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AN INEQUALITY FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract: In this paper, an inequality for the polar derivative of a polynomial with restricted zeros is obtained, which refines and generalizes some well known polynomial inequalities.

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1. Introduction

Let $P(z)=\sum_{j=1}^{n} a_j z^j$ be a polynomial of degree n having all its zeros in $|z|\le 1$, then

(1) $$\underset{|z|=1}{Max}|P^{/}(z)|\ge \frac{n}{2}\underset{|z|=1}{Max}|P(z)|.$$

Inequality (1) is due to P.Turan [10].

As a generalization of (1), Malik [8] considered the class of polynomials $P(z)$ of degree n having all the zeros in $|z|\le k$ where $k\le 1$ and proved that

(2) $$\underset{|z|=1}{Max}|P^{/}(z)|\ge \frac{n}{1+k}\underset{|z|=1}{Max}|P(z).$$

The result is sharp and equality holds for the polynomial $P(z)=(z+k)^n$.

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The case $k \geq 1$ was considered by Govil [7] who showed that if $P(z)$ is a polynomial of degree n having all the zeros in $|z| \leq k$ where $k \geq 1$ then

(3) $$\underset{|z|=1}{Max} |P^{/}(z)| \geq \frac{n}{1+k^n} \underset{|z|=1}{Max} |P(z)|.$$

Equality in (3) holds for the polynomial $P(z) = z^n + k^n$.

The polar derivative $D_\alpha P(z)$ of a polynomial $P(z)$ of degree n with respect to a point α ∈ C is defined as

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

(for reference see [9])

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$, and it generalizes the ordinary derivative in the sense that

$$\lim_{n\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly for $|z| \leq R,\ R > 0$.

Aziz and Rather obtained several sharp results concerning the maximum modulus of $D_\alpha P(z)$ on $|z| = 1$. Among other things, they [3] established the following extension of inequality (3) to the polar derivative of a polynomial.

Theorem A. *If $P(z)$ is a polynomial of degree n having all the zeros in $|z| \leq k$ where $k \geq 1$, then for every real or complex number α with $|\alpha| \geq k$,*

(4) $$\underset{|z|=1}{Max} |D_\alpha P(z)| \geq \frac{n(|\alpha| - k)}{1+k^n} \underset{|z|=1}{Max} |P(z)|.$$

In this paper, we present the following result, which is a refinement as well as generalization of the Theorem A.

2. Preliminaries

For the proof of the Theorem, we need the following Lemmas.

Lemma 1. *If* $P(z)=\sum_{j=1}^{n} a_j z^j$ *is a polynomial of degree n, then*

$$\underset{|z|=R>1}{Max}|P(z)| \le R^n \underset{|z|=1}{Max}|P(z)| - |a_1| \left[\left\{\frac{(c_n+1)(R^n-1)}{n}\right\} - \left\{\frac{R^{n-2}-1}{n-2}\right\}\right], \qquad n>2$$

$$\le R^2 \underset{|z|=1}{Max}|P(z)| - |a_1| \left\{\frac{(R-1)(R+1-\sqrt{2})}{\sqrt{2}}\right\} \qquad \textit{for} \quad n=2.$$

where $c_2=\sqrt{2}-1,\ c_3=\frac{1}{\sqrt{2}}$ *and for* $n\ge 4$, c_n *is the unique positive root of the equation*

$$16n-8(3n+2)x^2-16x^3+(n+4)x^4=0$$

lying in (0, 1).

The above lemma is due to Bhat [4].

Frapper [6] showed that the coefficient c_n appearing in the above Lemma is given by

$$c_n=\frac{2n}{n-4}\sqrt{\frac{2(n+2)}{n}}-1 \ \text{ for } \ n\ge 4 .$$

Lemma 2. *If* $P(z)=\sum_{j=1}^{n} a_j z^j$ *, is a polynomial of degree n having no zero in* $|z|\le 1$ *and* $m=\underset{|z|=1}{Min}|Pz)|$ *, then for* $R\ge 1$ *,*

$$\underset{|z|=R}{Max}|P(z)| \le \frac{(R^n+1)}{2}\underset{|z|=1}{Max}|P(z)| - \frac{(R^n-1)}{2}m - \frac{2|a_1|}{n+1}\left[\frac{(R^n-1)}{n}-(R-1)\right]$$

$$-2|a_2|\left[\left\{\frac{(R^n-1)-n(R-1)}{n(n-1)}\right\}-\left\{\frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}\right\}\right] \ \textit{for } n>3.$$

and

$$\underset{|z|=R}{Max}|P(z)| \le \frac{(R^n+1)}{2}\underset{|z|=1}{Max}|P(z)| - \frac{(R^n-1)}{2}m - \frac{2|a_1|}{n+1}\left[\frac{(R^n-1)}{n}-(R-1)\right]$$

$$-\frac{2|a_2|}{n(n-1)}(R-1)^n \qquad \textit{for } n=3.$$

The above result is a special case of a result due to Dewan, Singh and Mir [5] with $K=1$.

