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HERMITE-HADAMARD TYPE INEQUALITY FOR SUGENO INTEGRALS USING GENERAL (α, m, r)- CONVEX FUNCTIONS

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Abstract. The aim of this paper is to obtain, Hermite-Hadamard type inequality for product of convex function using Sugeno integral which is based on general (α, m, r)-convex function. Some examples are also given.

Keywords: Hermite-Hadamard type inequality for product of convex function; Sugeno integrals; general (α, m, r)-convex function.

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1. Introduction

M. Sugeno [17] has introduced the theory of fuzzy measures and fuzzy integrals which has a wide applications in systems and control theory. Since then many authors have studied the various properties and applications on fuzzy integrals. In [15] Relescu and Admas proposed the equivalent definition of fuzzy integrals.

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Recently [1]-[9] authors have generalized the Sugeno integral. Recent literature reveals the integral inequalities for Sugeno integral such as Berwald type inequality [10], Barnes-Godunova-Levin type inequality [11], Hermite-Hadamard type inequality [12], general Minkowski type inequality [13], Cauchy-Schwarz type inequality [14], Sandor type inequality [16], etc.

The main objective of this paper is to study the Hermite-Hadamard type inequality for product of convex functions in fuzzy context.

2. **Preliminaries**

In this section we give some basic definitions and properties of the fuzzy integral see [17], [22].

Definition 2.1. [18] *Let $I \subseteq \mathbb{R}$ be an interval, $\lambda \in [0,1]$. A function $f : I \longrightarrow R$ is said to be convex on I if*

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y) \tag{1}$$

for all $x, y \in I$. If the above inequality reverse, then we say that the function f is concave on I.

Let X be a nonempty set and let $P(X) = \{A|A : X \to [0,1]\}$ be the class of all subsets of X.

Definition 2.2. [22] *Let σ-algebra $\wp$ be a nonempty subclass of $P(X)$ with the following properties:*

(1) $X, \phi \in \wp$.
(2) If $A \in \wp$, then $A^c \in \wp$.
(3) If $\{A_n\} \in \wp$, then $\bigcup_{n=1}^{\infty} A_n \in \wp$.

Let $\wp$ be a σ-algebra of subsets of X and $\mu : \wp \longrightarrow [0,\infty)$ be a non-negative, extended real valued set function. We say that μ is a fuzzy measure if it satisfies:

(1) $\mu(\phi) = 0$.
(2) $E, F \in \wp$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$.
(3) $\{E_n\} \subset \wp, E_1 \subset E_2 \subset ...$, imply $\lim_{n\longrightarrow\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$.
(4) $\{E_n\} \subset \wp$, $E_1 \supset E_2 \supset ...$, $\mu(E_1) < \infty$, imply $\lim_{n\longrightarrow\infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$.

If f is non-negative real-valued function defined on X, we denote the set $\{x \in X : f(x) \geq \alpha\} = \{x \in X : f \geq \alpha\}$ by F_α for $\alpha \geq 0$. Note that if $\alpha \leq \beta$ then $F_\beta \subset F_\alpha$.

Let $(X, \wp, \mu)$ be a fuzzy measure space, we denote M^+ the set of all non-negative measurable functions with respect to $\wp$.

Definition 2.3. *(Sugeno* [17]*). Let $(X, \wp, \mu)$ be a fuzzy measure space, $f \in M^+$ and $A \in \wp$, the Sugeno integral of f on A, with respect to the fuzzy measure μ, is defined as*

$$(s)\int_A f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(A \cap F_\alpha)],$$

when $A = X$,

$$(s)\int_X f d\mu = \bigvee_{\alpha \geq 0} [\alpha \wedge \mu(F_\alpha)],$$

where $\bigvee$ and $\wedge$ denote the operations sup and inf on $[0, \infty)$, respectively.

Now we give some basic properties of fuzzy integral given in [21].

Proposition 2.1. *Let $(X, \wp, \mu)$ be fuzzy measure space, $A, B \in \wp$ and $f, g \in M^+$ then:*

(1) $(s)\int_A f d\mu \leq \mu(A)$.

(2) $(s)\int_A k d\mu = k \wedge \mu(A)$, *$k$ non-negative constant.*

(3) $(s)\int_A f d\mu \leq (s)\int_A g d\mu$, *for $f \leq g$.*

(4) $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Longrightarrow (s)\int_A f d\mu \geq \alpha$.

(5) $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Longrightarrow (s)\int_A f d\mu \leq \alpha$.

