M_i^i(g,w). \end{aligned}$$

Let $g, w\colon [a,b] \to \mathbb{R}$ be positive integrable functions and let $W = \int_a^b w(x)\mathrm{d}x$ and $\bar{g} = \frac{1}{W}\int_a^b w(x)g(x)\mathrm{d}x$. We define a new weight function $p\colon [a,b] \to \mathbb{R}$ with $p(x) = w(x)/W$. This is a positive, integrable function such that $\int_a^b p(x)\mathrm{d}x = 1$.

Since the function $f\colon(0,\infty)\to\mathbb{R}$, $f(x)=x^p$ is convex for all $p\in(-\infty,0)\cup(1,\infty)$, the conditions from Theorem 3.1 are satisfied, and we easily get the following result:

Theorem 4.1. *Let w,g,f be as stated above. Then for all $p\in(-\infty,0)\cup(1,\infty)$ and for all $t\in[0,1]$ we have*

$$
\begin{aligned}
0 &\le tM_p^p(g,w)+(1-t)\bar{g}^p-H_p(t;g)\\
&\le pt(1-t)(M_p^p(g,w)-\bar{g}M_{p-1}^{p-1}(g,w))
\end{aligned}
\tag{21}
$$

and

$$
\begin{aligned}
0 &\le M_p^p(g,w)-F_p(t;g)\\
&\le 2pt(1-t)(M_p^p(g,w)-\bar{g}M_{p-1}^{p-1}(g,w)).
\end{aligned}
\tag{22}
$$

In particular, if we choose $t=\frac{1}{2}$, we get

$$
\begin{aligned}
0 &\le A(M_p^p(g,w),\bar{g}^p)-H_p(\frac{1}{2};g)\\
&\le \frac{p}{4}(M_p^p(g,w)-\bar{g}M_{p-1}^{p-1}(g,w))
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
0 &\le M_p^p(g,w)-F_p(\frac{1}{2};g)\\
&\le \frac{p}{2}(M_p^p(g,w)-\bar{g}M_{p-1}^{p-1}(g,w)).
\end{aligned}
\tag{24}
$$

where $A(a,b)=\frac{a+b}{2}$ is the arithmetic mean of the numbers a and b.

Conflict of Interests

The authors declare that there is no conflict of interests.

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