Lemma 3. *If* $P(z)=\sum_{j=1}^{n} a_j z^j$, $a_n \neq 0$, *is a polynomial of degree n having all its zeros in* $|z|\leq 1$, *then*

$$\underset{|z|=1}{Max}|P^{/}(z)|\geq \frac{n}{2}\left\{\underset{|z|=1}{Max}|P(z)|+\underset{|z|=1}{Min}|P(z)|\right\}.$$

This Lemma is due to A.Aziz and Dawood [1].

3. Main results

Theorem 1.1. *Let* $P(z)=\sum_{j=1}^{n} a_j z^j$, $a_n a_0 \neq 0$ *be a polynomial of degree* $n \geq 3$, *having all its zeros in* $|z|\leq k,\ k\geq 1$, *then for every real or complex number* α *with* $|\alpha|\geq k$,

$$(5)\quad \underset{|z|=1}{Max}|D_\alpha P(z)|\geq \frac{n(|\alpha|-k)}{1+k^n}\left\{\underset{|z|=1}{Max}|P(z)|+\underset{|z|=k}{Min}|P(z)|+2\frac{|a_{n-1}|}{(n+1)}\left[\frac{k^n-1}{n}-(k-1)\right]\right.$$
$$+2|a_{n-2}|\left[\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\right\}-\left\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right]\Bigg\}$$
$$+\frac{|(n-1)a_1+2\alpha a_2|}{k^{n-1}}\left[\left\{\frac{(c_{n-1}+1)(k^{n-1}-1)}{n-1}\right\}-\left\{\frac{k^{n-3}-1}{n-3}\right\}\right]\quad \textit{for } n>3.$$

and

$$(6)\qquad \underset{|z|=1}{Max}|D_\alpha P(z)|\geq \frac{n(|\alpha|-k)}{1+k^n}k^{n-3}\left\{\underset{|z|=1}{Max}|P(z)|+\underset{|z|=k}{Min}|P(z)|+2\frac{|a_{n-1}|}{(n+1)}\left[\frac{k^n-1}{n}-(k-1)\right]\right.$$
$$+\frac{1}{k^2}\left[\frac{2|a_{n-2}|(k-1)^n}{n(n-1)}\right]\Bigg\}+\frac{1}{k^2}\left[|(n-1)a_1+2\alpha a_2|\left\{\frac{(k-1)(k+1-\sqrt{2})}{\sqrt{2}}\right\}\right]\quad \textit{for } n=3.$$

where $c_2=\sqrt{2}-1,\ c_3=\frac{1}{\sqrt{2}}$ *and* $c_n=\frac{2n}{n-4}\sqrt{\frac{2(n+2)}{n}}-1$, *for* $n\geq 4$.

Proof.

By hypothesis, all the zeros of $P(z)$ lie in $|z|\leq k$, therefore the polynomial $F(z)=P(kz)$ has all its zeros in $|z|\leq 1$. Applying Lemma 3 to the polynomial $F(z)$, we get

$$(7)\qquad \underset{|z|=1}{Max}|F^{/}(z)|\geq \frac{n}{2}\left\{\underset{|z|=1}{Max}|F(z)|+\underset{|z|=1}{Min}|F(z)|\right\}.$$

Let $G(z) = z^n \overline{F(1/\bar{z})}$, then all the zeros of $G(z)$ lie $|z| \geq 1$. Moreover, it can be easily verified that for $|z| = 1$,

(8) $$|G'(z)| = |nF(z) - zF'(z)| \quad \text{and} \quad |F'(z)| = |nG(z) - zG'(z)|.$$