(6) $(s)\int_A f d\mu > \alpha \Longleftrightarrow$ *there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha$.*

(7) $(s)\int_A f d\mu < \alpha \Longleftrightarrow$ *there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$.*

Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$. Then from (4) *and* (5) *of Proposition 2.1, we have $F(\alpha) = \alpha \Longrightarrow (s)\int_A f d\mu = \alpha$. Thus, the fuzzy integral can be calculated by solving the equation $F(\alpha) = \alpha$.*

Definition 2.4. [16] *Let $I \subseteq \mathbb{R}$ be an interval, $\lambda, \alpha, m \in [0, 1]$, $r \in \mathbb{R}$. t be a continuous and monotonous function on $\mathbb{R}$. A function $f : I \longrightarrow R$ is said to be general (α, m, r)-convex on I if*

$$f\left([\lambda x^r + m(1-\lambda)y^r]^{1/r}\right) \leq t^{-1}\left([\lambda^\alpha (t \circ f)^r(x) + m(1-\lambda^\alpha)(t \circ f)^r(y)]^{1/r}\right), \quad r \neq 0, \tag{2}$$

or

$$f\left(x^{\lambda}y^{m(1-\lambda)}\right)\le t^{-1}\left((t\circ f)^{\lambda^{\alpha}}(x)(t\circ f)^{m(1-\lambda^{\alpha})}(y)\right),\quad r=0, \tag{3}$$

for all $x,y\in I$. If the above inequalities reverse, then we say that the function f is general (α,m,r)-concave function on I.

Remark 2.1. [16] *If, in Definition* (2.4)*, $t=id$ (i.e., $t(x)=x$ for any $x\in I$), then one obtains the definition of (α,m,r)-convexity.*
If, in Definition (2.4)*, $\alpha,m=1$, then one obtains the definition of general r-mean convexity.*
If, in Definition (2.4)*, $\alpha,m=1$, $t=id$, then one obtains the definition of r-mean convexity* [20].
If, in Definition (2.4)*, $r=1$, then one obtain the definition of general (α,m)-convexity.*
If, in Definition (2.4)*, $r=1$, and $t=id$, then one obtains the definition of (α,m)- convexity* [18].
If $(\alpha,m,r)\in\{(0,0,1),(\alpha,0,1),(1,0,1),(1,m,1),(1,1,1),(\alpha,1,1)\}$ and $t=id$ in Definition (2.4)*, one obtain the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex and α-convex respectively.*

3. Main Results

Hermite-Hadamard type inequality for product of convex function was established by B. G. Pachpatte [19] which is as follows.

Theorem 3.1. *Let f, g be real valued, nonnegative and convex function on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\le\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b), \tag{4}$$

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b), \tag{5}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Now consider an example.

Example 3.1. *Consider* $X=[0,1]$ *and let* μ *be the Lebesgue measure on* X. *Now we take the function* $f(x)=\frac{1}{5}x^2$ *and* $g(x)=\frac{1}{5}x^2$ *and* $t(x)=\sqrt{x}$, *then* $f(x),g(x)$ *are general* $(1/2,1/3,2)$*-convex function. Then*

$$\begin{aligned}\frac{1}{5}x^2 = f\left(\left[x^2.1^2+\frac{1}{3}(1-x^2).0^2\right]^{1/2}\right) &\leq \left(\left[x.\frac{1}{5}+\frac{1}{3}(1-x).0\right]^{1/2}\right)^2\\ &=\frac{1}{5}x.\\ \frac{1}{5}x^2 = g\left(\left[x^2.1^2+\frac{1}{3}(1-x^2).0^2\right]^{1/2}\right) &\leq \left(\left[x.\frac{1}{5}+\frac{1}{3}(1-x).0\right]^{1/2}\right)^2\\ &=\frac{1}{5}x.\end{aligned}$$

For $x\in[0,1]$, *from a simple calculation we get*

$$(s)\int_0^1 \frac{1}{25}x^4 d\mu = 0.034727.$$

Also, $\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b)=0.01333.$

This proves that right hand side of (4) is not satisfied for Sugeno integral.

Now we give the Hermite-Hadamard type inequalities for product of convex function via Sugeno integral using general (α,m,r)-convex function.

Theorem 3.2. *Let* $(\alpha,m)\in(0,1]^2$, $r\in\mathbb{R}$, *and* $r\neq 0$, *let* t *be a continuous and monotonous function, let* $f,g:[0,1]\longrightarrow[0,\infty)$ *be general* (α,m,r)*- convex functions and let* μ *be the Lebesgue measure on* $\mathbb{R}$. *Then*

