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SHARP BOUNDS FOR SÁNDOR-YANG MEANS IN TERMS OF LEHMER MEANS

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Abstract. In the article, the authors prove that the double inequalities $L_0(a,b) < S_{AQ}(a,b) < L_{1/6}(a,b)$, $L_0(a,b) < S_{QA}(a,b) < L_{1/3}(a,b)$ holds for all $a,b > 0$ with $a \neq b$, where $L_p(a,b) = \left(a^{p+1}+b^{p+1}\right)/\left(a^p+b^p\right)$ is the pth Lehmer mean, and $S_{AQ}(a,b)$, $S_{QA}(a,b)$ are the Sándor-Yang means, respectively.

Keywords: Lehmer mean; Sándor-Yang mean; inequality.

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1. Introduction

For $p \in \mathbb{R}$ and $a,b > 0$,the Sándor-Yang means $S_{AQ}(a,b)$ and $S_{QA}(a,b)$[2] ,and Lehmer mean $L_p(a,b)$[1] are defined by

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$$S_{AQ}(a,b) = Q(a,b)\mathrm{e}^{\frac{A(a,b)}{T(a,b)}-1}, \tag{1.1}$$

$$S_{QA}(a,b) = A(a,b)\,\mathrm{e}^{\frac{Q(a,b)}{M(a,b)}-1} \tag{1.2}$$

and

$$L_p(a,b) = \frac{a^{p+1}+b^{p+1}}{a^p+b^p} \tag{1.3}$$

where $Q(a,b) = \sqrt{(a^2+b^2)/2}$, $A(a,b) = (a+b)/2$, $T(a,b) = (a-b)/[2\arctan((a-b)/(a+b))]$ and $M(a,b) = (a-b)/[2\operatorname{arcsin}h((a-b)/(a+b))]$ are respectively the quadratic, arithmetic, second Seiffert and Neuman-Sándor means of a and b.

Recently, the inequalities and sharp bounds for the bivariate means have attracted the attention of many researchers. In particular, many remarkable inequalities for the Sándor-Yang mean and the Lehmer mean can be found in the literature [2, 3, 4, 5, 7, 10, 11].

Xu[6] find the best possible parameters $\alpha_1 \le 2/3$, $\beta_1 \ge (1+\sqrt{2})[(1+\sqrt{2})^{\sqrt{2}} - \mathrm{e}]/\mathrm{e} = 0.6747\cdots$, $\alpha_2 \le 1/3, \beta_2 \ge (\sqrt{2}\mathrm{e}^{\frac{\pi}{4}-1}-1)/(\sqrt{2}-1) = 0.3405\cdots$, $\alpha_3 \le (1+\sqrt{2})^{\sqrt{2}}/\mathrm{e}-1 = 0.2794\cdots, \beta_3 \ge 1/3$, $\alpha_4 \le \sqrt{2}\mathrm{e}^{\frac{\pi}{4}-1}-1 = 0.1410\cdots, \beta_4 \ge 1/6$ such that the double inequalities

$$\alpha_1 Q(a,b) + (1-\alpha_1)A(a,b) < S_{QA}(a,b) < \beta_1 Q(a,b) + (1-\beta_1)A(a,b),$$

$$\alpha_2 Q(a,b) + (1-\alpha_2)A(a,b) < S_{AQ}(a,b) < \beta_2 Q(a,b) + (1-\beta_2)A(a,b),$$

$$\alpha_3 C(a,b) + (1-\alpha_3)A(a,b) < S_{QA}(a,b) < \beta_3 C(a,b) + (1-\beta_3)A(a,b),$$

$$\alpha_4 C(a,b) + (1-\alpha_4)A(a,b) < S_{AQ}(a,b) < \beta_4 C(a,b) + (1-\beta_4)A(a,b)$$

holds for all $a,b > 0$ with $a \ne b$.

In [8, 9], the authors proved that the double inequalities

$$L_0(a,b) < M(a,b) < L_{1/6}(a,b),$$

$$L_0(a,b) < T(a,b) < L_{1/3}(a,b),$$

holds for all $a,b > 0$ with $a \ne b$.

The main purpose of this paper is to present the best possible parameters $\lambda_1, \lambda_2, \mu_1$ and μ_2 such that the double inequalities

$$L_{\lambda_1}(a,b) < S_{AQ}(a,b) < L_{\mu_1}(a,b), L_{\lambda_2}(a,b) < S_{QA}(a,b) < L_{\mu_2}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$.

