ALTERNATIVE PROOFS OF THE GENERALIZED REVERSE YOUNG INEQUALITIES

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1. Main results

We start from the famous formula
\[ e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{r \to 0} (1 + rx)^{1/r}. \]

In this article, we consider the inverse function of \( r \)-exponential function \( \exp_r(x) \equiv (1 + rx)^{1/r} \), namely \( r \)-logarithmic function defined by \( \ln_r x \equiv \frac{x^r - 1}{r} \) for \( x \geq 0 \) and a real number \( r \neq 0 \).

Lemma 1.1. \( \ln_r x \) is a monotone increasing function in \( r \).
Proof. In the inequality \( \log t \leq t - 1 \) for \( t > 0 \), we set \( t = x^{-r} \), we obtain the following
\[
\frac{\partial \ln x}{\partial r} = \frac{x^r (\log x^r - 1 + x^{-r})}{r^2} \geq 0.
\]
\( \square \)

Lemma 1.1. implies the following lemma.

Lemma 1.2. Let \( r, v, t \) be real numbers with \( r \neq 0 \) and \( t > 0 \).

(i) For \( 0 < r < v - \frac{1}{2} \) or \( v - \frac{1}{2} < 0 < r \), we have
\[
(1) \quad \left( v - \frac{1}{2} \right) \frac{t^r - 1}{r} \leq t^{v - \frac{1}{2}} - 1.
\]

(ii) For \( 0 < r < v \) or \( v < 0 < r \), we have
\[
(2) \quad v \frac{t^r - 1}{r} \leq t^v - 1.
\]

Applying (i) and (ii) of Lemma, we can derive respectively [1, Theorem 1] and [1, Theorem 3] without using the supplemental Young’s inequality given in [1, Lemma 5] which used to prove [1, Theorem 1] and [1, Theorem 3].

Theorem 1.1. ([1, Theorem 1]) Let \( v, t \) be real numbers with \( t > 0 \), and \( n \in \mathbb{N} \) with \( n \geq 2 \).

(i) For \( v \notin \left[ \frac{1}{2}, \frac{2^{n-1} + 1}{2n} \right] \), we have
\[
(3) \quad (1 - v) + vt \leq t^v + (1 - v) \left( 1 - \sqrt{t} \right)^2 + (2v - 1) \sqrt{t} \sum_{k=2}^{n} 2^{k-2} \left( \frac{v}{\sqrt{t}} - 1 \right)^2.
\]

(ii) For \( v \notin \left[ \frac{2^{n-1} - 1}{2n}, \frac{1}{2} \right] \), we have
\[
(4) \quad (1 - v)t + v \leq t^{1-v} + v \left( \sqrt{t} - 1 \right)^2 + (1 - 2v) \sqrt{t} \sum_{k=2}^{n} 2^{k-2} \left( \frac{v}{\sqrt{t}} - 1 \right)^2.
\]

Proof.

(i) Direct calculations imply
\[
(1 - v) + vt - (1 - v) \left( 1 - \sqrt{t} \right)^2 - (2v - 1) \sqrt{t} \sum_{k=2}^{n} 2^{k-2} \left( \frac{v}{\sqrt{t}} - 1 \right)^2
\]
\[
= \sqrt{t} + \sqrt{t} \left( v - \frac{1}{2} \right) 2^n \left( \frac{v}{\sqrt{t}} - 1 \right).
\]

Thus the inequality (3) is equivalent to the inequality
\[
(5) \quad \left( v - \frac{1}{2} \right) 2^n \left( \frac{v}{\sqrt{t}} - 1 \right) \leq t^{v - \frac{1}{2}} - 1.
\]
This inequality is true by (i) of Lemma 1.2. with \( r = \frac{1}{2n} \), since the conditions \( 0 < r < v - \frac{1}{2} \) or \( v - \frac{1}{2} < 0 < r \) are satisfied in the case of \( v \notin \left[ \frac{1}{2}, \frac{2^n - 1}{2n} \right] \) in (i) of Lemma 1.2.

(ii) Exchanging \( 1 - v \) with \( v \) in (i) of Lemma 1.2., we have

\[(1 - v) + vt \leq v \sum_{k=1}^{n} 2^{k-1} \left( 1 - \frac{2^k}{\sqrt{t}} \right)^2 \]

for \( v < \frac{1}{2} - r \) or \( \frac{1}{2} < v \). Exchanging \( 1 - v \) with \( v \) in the inequality (6), we have

\[(1 - v) t + v \leq t^{1-v} + (1 - v) \sum_{k=1}^{n} 2^{k-1} \left( 1 - \frac{2^k}{\sqrt{t}} \right)^2 \]

\[v2^n \left( \frac{2^n}{\sqrt{t}} - 1 \right) \leq t^v - 1 \]

This inequality is true by (ii) of Lemma 1.2. with \( r = \frac{1}{2n} \), since the conditions \( 0 < r < v \) or \( v < 0 < r \) are satisfied in the case of \( v \notin \left[ 0, \frac{1}{2n} \right] \) in (ii) of Lemma 1.2.
(ii) Exchanging $1 - v$ with $v$ in (ii) of Lemma 1.2., we have

$$\frac{(1 - v)^{t^r - 1}}{r} \leq t^{1 - v} - 1$$

for $0 < r < 1 - v$ or $1 - v < 0 < r$. Exchanging $1 - v$ with $v$ in the inequality (12), we also have

$$\frac{(1 - v)^{2^n(\frac{v}{2^n} - 1)}}{r} \leq t^{1 - v} - 1$$

This inequality is true by the inequality (13) with $r = \frac{1}{2^n}$, since the conditions $0 < r < 1 - v$ or $1 - v < 0 < r$ are satisfied in the case of $v \notin \left[\frac{2^n - 1}{2^n}, 1\right]$ in (ii) of Lemma 1.2.