Case 1*: If* $(t\circ f)^r(1)-m(t\circ f)^r(0)>0$ *and* $(t\circ g)^r(1)-m(t\circ g)^r(0)>0$, *then*

$$(s)\int_0^1 f(x)g(x)d\mu \leq min\{\beta,1\},$$

where β *is given by*

$$\begin{aligned}&1-\left(\frac{t^r(\beta)-m(t\circ g)^r(0)}{(t\circ g)^r(1)-m(t\circ g)^r(0)}\right)^{1/\alpha r}-\left(\frac{t^r(\beta)-m(t\circ f)^r(0)}{(t\circ f)^r(1)-m(t\circ f)^r(0)}\right)^{1/\alpha r}\\ &+\left(\frac{t^r(\beta)-m(t\circ f)^r(0)}{(t\circ f)^r(1)-m(t\circ f)^r(0)}\right)^{1/\alpha r}.\left(\frac{t^r(\beta)-m(t\circ g)^r(0)}{(t\circ g)^r(1)-m(t\circ g)^r(0)}\right)^{1/\alpha r}=\beta. \qquad (6)\end{aligned}$$

Case 2: *If* $(t\circ f)^r(1)-m(t\circ f)^r(0)=0$ *and* $(t\circ g)^r(1)-m(t\circ g)^r(0)=0$, *then*

$$(s)\int_0^1 f(x)g(x)d\mu \le min\{m^{2/r}f(0)g(0),1\}.$$

Case 3: *If* $(t\circ f)^r(1)-m(t\circ f)^r(0)<0$ *and* $(t\circ g)^r(1)-m(t\circ g)^r(0)<0$, *then*

$$(s)\int_0^1 f(x)g(x)d\mu \le min\{\beta,1\},$$

where β *satisfies the following equation*

$$\begin{aligned}&\beta^{\alpha r}\big(((t\circ f)^r(1)-m(t\circ f)^r(0))((t\circ g)^r(1)-m(t\circ g)^r(0))\big)\\ &-(t^r(\beta)-m(t\circ f)^r(0))(t^r(\beta)-m(t\circ g)^r(0))=0. \qquad (7)\end{aligned}$$

Proof: As f,g are general (α,m,r)-convex function for $x\in[0,1]$, we have

$$\begin{aligned}f(x)=&f\big([x^r.1^r+m(1-x^r).0^r]^{1/r}\big)\\ \le&t^{-1}\big([x^{\alpha r}(t\circ f)^r(1)+m(1-x^{\alpha r})(t\circ f)^r(0)]^{1/r}\big)\\ =&h_1(x).\\ g(x)=&g\big([x^r.1^r+m(1-x^r).0^r]^{1/r}\big)\\ \le&t^{-1}\big([x^{\alpha r}(t\circ g)^r(1)+m(1-x^{\alpha r})(t\circ g)^r(0)]^{1/r}\big)\\ =&h_2(x).\end{aligned}$$

By Proposition 2.1, we have

$$\begin{aligned}(s)\int_0^1 f(x)g(x)d\mu=&(s)\int_0^1 f\big([x^r.1^r+m(1-x^r).0^r]^{1/r}\big).g\big([x^r.1^r+m(1-x^r).0^r]^{1/r}\big)d\mu\\ \le&(s)\int_0^1 t^{-1}\big([x^{\alpha r}(t\circ f)^r(1)+m(1-x^{\alpha r})(t\circ f)^r(0)]^{1/r}\big).\\ &t^{-1}\big([x^{\alpha r}(t\circ g)^r(1)+m(1-x^{\alpha r})(t\circ g)^r(0)]^{1/r}\big)d\mu\\ =&(s)\int_0^1 h_1(x)h_2(x)d\mu. \qquad (8)\end{aligned}$$

To calculate the right hand side of (8), we consider the distribution function F given by

$$F(\beta)=\mu([0,1]\cap\{h_1(x)h_2(x)\ge\beta\})$$

$$=\mu([0,1]\cap\{h_1(x)\geq\beta\}).\mu([0,1]\cap\{h_2(x)\geq\beta\})$$

$$=\mu\left([0,1]\cap\left\{x|t^{-1}\left([x^{\alpha r}(t\circ f)^r(1)+m(1-x^{\alpha r})(t\circ f)^r(0)]^{1/r}\right)\geq\beta\right\}\right).$$

$$\mu\left([0,1]\cap\left\{x|t^{-1}\left([x^{\alpha r}(t\circ g)^r(1)+m(1-x^{\alpha r})(t\circ g)^r(0)]^{1/r}\right)\geq\beta\right\}\right). \tag{9}$$