Since $G(z)$ does not vanish in $|z| < 1$, it follows by a simple argument (see [2, inequality (9)]) that

$$|G'(z)| \leq |nG(z) - zG'(z)| \qquad \text{for} \quad |z| = 1.$$

This gives with the help of (8),

(9) $$|nF(z) - zF'(z)| \leq |F'(z)| \qquad \text{for} \quad |z| = 1.$$

Now for every real or complex number α with $|\alpha| \geq k$, we have

$$\left|D_{\frac{\alpha}{k}} F(z)\right| = \left|nF(z) - zF'(z) + \frac{\alpha}{k} F'(z)\right|$$

$$\geq \left|\frac{\alpha}{k}\right| |F'(z)| - |nF(z) - zF'(z)|.$$

Using (9), we get

$$\left|D_{\frac{\alpha}{k}} F(z)\right| \geq \left|\frac{\alpha}{k}\right| |F'(z)| - |F'(z)| = \frac{|\alpha| - k}{k} |F'(z)| \qquad \text{for} \quad |z| = 1.$$

This further gives

(10) $$\underset{|z|=1}{Max}\left|D_{\frac{\alpha}{k}} F(z)\right| \geq \frac{|\alpha| - k}{k} \underset{|z|=1}{Max}|F'(z)|.$$

Together (7) and (10) yield,

$$\underset{|z|=1}{Max}\left|D_{\frac{\alpha}{k}} F(z)\right| \geq \frac{n}{2} \frac{(|\alpha| - k)}{k} \left\{\underset{|z|=1}{Max}|F'(z)| + \underset{|z|=1}{Min}|F'(z)|\right\}.$$

Replacing $F(z)$ by $P(kz)$, we get

$$\underset{|z|=1}{Max}\left|D_{\frac{\alpha}{k}} P(kz)\right| \geq \frac{(|\alpha| - k)}{k} \frac{n}{2} \left\{\underset{|z|=1}{Max}|P(kz)| + \underset{|z|=1}{Min}|P(kz)|\right\}.$$

This can be written as

$$\max_{|z|=1}\left|nP(kz)+\left(\frac{\alpha}{k}-z\right)k\,P'(kz)\right|\geq\frac{n(|\alpha|-k)}{2k}\left\{\max_{|z|=1}|P(kz)|+\min_{|z|=1}|P(kz)|\right\}.$$

or equivalently,

(11) $$\max_{|z|=k}|D_\alpha P(z)|\geq\frac{n(|\alpha|-k)}{2k}\left\{\max_{|z|=k}|P(z)|+\min_{|z|=k}|P(z)|\right\}.$$

Let $Q(z)=z^n\overline{P(1/\overline{z})}$. Since all the zeros of the polynomial $P(z)$ lie in $|z|\leq k,\ k\geq 1$, all the zeros of $Q(z)$ lie in $|z|\geq\frac{1}{k},\ k\geq 1$ and therefore the polynomial $Q\left(\frac{z}{k}\right)$ have all its zeros in $|z|\geq 1$. Applying Lemma 2 to the polynomial $Q\left(\frac{z}{k}\right)$, with $R=k\geq 1$, we get

$$\max_{|z|=k}\left|Q\left(\frac{z}{k}\right)\right|\leq\frac{(k^n+1)}{2}\max_{|z|=1}\left|Q\left(\frac{z}{k}\right)\right|-\frac{(k^n-1)}{2}\min_{|z|=1}\left|Q\left(\frac{z}{k}\right)\right|-\frac{2|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]$$
$$-2|a_{n-2}|\left[\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\right\}-\left\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right],\ \text{for n > 3}$$

and

$$\max_{|z|=k}\left|Q\left(\frac{z}{k}\right)\right|\leq\frac{(k^n+1)}{2}\max_{|z|=1}\left|Q\left(\frac{z}{k}\right)\right|-\frac{(k^n-1)}{2}\min_{|z|=1}\left|Q\left(\frac{z}{k}\right)\right|-\frac{2|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]$$
$$-\frac{2|a_{n-2}|}{n(n-1)}(k-1)^n,\qquad \text{for n = 3.}$$