By theory of Kubo-Ando [2], we have the following corollary from Lemma 1.2.

**Corollary 1.3.** Let $r, v, t$ be real numbers with $r \neq 0$ and $t > 0$. For $\alpha \in \mathbb{R}$, a positive definite matrix $A$ and a positive semidefinite matrix $B$, we define $A_{\alpha} \equiv A^{1/2} (A^{-1/2}BA^{-1/2})^{\alpha} A^{1/2}$. Then we have the following matrix inequalities.

(i) For $0 < r < v - \frac{1}{2}$ or $v - \frac{1}{2} < 0 < r$, we have

$$\left(v - \frac{1}{2}\right) \frac{A_{v^r} - A}{r} \leq A_{v^{r-1/2}} - A.$$

(ii) For $0 < r < v$ or $v < 0 < r$, we have

$$\frac{v A_{v^r} - A}{r} \leq A_{v^r} - A.$$

2. Additional results

The methods in previous section are applicable to obtain the inequalities in the following propositions.

**Proposition 2.1.** Let $v, t$ be real numbers with $t > 0$, and $n \in \mathbb{N}$.

(i) For $v \in \left[0, \frac{1}{2^n}\right]$, we have

$$1 - v + vt \geq t^v + v \sum_{k=1}^{n} 2^{k-1} \left(1 - \frac{2^k}{2^n}\right)^2$$

If $\alpha \in [0, 1]$, $A_{\alpha} \equiv A^{1/2} (A^{-1/2}BA^{-1/2})^{\alpha} A^{1/2}$ is called $\alpha$-weighted geometric mean.
(ii) For \( v \in \left[ \frac{2n-1}{2^{n-1}}, 1 \right] \), we have

\[
(1 - v)t + v \geq t^{1-v} + (1 - v) \sum_{k=1}^{n} 2^{k-1} \left( 1 - \frac{v}{\sqrt[k]{t}} \right)^2
\]

**Proof.**

(i) By Lemma 1.1., we have

\[
v \frac{t^r - 1}{r} \geq t^r - 1, \quad (0 \leq v \leq r)
\]

which implies

\[
v 2^n \left( t^{\frac{1}{2^n}} - 1 \right) \geq t^v - 1, \quad \text{for} \quad v \in \left[ 0, \frac{1}{2^n} \right]
\]

by putting \( r = \frac{1}{2^n} \). Since we have the identity (11), the inequality (17) is equivalent to the inequality (19).

(ii) Exchanging \( 1 - v \) with \( v \), the inequality (19) becomes

\[
(1 - v) 2^n \left( t^{\frac{1}{2^n}} - 1 \right) \geq t^{1-v} - 1, \quad \text{for} \quad v \in \left[ \frac{2n-1}{2^n}, 1 \right].
\]

Then the inequality (17) is also changed to the inequality (18), which is true by the inequality (20).

\[\square\]

**Proposition 2.2.** Let \( v, t \) be real numbers with \( t > 0 \), and \( n \in \mathbb{N} \) with \( n \geq 2 \).

(i) For \( v \in \left[ \frac{1}{2}, \frac{2n-1+1}{2^n} \right] \), we have

\[
(1 - v) + vt \geq t^v + (1 - v) \left( 1 - \sqrt[k]{t} \right)^2 + (2v - 1) \sqrt[k]{t} \sum_{k=2}^{n} 2^{k-2} \left( \frac{v}{\sqrt[k]{t}} - 1 \right)^2
\]

(ii) For \( v \in \left[ \frac{2n-1-1}{2^n}, \frac{1}{2} \right] \), we have

\[
(1 - v)t + v \geq t^{1-v} + v \left( \sqrt[k]{t} - 1 \right)^2 + (1 - 2v) \sqrt[k]{t} \sum_{k=2}^{n} 2^{k-2} \left( \frac{v}{\sqrt[k]{t}} - 1 \right)^2
\]

**Proof.**

(i) By Lemma 1.1., we have

\[
\left( v - \frac{1}{2} \right) \frac{t^r - 1}{r} \geq t^{v - \frac{1}{2}} - 1, \quad \left( 0 \leq v - \frac{1}{2} \leq r \right),
\]
which implies

$$\left(v - \frac{1}{2}\right) 2^n \left(t^{\frac{1}{2n}} - 1 \right) \geq t^{\frac{1}{2} - \frac{1}{2}} - 1, \quad \text{for} \quad v \in \left[\frac{1}{2}, \frac{2^{n-1} + 1}{2^n}\right]$$

by putting $r = \frac{1}{2n}$. Since we have the identity (5), the inequality (21) is equivalent to the inequality (23).

(ii) Exchanging $1 - v$ with $v$, the inequality (23) becomes

$$\left(\frac{1}{2} - v\right) 2^n \left(t^{\frac{1}{2n}} - 1 \right) \geq t^{\frac{1}{2} - v} - 1, \quad \text{for} \quad v \in \left[\frac{2^{n-1} - 1}{2^n}, \frac{1}{2}\right].$$

Then the inequality (21) is also changed to the inequality (22), which is true by the inequality (24).

□

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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**References**