Case 1: If $(t\circ f)^r(1)-m(t\circ f)^r(0)>0$ and $(t\circ g)^r(1)-m(t\circ g)^r(0)>0$, then from (9), we have

$$F(\beta)=\mu\left([0,1]\cap\left\{x|x\geq\left(\frac{t^r(\beta)-m(t\circ f)^r(0)}{(t\circ f)^r(1)-m(t\circ f)^r(0)}\right)^{1/\alpha r}\right\}\right).$$

$$\mu\left([0,1]\cap\left\{x|x\geq\left(\frac{t^r(\beta)-m(t\circ g)^r(0)}{(t\circ g)^r(1)-m(t\circ g)^r(0)}\right)^{1/\alpha r}\right\}\right)$$

$$=\mu\left(\left(\frac{t^r(\beta)-m(t\circ f)^r(0)}{(t\circ f)^r(1)-m(t\circ f)^r(0)}\right)^{1/\alpha r},1\right).$$

$$\mu\left(\left(\frac{t^r(\beta)-m(t\circ g)^r(0)}{(t\circ g)^r(1)-m(t\circ g)^r(0)}\right)^{1/\alpha r},1\right)$$

$$=\left(1-\left(\frac{t^r(\beta)-m(t\circ f)^r(0)}{(t\circ f)^r(1)-m(t\circ f)^r(0)}\right)^{1/\alpha r}\right).$$

$$\left(1-\left(\frac{t^r(\beta)-m(t\circ g)^r(0)}{(t\circ g)^r(1)-m(t\circ g)^r(0)}\right)^{1/\alpha r}\right), \tag{10}$$

and the solution of the (10) is $F(\beta)=\beta$, given by (6). By Proposition 2.1, we have

$$(s)\int_0^1 f(x)g(x)d\mu\leq min\{\beta,1\}.$$

Case 2: If $(t\circ f)^r(1)-m(t\circ f)^r(0)=0$ and $(t\circ g)^r(1)-m(t\circ g)^r(0)=0$, then from (9), we have

$$F(\beta)=\mu([0,1]\cap\{x|m^{1/r}f(0)\geq\beta\}).\mu([0,1]\cap\{x|m^{1/r}g(0)\geq\beta\})$$

$$=m^{2/r}f(0)g(0), \tag{11}$$

and the solution of (11) is $F(\beta)=\beta$. By Proposition 2.1, we have

$$(s)\int_0^1 f(x)g(x)d\mu\leq min\{m^{2/r}f(0)g(0),1\}.$$

Case 3: If $(t\circ f)^r(1)-m(t\circ f)^r(0)<0$ and $(t\circ g)^r(1)-m(t\circ g)^r(0)<0$, then from (9), we have

$$
\begin{aligned}
F(\beta)=&\mu\left([0,1]\cap\left\{x|x\leq\left(\frac{t^r(\beta)-m(t\circ f)^r(0)}{(t\circ f)^r(1)-m(t\circ f)^r(0)}\right)^{1/\alpha r}\right\}\right).\\
&\mu\left([0,1]\cap\left\{x|x\leq\left(\frac{t^r(\beta)-m(t\circ g)^r(0)}{(t\circ g)^r(1)-m(t\circ g)^r(0)}\right)^{1/\alpha r}\right\}\right)\\
=&\mu\left(0,\left(\frac{t^r(\beta)-m(t\circ f)^r(0)}{(t\circ f)^r(1)-m(t\circ f)^r(0)}\right)^{1/\alpha r}\right).\\
&\mu\left(0,\left(\frac{t^r(\beta)-m(t\circ g)^r(0)}{(t\circ g)^r(1)-m(t\circ g)^r(0)}\right)^{1/\alpha r}\right)\\
=&\left(\frac{t^r(\beta)-m(t\circ f)^r(0)}{(t\circ f)^r(1)-m(t\circ f)^r(0)}\right)^{1/\alpha r}.\\
&\left(\frac{t^r(\beta)-m(t\circ g)^r(0)}{(t\circ g)^r(1)-m(t\circ g)^r(0)}\right)^{1/\alpha r}, \qquad (12)
\end{aligned}
$$

and the solution of the (12) is $F(\beta)=\beta$, given by (7). By Proposition 2.1, we have

$$(s)\int_0^1 f(x)g(x)d\mu\leq min\{\beta,1\}.$$

Remark 3.1. *If $\alpha=0$ in Theorem 3.2, then we have*

$$(s)\int_0^1 f(x)g(x)d\mu\leq min\{f(1)g(1),1\}.$$

Example 3.2. *Consider $X=[0,1]$ and let μ be the Lebesgue measure on X. If we take the functions $f(x)=x^2$ and $g(x)=x^3$, $t(x)=\sqrt{x}$ then $f(x),g(x)$ are a general $(1/2,1/3,2)$-convex function. In fact*

$$
\begin{aligned}
x^2=&f\left(\left[x^2.1^2+\frac{1}{3}(1-x^2).0^2\right]^{1/2}\right)\\
\leq&\left(\left[x+\frac{1}{3}(1-x).0\right]^{1/2}\right)^2=x.\\
x^3=&g\left(\left[x^2.1^2+\frac{1}{3}(1-x^2).0^2\right]^{1/2}\right)\\
\leq&\left(\left[x+\frac{1}{3}(1-x).0\right]^{1/2}\right)^2=x.
\end{aligned}
$$

Then by Theorem 3.2, we have

$$0.2451 = (s)\int_0^1 x^5 d\mu \leq min\{0.3819, 1\} = 0.3819. \tag{13}$$

Now the following theorem is the general case of Theorem3.2.