Using the fact that $|P(z)|=|Q(z)|$ for |z|=1, it follows that

$$\max_{|z|=1}|P(z)|\leq\frac{(k^n+1)}{2k^n}\max_{|z|=k}|P(z)|-\frac{(k^n-1)}{2k^n}\min_{|z|=k}|P(z)|-\frac{2|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]$$
$$-2|a_{n-2}|\left[\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\right\}-\left\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right],\ \text{for n > 3}$$

and

$$\max_{|z|=1}|P(z)|\leq\frac{(k^n+1)}{2k^n}\max_{|z|=k}|P(z)|-\frac{(k^n-1)}{2k^n}\min_{|z|=k}|P(z)|-\frac{2|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]$$
$$-\frac{2|a_{n-2}|}{n(n-1)}(k-1)^n,\qquad \text{for n = 3,}$$

which implies,

$$(12)\ \underset{|z|=k}{Max}|P(z)| \geq \frac{2k^n}{(k^n+1)}\underset{|z|=1}{Max}|P(z)| + \frac{(k^n-1)}{(k^n+1)}\underset{|z|=k}{Min}|P(z)| + \frac{4k^n}{(k^n+1)}\frac{|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]$$

$$+\frac{4k^n}{(k^n+1)}|a_{n-2}|\left[\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\right\}-\left\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right], \text{ for n > 3}$$

and

$$(13)\qquad \underset{|z|=k}{Max}|P(z)| \geq \frac{2k^n}{(k^n+1)}\underset{|z|=1}{Max}|P(z)| + \frac{(k^n-1)}{(k^n+1)}\underset{|z|=k}{Min}|P(z)| + \frac{4k^n}{(k^n+1)}\frac{|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]$$

$$+\frac{4k^n}{(k^n+1)}\frac{|a_{n-2}|}{n(n-1)}(k-1)^n, \qquad \text{for n = 3.}$$

Combining (11), (12) and (13), we get

$$\underset{|z|=k}{Max}|D_\alpha P(z)| \geq \frac{n(|\alpha|-k)}{2k}\left\{\frac{2k^n}{(k^n+1)}\underset{|z|=1}{Max}|P(z)| + \frac{(k^n-1)}{(k^n+1)}\underset{|z|=k}{Min}|P(z)| + \frac{4k^n}{(k^n+1)}\frac{|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]\right.$$

$$\left.+\frac{4k^n}{(k^n+1)}|a_{n-2}|\left[\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\right\}-\left\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right]+\underset{|z|=k}{Min}|P(z)|\right\}, \text{for n > 3}$$

and

$$\underset{|z|=k}{Max}|D_\alpha P(z)| \geq \frac{n(|\alpha|-k)}{2k}\left\{\frac{2k^n}{(k^n+1)}\underset{|z|=1}{Max}|P(z)| - \frac{(k^n-1)}{(k^n+1)}\underset{|z|=k}{Min}|P(z)| + \frac{4k^n}{(k^n+1)}\frac{|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]\right.$$

$$\left.+\frac{4k^n}{(k^n+1)}\frac{|a_{n-2}|}{n(n-1)}(k-1)+\underset{|z|=k}{Min}|P(z)|\right\}, \qquad \text{for n = 3.}$$

This gives

$$(14)\ \underset{|z|=k}{Max}|D_\alpha P(z)| \geq \frac{n(|\alpha|-k)k^{n-1}}{(k^n+1)}\left\{\underset{|z|=1}{Max}|P(z)| + \underset{|z|=k}{Min}|P(z)| + \frac{2|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]\right.$$

$$\left.+2|a_{n-2}|\left[\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\right\}-\left\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right]\right\}, \text{ for n > 3.}$$

and

$$(15)\ \underset{|z|=k}{Max}|D_\alpha P(z)| \geq \frac{n(|\alpha|-k)k^{n-1}}{(k^n+1)}\left\{\underset{|z|=1}{Max}|P(z)| + \underset{|z|=k}{Min}|P(z)| + \frac{2|a_{n-1}|}{(n+1)}\left[\frac{(k^n-1)}{n}-(k-1)\right]\right.$$

$$\left.+\frac{2|a_{n-2}|}{n(n-1)}(k-1)^n\right\}, \qquad \text{for n = 3.}$$

Since $D_\alpha P(z)$ is a polynomial of degree $n-1$ and $k \geq 1$, applying Lemma 1, we obtain

$$\underset{|z|=k}{Max}\left|D_\alpha P(z)\right| \leq k^{n-1}\underset{|z|=1}{Max}\left|D_\alpha P(z)\right| - |\,(n-1)a_1 + 2\alpha\, a_2\,|\left[\left\{\frac{(c_{n-1}+1)(k^{n-1}-1)}{n-1}\right\} - \left\{\frac{k^{n-3}-1}{n-3}\right\}\right],\;\; n>3$$

and

$$\underset{|z|=k}{Max}\left|D_\alpha P(z)\right| \leq k^{2}\underset{|z|=1}{Max}\left|D_\alpha P(z)\right| - |\,(n-1)a_1 + 2\alpha\, a_2\,|\left[\left\{\frac{(k-1)(k+1-\sqrt{2})}{\sqrt{2}}\right\}\right],\;\; n=3\,.$$