2. Main results

Theorem 2.1. The double inequality

$$L_{\lambda_1}(a,b) < S_{AQ}(a,b) < L_{\mu_1}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq 0$ and $\mu_1 \geq 1/6$.

Proof Since $L_p(a,b)$ and $S_{AQ}(a,b)$ are symmetric and homogenous of degree one, without loss of generality, we assume that $a = x > 1$ and $b = 1$. Let $p \in \mathbb{R}$, then (1.1) and (1.3) lead to

$$\begin{aligned} & \log\left[S_{AQ}(a,b)\right] - \log\left[L_p(a,b)\right] \\ & = \frac{1}{2}\log\left(\frac{x^2+1}{2}\right) + \frac{x+1}{x-1}\arctan\left(\frac{x-1}{x+1}\right) - \log\left(\frac{x^{p+1}+1}{x^p+1}\right) - 1. \end{aligned} \tag{2.1}$$

Let

$$F(x) = \frac{1}{2}\log\left(\frac{x^2+1}{2}\right) + \frac{x+1}{x-1}\arctan\left(\frac{x-1}{x+1}\right) - \log\left(\frac{x^{p+1}+1}{x^p+1}\right) - 1. \tag{2.2}$$

Then simple computations lead to

$$F\left(1^+\right) = 0, \tag{2.3}$$

$$F'(x) = \frac{1}{(x-1)^2}F_1(x), \tag{2.4}$$

where

$$F_1(x) = \frac{(x-1)\left[x^{2p} - px^{p+1} + 2(p+1)x^p - px^{p-1} + 1\right]}{\left(x^p+1\right)\left(x^{p+1}+1\right)} - 2\arctan\left(\frac{x-1}{x+1}\right),$$

$$F_1\left(1\right) = 0, \tag{2.5}$$

$$F_1'(x) = -\frac{x-1}{x^2\left(x+1\right)^2\left(x^p+1\right)^2\left(x^{p+1}+1\right)^2}f(x), \tag{2.6}$$

where

$$f(x)=x^{4p+3}+x^{4p+2}-p(p-1)x^{3p+5}+2p(p+1)x^{3p+4}-2p(p+1)x^{3p+3}$$
$$+2\left(p^2+p+2\right)x^{3p+2}-p(p+3)x^{3p+1}+px^{2p+5}+5px^{2p+4}-4(p+1)x^{2p+3}$$
$$+4(p+1)x^{2p+2}-5px^{2p+1}-px^{2p}+p(p+3)x^{p+4}-2\left(p^2+p+2\right)x^{p+3}$$
$$+2p(p+1)x^{p+2}-2p(p+1)x^{p+1}+p(p-1)x^{p}-x^3-x^2. \tag{2.7}$$

We divide the proof into four cases.

Case 1 $p=0$. Then (2.7) becomes

$$f(x)=8x^2(1-x)<0 \tag{2.8}$$

for $x>1$.

Therefore,

$$S_{AQ}(a,b)>L_0(a,b)$$

follows easily from (2.1)–(2.6) and (2.8).

Case 2 $p>0$. Then (1.1) and (1.3) lead to

$$\lim_{x\to+\infty}\frac{L_p(x,1)}{S_{AQ}(x,1)}=\lim_{x\to+\infty}\frac{\sqrt{2}\left(x^{p+1}+1\right)}{\sqrt{x^2+1}\,(x^p+1)\,e^{\frac{(x+1)\arctan\left(\frac{x-1}{x+1}\right)}{x-1}-1}}=\frac{\sqrt{2}}{e^{\frac{\pi}{4}-1}}>1. \tag{2.9}$$

Inequality (2.9) implies that there exists large enough $X_1=X_1(p)>1$ such that $S_{AQ}(x,1)<L_p(x,1)$ for all $x\in(X_1,+\infty)$.

Case 3 $p=1/6$. Then (2.7) lead to

$$f(x)=\frac{1}{36}x(x^{1/3}-1)^3(x^{1/6}+1)^2(5x^4-4x^{23/6}+18x^{11/3}-14x^{7/2}+40x^{10/3}-30x^{19/6}+84x^3$$
$$-48x^{17/6}+148x^{8/3}-68x^{5/2}+234x^{7/3}-58x^{13/6}+262x^2-58x^{11/6}+234x^{5/3}$$
$$-68x^{3/2}+148x^{4/3}-48x^{7/6}+84x-30x^{5/6}+40x^{2/3}-14x^{1/2}+18x^{1/3}-4x^{1/6}+5)$$
$$\geq\frac{1}{36}x(x^{1/3}-1)^3(x^{1/6}+1)^2(x^4+4x^{11/3}+10x^{10/3}+36x^3+80x^{8/3}+176x^{7/3}+204x^2$$
$$+166x^{5/3}+100x^{4/3}+54x+26x^{2/3}+14x^{1/3}+5)>0 \tag{2.10}$$

for $x>1$.