Theorem 3.3. *Let $(\alpha, m) \in (0,1]^2$, $r \in \mathbb{R}$ and $r \neq 0$, let t be a continuous and monotonous function, let $f, g : [a,b] \longrightarrow [0,\infty)$ be general (α, m, r)-convex functions and let μ be the Lebesgue measure on $\mathbb{R}$. Then*

Case 1*: If $(t \circ f)^r(b) - m(t \circ f)^r(a) > 0$ and $(t \circ g)^r(b) - m(t \circ g)^r(a) > 0$, then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where β is given by

$$\begin{aligned} b^2 &- b\left((b^r - ma^r)\left(\frac{t^r(\beta) - m(t \circ g)^r(a)}{(t \circ g)^r(b) - m(t \circ g)^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r} \\ &- b\left((b^r - ma^r)\left(\frac{t^r(\beta) - m(t \circ f)^r(a)}{(t \circ f)^r(b) - m(t \circ f)^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r} \\ &+ \left((b^r - ma^r)\left(\frac{t^r(\beta) - m(t \circ f)^r(a)}{(t \circ f)^r(b) - m(t \circ f)^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}. \end{aligned}$$

$$\left((b^r - ma^r)\left(\frac{t^r(\beta) - m(t \circ g)^r(a)}{(t \circ g)^r(b) - m(t \circ g)^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}. \tag{14}$$

Case 2*: If $(t \circ f)^r(b) - m(t \circ f)^r(a) = 0$ and $(t \circ g)^r(b) - m(t \circ g)^r(a) = 0$, then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{m^{2/r} f(a)g(a), b-a\}. \tag{15}$$

Case 3*: If $(t \circ f)^r(b) - m(t \circ f)^r(a) < 0$ and $(t \circ g)^r(b) - m(t \circ g)^r(a) < 0$, then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where β is given by

$$\left((b^r - ma^r)\left(\frac{t^r(\beta) - m(t \circ f)^r(a)}{(t \circ f)^r(b) - m(t \circ f)^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}.$$

$$\left((b^r - ma^r)\left(\frac{t^r(\beta) - m(t \circ g)^r(a)}{(t \circ g)^r(b) - m(t \circ g)^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}$$

$$- a\left((b^r - ma^r)\left(\frac{t^r(\beta) - m(t \circ f)^r(a)}{(t \circ f)^r(b) - m(t \circ f)^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}$$

$$-a\left((b^r-ma^r)\left(\frac{t^r(\beta)-m(t\circ g)^r(a)}{(t\circ g)^r(b)-m(t\circ g)^r(a)}\right)^{1/\alpha}+ma^r\right)^{1/r}+a^2=\beta. \tag{16}$$

Proof: As f,g are general (α,m,r)-convex functions for $x\in[a,b]$, we have

$$\begin{aligned}
f(x)=&f\left(\left[m\left(1-\frac{x^r-ma^r}{b^r-ma^r}\right)a^r+\frac{x^r-ma^r}{b^r-ma^r}.b^r\right]^{1/r}\right)\\
\leq& t^{-1}\left(\left[m\left(1-\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}\right)(t\circ f)^r(a)+\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}(t\circ f)^r(b)\right]^{1/r}\right)\\
=&h_1(x)\\
g(x)=&g\left(\left[m\left(1-\frac{x^r-ma^r}{b^r-ma^r}\right)a^r+\frac{x^r-ma^r}{b^r-ma^r}.b^r\right]^{1/r}\right)\\
\leq& t^{-1}\left(\left[m\left(1-\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}\right)(t\circ g)^r(a)+\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}(t\circ g)^r(b)\right]^{1/r}\right)\\
=&h_2(x).
\end{aligned}$$