where c_n is defined as in the Theorem. These in conjunction with (14) and (15) yields,

$$\underset{|z|=1}{Max}\,|\,D_\alpha P(z)\,| \geq \frac{n(|\,\alpha\,|-k)}{1+k^n}\left\{\underset{|z|=1}{Max}\,|\,P(z)\,| + \underset{|z|=k}{Min}\,|\,P(z)\,| + 2\frac{|\,a_{n-1}\,|}{(n+1)}\left[\frac{k^n-1}{n} - (k-1)\right]\right.$$

$$\left. + 2\,|\,a_{n-2}\,|\left[\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\right\} - \left\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right]\right\}$$

$$+\frac{|\,(n-1)\,a_1 + 2\alpha\, a_2\,|}{k^{n-1}}\left[\left\{\frac{(c_{n-1}+1)(k^{n-1}-1)}{n-1}\right\} - \left\{\frac{k^{n-3}-1}{n-3}\right\}\right], \text{ for n > 3.}$$

and

$$\underset{|z|=1}{Max}\,|\,D_\alpha P(z)\,| \geq \frac{n(|\,\alpha\,|-k)}{1+k^n}\left\{\underset{|z|=1}{Max}\,|\,P(z)\,| + \underset{|z|=k}{Min}\,|\,P(z)\,| + 2\frac{|\,a_2\,|}{(n+1)}\left[\frac{k^n-1}{n} - (k-1)\right]\right.$$

$$\left. + \frac{1}{k^2}\left[\frac{2\,|\,a_1\,|\,(k-1)^n}{n(n-1)}\right]\right\} + \frac{1}{k^2}\left[\left|(n-1)a_1 + 2\alpha\, a_2\right|\left\{\frac{(k-1)(k+1-\sqrt{2})}{\sqrt{2}}\right\}\right] \text{ for n = 3,}$$

where c_n is defined as in the Theorem. This completes the proof of Theorem 1.

Remark 1. Since $k \geq 1$ and n ≥ 3, it follows that $\dfrac{k^{n-1}-1}{n-1} \geq \dfrac{k^{n-3}-1}{n-3}$ and $\dfrac{k^n-1}{n} > k-1$. Therefore the above Theorem 1 is a refinement of Theorem A.

If we divide the both sides of (5) and (6) by $|\,\alpha\,|$ and let $|\,\alpha\,| \to \infty$, we get the following result, which is a refinement of the inequality (3).

Corollary 1. *Let* $P(z) = \sum_{j=1}^{n} a_j z^j$, $a_n a_0 \neq 0$ *be a polynomial of degree n ≥ 3, having all its zeros in* $|\,z\,| \leq k,\; k \geq 1$*, then for every real or complex number α with* $|\,\alpha\,| \geq k$,

(16) $$\underset{|z|=1}{Max}|P'(z)| \geq \frac{n}{1+k^n}\left\{\underset{|z|=1}{Max}|P(z)| + \underset{|z|=k}{Min}|P(z)| + 2\frac{|a_{n-1}|}{(n+1)}\left[\frac{k^n-1}{n} - (k-1)\right]\right.$$
$$+ 2|a_{n-2}|\left[\left[\left\{\frac{(k^n-1)-n(k-1)}{n(n-1)}\right\} - \left\{\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}\right\}\right]\right\}$$
$$+ \frac{2|a_2|}{k^{n-1}}\left[\left\{\frac{(c_{n-1}+1)(k^{n-1}-1)}{n-1}\right\} - \left\{\frac{k^{n-3}-1}{n-3}\right\}\right] \quad \textit{for } n > 3.$$

and

(17) $$\underset{|z|=1}{Max}|P'(z)| \geq \frac{n}{1+k^n}\left\{\underset{|z|=1}{Max}|P(z)| + \underset{|z|=k}{Min}|P(z)| + 2\frac{|a_2|}{(n+1)}\left[\frac{k^n-1}{n} - (k-1)\right]\right.$$
$$+ \frac{1}{k^2}\left[\frac{2|a_1|(k-1)^n}{n(n-1)}\right]\right\} + \frac{2}{k^2}|a_2|\left\{\frac{(k-1)(k+1-\sqrt{2})}{\sqrt{2}}\right\} \quad \textit{for } n = 3,$$

where c_n *is defined as in the Theorem.*

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