From (2.4)–(2.6) and (2.10) we clearly see that $F(x)$ is strictly decreasing on $(1,+\infty)$.

Therefore,

$$S_{AQ}(a,b) < L_{1/6}(a,b)$$

follows from (2.1)–(2.3) and the monotonicity of $F(x)$.

Case 4 $p < 1/6$. Let $x > 0$ and $x \to 0$, then making use of (1.1) and (1.3) together with the Taylor expansion we get

$$\begin{aligned} & S_{AQ}(1,1+x) - L_p(1,1+x) \\ &= \sqrt{\frac{1+(1+x)^2}{2}} \mathrm{e}^{\frac{(2+x)\arctan\left(\frac{x}{2+x}\right)}{x} - 1} - \frac{1+(1+x)^{p+1}}{1+(1+x)^p} \\ &= \frac{1-6p}{24} x^2 + o\left(x^2\right). \end{aligned} \tag{2.11}$$

Equation (2.11) implies that there exists small enough $\delta_1 = \delta_1(p) > 0$ such that $S_{AQ}(1,1+x) > L_p(1,1+x)$ for all $x \in (0, \delta_1)$.

Theorem 2.2. The double inequality

$$L_{\lambda_2}(a,b) < S_{QA}(a,b) < L_{\mu_2}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_2 \leq 0$ and $\mu_2 \geq 1/3$.

Proof Since $L_p(a,b)$ and $S_{QA}(a,b)$ are symmetric and homogenous of degree one, without loss of generality, we assume that $a = x > 1$ and $b = 1$. Let $p \in \mathbb{R}$, then (1.2) and (1.3) lead to

$$\begin{aligned} & \log\left[S_{QA}(a,b)\right] - \log\left[L_p(a,b)\right] \\ &= \log\left(\frac{x+1}{2}\right) + \frac{\sqrt{2\left(x^2+1\right)}\operatorname{arcsin}h\left(\frac{x-1}{x+1}\right)}{x-1} - \log\left(\frac{x^{p+1}+1}{x^p+1}\right) - 1. \end{aligned} \tag{2.12}$$

Let

$$G(x) = \log\left(\frac{x+1}{2}\right) + \frac{\sqrt{2\left(x^2+1\right)}\operatorname{arcsin}h\left(\frac{x-1}{x+1}\right)}{x-1} - \log\left(\frac{x^{p+1}+1}{x^p+1}\right) - 1. \tag{2.13}$$

Then elaborated computations lead to

$$G(1^+) = 0, \tag{2.14}$$

$$G'(x) = \frac{x+1}{\sqrt{x^2+1}\,(x-1)^2} G_1(x), \tag{2.15}$$

where

$$G_1(x) = \frac{\sqrt{x^2+1}\,(x-1)\left[x^{2p} - px^{p+1} + 2(p+1)x^p - px^{p-1} + 1\right]}{(x+1)\,(x^p+1)\,(x^{p+1}+1)} - \sqrt{2}\operatorname{arcsin}h\left(\frac{x-1}{x+1}\right).$$

$$G_1(1) = 0, \tag{2.16}$$

$$G_1'(x) = -\frac{x-1}{x^2\sqrt{x^2+1}\,(x+1)^2\,(x^p+1)^2\,(x^{p+1}+1)^2}\,g(x), \tag{2.17}$$

where

$$\begin{aligned} g(x) = {} & 3x^{4p+3} + x^{4p+2} - p\,(p-1)\,x^{3p+6} + p\,(p+4)\,x^{3p+5} - 3(p+1)x^{3p+4} + 3\,(p+2)\,x^{3p+3} \\ & + \left(p^2 - 2p + 5\right)x^{3p+2} - p\,(p+3)\,x^{3p+1} + px^{2p+6} + 7px^{2p+5} - (p+7)\,x^{2p+4} \\ & + (p+7)\,x^{2p+2} - 7px^{2p+1} - px^{2p} + p\,(p+3)\,x^{p+5} - \left(p^2 - 2p + 5\right)x^{p+4} \\ & - 3\,(p+2)\,x^{p+3} + 3(p+1)x^{p+2} - p\,(p+4)\,x^{p+1} + p\,(p-1)\,x^p - x^4 - 3x^3. \end{aligned} \tag{2.18}$$

We divide the proof into four cases.