By Proposition 2.1, we have

$$\begin{aligned}
(s)\int_a^b f(x)g(x)d\mu =& (s)\int_a^b f\left(\left[m\left(1-\frac{x^r-ma^r}{b^r-ma^r}\right)a^r+\frac{x^r-ma^r}{b^r-ma^r}.b^r\right]^{1/r}\right).\\
& g\left(\left[m\left(1-\frac{x^r-ma^r}{b^r-ma^r}\right)a^r+\frac{x^r-ma^r}{b^r-ma^r}.b^r\right]^{1/r}\right)d\mu\\
\leq (s)\int_a^b & t^{-1}\left(\left[m\left(1-\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}\right)(t\circ f)^r(a)+\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}(t\circ f)^r(b)\right]^{1/r}\right).\\
& t^{-1}\left(\left[m\left(1-\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}\right)(t\circ g)^r(a)+\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}(t\circ g)^r(b)\right]^{1/r}\right)d\mu
\end{aligned}$$

$$=(s)\int_a^b h_1(x)h_2(x)d\mu. \tag{17}$$

To calculate right hand side of (17), we consider the distribution function F given by

$$\begin{aligned}
F(\beta)=&\mu\Big([a,b]\cap\{h_1(x)h_2(x)\geq\beta\}\Big)\\
=&\mu\Big([a,b]\cap\{h_1(x)\geq\beta\}\Big).\mu\Big([a,b]\cap\{h_2(x)\geq\beta\}\Big)\\
=&\mu\left([a,b]\cap\left\{x|t^{-1}\left(\left[m\left(1-\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}\right)(t\circ f)^r(a)+\right.\right.\right.\right.
\end{aligned}$$

$$\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}(t\circ f)^r(b)\Big]^{1/r}\Big)\geq\beta\Big\}\Big).$$

$$\mu\left([a,b]\cap\left\{x|t^{-1}\left(\left[m\left(1-\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}\right)(t\circ g)^r(a)+\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left(\frac{x^r-ma^r}{b^r-ma^r}\right)^{\alpha}(t\circ g)^r(b)\right]^{1/r}\right)\geq\beta\right\}\right). \tag{18}$$

Case 1: If $(t\circ f)^r(b)-m(t\circ f)^r(a)>0$ and

$(t\circ g)^r(b)-m(t\circ g)^r(a)>0$, then from (18) we have

$$\begin{aligned}
F(\beta)=&\mu\left([a,b]\cap\left\{x|x\geq\left((b^r-ma^r)\left(\frac{t^r(\beta)-m(t\circ f)^r(a)}{(t\circ f)^r(b)-m(t\circ f)^r(a)}\right)^{1/\alpha}+ma^r\right)^{1/r}\right\}\right).\\
&\mu\left([a,b]\cap\left\{x|x\geq\left((b^r-ma^r)\left(\frac{t^r(\beta)-m(t\circ g)^r(a)}{(t\circ g)^r(b)-m(t\circ g)^r(a)}\right)^{1/\alpha}+ma^r\right)^{1/r}\right\}\right)\\
=&\mu\left(\left((b^r-ma^r)\left(\frac{t^r(\beta)-m(t\circ f)^r(a)}{(t\circ f)^r(b)-m(t\circ f)^r(a)}\right)^{1/\alpha}+ma^r\right)^{1/r},b\right).\\
&\mu\left(\left((b^r-ma^r)\left(\frac{t^r(\beta)-m(t\circ g)^r(a)}{(t\circ g)^r(b)-m(t\circ g)^r(a)}\right)^{1/\alpha}+ma^r\right)^{1/r},b\right)\\
=&\left(b-\left((b^r-ma^r)\left(\frac{t^r(\beta)-m(t\circ f)^r(a)}{(t\circ f)^r(b)-m(t\circ f)^r(a)}\right)^{1/\alpha}+ma^r\right)^{1/r}\right).\\
&\left(b-\left((b^r-ma^r)\left(\frac{t^r(\beta)-m(t\circ g)^r(a)}{(t\circ g)^r(b)-m(t\circ g)^r(a)}\right)^{1/\alpha}+ma^r\right)^{1/r}\right),
\end{aligned} \tag{19}$$

and the solution of the above equation $F(\beta)=\beta$, given in (14). By Proposition 2.1, we get

$$(s)\int_a^b f(x)g(x)d\mu\leq min\{\beta,b-a\},$$

The proof Case(2) and Case(3) can be given similarly, so we omit details.

Remark 3.2. *If $\alpha=0$ in Theorem 3.3, then we have*

$$(s)\int_a^b f(x)g(x)d\mu\leq min\{f(b)g(b),b-a\}.$$

Remark 3.3. *Let $(\alpha,m)\in[0,1]^2$, $r\in\mathbb{R}$, and $r\neq 0$, $t=id$, let $f,g:[a,b]\longrightarrow[0,\infty)$ be an (α,m,r) -convex functions, and let μ be the Lebesgue measure on $\mathbb{R}$. Then*

***Case 1**: If $f^r(b)-mf^r(a)>0$ and $g^r(b)-mg^r(a)>0$, then*

$$(s)\int_a^b f(x)g(x)d\mu\leq min\{\beta,b-a\},$$

where β *is given by*

$$b^2 - b\left((b^r - ma^r)\left(\frac{\beta^r - mg^r(a)}{g^r(b) - mg^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}$$

$$- b\left((b^r - ma^r)\left(\frac{\beta^r - mf^r(a)}{f^r(b) - mf^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}$$

$$+ \left((b^r - ma^r)\left(\frac{\beta^r - mf^r(a)}{f^r(b) - mf^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}.$$

$$\left((b^r - ma^r)\left(\frac{\beta^r - mg^r(a)}{g^r(b) - mg^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r} = \beta. \tag{20}$$

Case 2: *If* $f^r(b) - mf^r(a) = 0$ *and* $g^r(b) - mg^r(a) = 0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{m^{2/r}f(a)g(a), b - a\}.$$

Case 3: *If* $f^r(b) - mf^r(a) < 0$ *and* $g^r(b) - mg^r(a) < 0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b - a\},$$

where β *is given by*

$$\left((b^r - ma^r)\left(\frac{\beta^r - mf^r(a)}{f^r(b) - mf^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}.$$

$$\left((b^r - ma^r)\left(\frac{\beta^r - mg^r(a)}{g^r(b) - mg^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}$$

$$- a\left((b^r - ma^r)\left(\frac{\beta^r - mf^r(a)}{f^r(b) - mf^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r}$$

$$- a\left((b^r - ma^r)\left(\frac{\beta^r - mg^r(a)}{g^r(b) - mg^r(a)}\right)^{1/\alpha} + ma^r\right)^{1/r} + a^2 = \beta. \tag{21}$$

Remark 3.4. *Let* $\alpha = m = 1$, $r \in \mathbb{R}$, *and* $r \neq 0$, *let* t *be a continuous and monotonous function, let* $f, g : [a, b] \longrightarrow [0, \infty)$ *be general r-mean convex functions, and let* μ *be the Lebesgue measure on* $\mathbb{R}$. *Then*

Case 1: *If* $(t \circ f)^r(b) - (t \circ f)^r(a) > 0$ *and* $(t \circ g)^r(b) - (t \circ g)^r(a) > 0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b - a\},$$

where β *is given by*

$$b^2 - b\left((b^r - a^r)\left(\frac{t^r(\beta) - (t \circ g)^r(a)}{(t \circ g)^r(b) - (t \circ g)^r(a)}\right) + a^r\right)^{1/r}$$

$$-b\left((b^r-a^r)\left(\frac{t^r(\beta)-(t\circ f)^r(a)}{(t\circ f)^r(b)-(t\circ f)^r(a)}\right)+a^r\right)^{1/r}$$

$$+\left((b^r-a^r)\left(\frac{t^r(\beta)-(t\circ f)^r(a)}{(t\circ f)^r(b)-(t\circ f)^r(a)}\right)+a^r\right)^{1/r}.$$

$$\left((b^r-a^r)\left(\frac{t^r(\beta)-(t\circ g)^r(a)}{(t\circ g)^r(b)-(t\circ g)^r(a)}\right)+a^r\right)^{1/r}=\beta. \tag{22}$$

Case 2: *If* $(t\circ f)^r(b)-(t\circ f)^r(a)=0$ *and* $(t\circ g)^r(b)-(t\circ g)^r(a)=0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu\leq min\{f(a)g(a),b-a\}.$$

Case 3: *If* $(t\circ f)^r(b)-(t\circ f)^r(a)<0$ *and* $(t\circ g)^r(b)-(t\circ g)^r(a)<0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu\leq min\{\beta,b-a\},$$

where β *is given by*

$$\left((b^r-a^r)\left(\frac{t^r(\beta)-(t\circ f)^r(a)}{(t\circ f)^r(b)-(t\circ f)^r(a)}\right)+a^r\right)^{1/r}.$$

$$\left((b^r-a^r)\left(\frac{t^r(\beta)-(t\circ g)^r(a)}{(t\circ g)^r(b)-(t\circ g)^r(a)}\right)+a^r\right)^{1/r}$$

$$-a\left((b^r-a^r)\left(\frac{t^r(\beta)-(t\circ f)^r(a)}{(t\circ f)^r(b)-(t\circ f)^r(a)}\right)+a^r\right)^{1/r}$$

$$-a\left((b^r-a^r)\left(\frac{t^r(\beta)-(t\circ g)^r(a)}{(t\circ g)^r(b)-(t\circ g)^r(a)}\right)+a^r\right)^{1/r}+a^2=\beta. \tag{23}$$