Case 1 $p = 0$. Then (2.18) becomes

$$g(x) = 16x^2\left(1 - x^2\right) < 0 \tag{2.19}$$

for $x > 1$.

Therefore,

$$S_{QA}(a,b) > L_0(a,b)$$

follows easily from (2.12)–(2.17) and (2.19).

Case 2 $p > 0$. Then (1.2) and (1.3) lead to

$$\lim_{x\to+\infty} \frac{L_p\,(x,1)}{S_{QA}\,(x,1)} = \lim_{x\to+\infty} \frac{2\left(x^{p+1}+1\right)}{(x+1)\,(x^p+1)\,\mathrm{e}^{\frac{\sqrt{2(x^2+1)}\operatorname{arcsin}h\left(\frac{x-1}{x+1}\right)}{x-1} - 1}} = \frac{2\mathrm{e}}{\left(1+\sqrt{2}\right)^{\sqrt{2}}} > 1. \tag{2.20}$$

Inequality (2.20) implies that there exists large enough $X_2 = X_2\,(p) > 1$ such that $S_{QA}(x,1) < L_p(x,1)$ for all $x \in (X_2, +\infty)$.

Case 3 $p = 1/3$. Then (2.18) lead to

$$\begin{aligned} g(x) = & \frac{1}{9}x^{1/3}(x^{2/3}-1)(x^{1/3}-1)^2(2x^{16/3}+7x^5+14x^{14/3}+37x^{13/3}+83x^4+155x^{11/3} \\ & +214x^{10/3}+233x^3+266x^{8/3}+233x^{7/3}+214x^2+155x^{5/3}+83x^{4/3}+37x \\ & +14x^{2/3}+7x^{1/3}+2) \end{aligned} \tag{2.21}$$

for $x > 1$.

From (2.15)–(2.17) and (2.21) we clearly see that $G(x)$ is strictly decreasing on $(1, +\infty)$.

Therefore,

$$S_{QA}(a,b) < L_{1/3}(a,b)$$

follows from (2.12)–(2.14) and the monotonicity of $G(x)$.

Case 4 $p < 1/3$. Let $x > 0$ and $x \to 0$, then making use of (1.2) and (1.3) together with the Taylor expansion we get

$$\begin{aligned} & S_{QA}(1,1+x) - L_p(1,1+x) \\ & = \left(1+\frac{x}{2}\right) e^{\frac{\sqrt{2\left(1+(1+x)^2\right)} \operatorname{arcsin} h\left(\frac{x}{2+x}\right)}{x}-1} - \frac{1+(1+x)^{p+1}}{1+(1+x)^p} \\ & = \frac{1-3p}{12}x^2 + o\left(x^2\right). \end{aligned} \tag{2.22}$$

Equation (2.22) implies that there exists small enough $\delta_2 = \delta_2(p) > 0$ such that $S_{QA}(1,1+x) > L_p(1,1+x)$ for all $x \in (0, \delta_2)$.

Let $x \in (0,1)$,$a = 1+x, b = 1-x$. Then Theorems 2.1-2.2 lead to Corollary 2.1 immediately.

Corollary 2.1. *The double inequality*

$$\begin{aligned} & 1 - \frac{1}{2}\log(1+x^2) < \frac{\arctan(x)}{x} < 1 - \frac{1}{2}\log(1+x^2) \\ & + \log\left[\sqrt[3]{1-x^2}(\sqrt[3]{1+x}+\sqrt[3]{1-x}) - \sqrt[6]{1-x^2}(\sqrt[3]{(1+x)^2}+\sqrt[3]{(1-x)^2}) - \sqrt{1-x^2}+2\right], \\ & \frac{1}{\sqrt{1+x^2}} < \frac{\operatorname{arcsin} h(x)}{x} < \frac{1}{\sqrt{1+x^2}} \\ & + \frac{\log\left[((1+x)\sqrt[3]{1+x}+(1-x)\sqrt[3]{1-x})/(\sqrt[3]{1+x}+\sqrt[3]{1-x})\right]}{\sqrt{1+x^2}} \end{aligned}$$

hold for all $x \in (0,1)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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