Remark 3.5. *Let* $\alpha=m=1$, $r\in\mathbb{R}$, *and* $r\neq 0$, $t=id$, *let* $f,g:[a,b]\longrightarrow[0,\infty)$ *be an r-mean convex functions and* μ *be the Lebesgue measure on* $\mathbb{R}$. *Then*

Case 1: *If* $f^r(b)-f^r(a)>0$ *and* $g^r(b)-g^r(a)>0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu\leq min\{\beta,b-a\},$$

where β *is given by*

$$b^2-b\left((b^r-a^r)\left(\frac{\beta^r-g^r(a)}{g^r(b)-g^r(a)}\right)+a^r\right)^{1/r}-b\left((b^r-a^r)\left(\frac{\beta^r-f^r(a)}{f^r(b)-f^r(a)}\right)+a^r\right)^{1/r}$$

$$+\left((b^r-a^r)\left(\frac{\beta^r-f^r(a)}{f^r(b)-f^r(a)}\right)+a^r\right)^{1/r}.\left((b^r-a^r)\left(\frac{\beta^r-g^r(a)}{g^r(b)-g^r(a)}\right)+a^r\right)^{1/r}=\beta. \tag{24}$$

Case 2: *If* $f^r(b) - f^r(a) = 0$ *and* $g^r(b) - g^r(a) = 0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{f(a)g(a), b-a\}.$$

Case 3: *If* $f^r(b) - f^r(a) < 0$ *and* $g^r(b) - g^r(a) < 0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where β *is given by*

$$\left((b^r - a^r)\left(\frac{\beta^r - f^r(a)}{f^r(b) - f^r(a)}\right) + a^r\right)^{1/r} . \left((b^r - a^r)\left(\frac{\beta^r - g^r(a)}{g^r(b) - g^r(a)}\right) + a^r\right)^{1/r}$$
$$- a\left((b^r - a^r)\left(\frac{\beta^r - f^r(a)}{f^r(b) - f^r(a)}\right) + a^r\right)^{1/r}$$
$$- a\left((b^r - a^r)\left(\frac{\beta^r - g^r(a)}{g^r(b) - g^r(a)}\right) + a^r\right)^{1/r} + a^2 = \beta.$$

Remark 3.6. *Let* $(\alpha, m) \in [0,1]^2$ *and* $r = 1$, *let* t *be a continuous and monotonous function, let* $f, g : [a,b] \longrightarrow [0,\infty)$ *be a general* (α, m)*-convex function, let* μ *be the Lebesgue measure on* $\mathbb{R}$. *Then*

Case 1: *If* $(t\circ f)(b) - m(t\circ f)(a) > 0$ *and* $(t\circ g)(b) - m(t\circ g)(a) > 0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where β *is given by*

$$(b-ma)^2 - (b-ma)^2\left(\frac{t(\beta) - m(t\circ g)(a)}{(t\circ g)(b) - m(t\circ g)(a)}\right)^{1/\alpha}$$
$$- (b-ma)^2\left(\frac{t(\beta) - m(t\circ f)(a)}{(t\circ f)(b) - m(t\circ f)(a)}\right)^{1/\alpha}$$
$$+ (b-ma)^2\left(\frac{t(\beta) - m(t\circ f)(a)}{(t\circ f)(b) - m(t\circ f)(a)}\right)^{1/\alpha}\left(\frac{t(\beta) - m(t\circ g)(a)}{(t\circ g)(b) - m(t\circ g)(a)}\right)^{1/\alpha} = \beta. \tag{25}$$

Case 2: *If* $(t\circ f)(b) - m(t\circ f)(a) = 0$ *and* $(t\circ g)(b) - m(t\circ g)(a) = 0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{m^2 f(a)g(a), b-a\}.$$

Case 3: *If* $(t\circ f)(b) - m(t\circ f)(a) < 0$ *and* $(t\circ g)(b) - m(t\circ g)(a) < 0$, *then*

$$(s)\int_a^b f(x)g(x)d\mu \leq min\{\beta, b-a\},$$

where β is given by

$$(b-ma)^2\left(\frac{t(\beta)-m(t\circ f)(a)}{(t\circ f)(b)-m(t\circ f)(a)}\right)^{1/\alpha}\cdot\left(\frac{t(\beta)-m(t\circ g)(a)}{(t\circ g)(b)-m(t\circ g)(a)}\right)^{1/\alpha}$$
$$+(ma-a)(b-ma)\left(\frac{t(\beta)-m(t\circ f)(a)}{(t\circ f)(b)-m(t\circ f)(a)}\right)^{1/\alpha}$$
$$+(ma-a)(b-ma)\left(\frac{t(\beta)-m(t\circ g)(a)}{(t\circ g)(b)-m(t\circ g)(a)}\right)^{1/\alpha}+(ma-a)^2=\beta. \tag{26}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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