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NEW INEQUALITIES INVOLVING CIRCULAR, INVERSE CIRCULAR, HYPERBOLIC, INVERSE HYPERBOLIC AND EXPONENTIAL FUNCTIONS

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Abstract. In this article, inequalities involving circular, inverse circular, hyperbolic, inverse hyperbolic and exponential functions are established. Obtained results provide new lower and upper bounds for the functions $x/tanx$, $x/arcsinx$, $arctanx/x$, $tanhx/x$ and $arcsinhx/x$.

Keywords: Jordan's inequality; circular-inverse circular; hyperbolic-inverse hyperbolic; exponential; lower-upper bound.

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1. Introduction

In recent years, many researchers have tried to obtain different bounds for functions of the type $\frac{f(x)}{x}$ or $\frac{x}{f(x)}$ where $f(x)$ is circular, inverse circular, hyperbolic or inverse hyperbolic function. The origin of this research is well-known Jordan's inequality [4, 12] which is stated as

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follows

$$\frac{2}{\pi} < \frac{sinx}{x} < 1 \; ; \; x \in (0,1). \tag{1.1}$$

The following inequalities

$$\left(\frac{sinh^{-1}x}{x}\right)^2 \leqslant \frac{tan^{-1}x}{x} \leqslant \left(\frac{sinh^{-1}x}{x\sqrt{1+x^2}}\right)^{1/2} \; ; \; x \in \mathbb{R}. \tag{1.2}$$

$$\left(\frac{tanh^{-1}u}{u}\right)^2 \leqslant \frac{sin^{-1}x}{x} \leqslant \left(\frac{tanh^{-1}u}{u(1-u^2)}\right)^{1/2} \tag{1.3}$$

where $|x| < 1, \; u = \sqrt{\frac{1}{2}(1-\sqrt{1-x^2})}$ and

$$\left(\frac{tan^{-1}v}{v}\right)^2 \leqslant \frac{sinh^{-1}x}{x} \leqslant \left(\frac{tan^{-1}v}{v(1+v^2)}\right)^{1/2} \; ; \; x \in \mathbb{R}, \; v = \sqrt{\frac{1}{2}(\sqrt{1+x^2}-1)} \tag{1.4}$$

are due to Edward Neuman [6].

M. Becker and E. L. Stark [11] proved the inequality

$$\frac{\pi^2-4x^2}{\pi^2} < \frac{x}{tanx} < \frac{\pi^2-4x^2}{8} \; ; \; 0 < x < \frac{\pi}{2}. \tag{1.5}$$

Many inequalities of these type and their refinements have been proved in [1-16] and the references therein. Motivated by these studies we aim to give natural exponential bounds for the functions mentioned above and to improve the bounds of (1.5). This paper is the continuation of author's earlier work [14].

2. Main Results

All the main results will be obtained by using l'Hôpital's Rule of Monotonicity [8] which is stated as follows

Lemma 1. (The monotone form of l'Hôpital's rule [8]) : *Let $f,g : [a,b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a,b) and $g' \neq 0$ in (a,b). If f'/g' is increasing (or decreasing) on (a,b), then the functions $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$ are also increasing (or decreasing) on (a,b). If f'/g' is strictly monotone, then the monotonicity in the conclusion is also strict.*

Now we state the Main results and their proofs.

Theorem 1. *If* $x \in (0,1)$ *then*

$$e^{-ax^2} < \frac{x}{sin^{-1}x} < e^{-x^2/6} \tag{2.1}$$

with the best possible constants $a \approx 0.451583$ *and* $1/6$.

Proof. Let $e^{-ax^2} < \frac{x}{sin^{-1}x} < e^{-bx^2}$,which implies that, $b < \frac{log(sin^{-1}x/x)}{x^2} < a$.

Then

$$f(x) = \frac{sin^{-1}x}{x} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = sin^{-1}x$ and $f_2(x) = x$, with $f_1(0) = f_2(0) = 0$. Differentiation gives

$$\frac{f_1'(x)}{f_2'(x)} = \frac{1}{\sqrt{1-x^2}},$$

which is strictly increasing in $(0,1)$. By lemma 1, $f(x)$ is strictly increasing in $(0,1)$. Thus for $x_1 < x_2$ in $(0,1)$ we have

$\frac{sin^{-1}x_1}{x_1} < \frac{sin^{-1}x_2}{x_2}$. Moreover, $x < sin^{-1}x$ in $(0,1)$, giving us $1 < \frac{sin^{-1}x_1}{x_1} < \frac{sin^{-1}x_2}{x_2} < \frac{\pi}{2}$, which implies that

$$0 < log\left(\frac{sin^{-1}x_1}{x_1}\right) < log\left(\frac{sin^{-1}x_2}{x_2}\right) < log(\pi/2) \approx 0.451583.$$

Now, $\frac{log(sin^{-1}x_1/x_1)}{log(sin^{-1}x_2/x_2)} < 1$ and $x_1^2 < x_2^2$.

We claim that, $\frac{log(sin^{-1}x_1/x_1)}{x_1^2} < \frac{log(sin^{-1}x_2/x_2)}{x_2^2}$.

For if , $\frac{log(sin^{-1}x_1/x_1)}{x_1^2} \geqslant \frac{log(sin^{-1}x_2/x_2)}{x_2^2}$ then

$x_1^2 \leqslant \frac{log(sin^{-1}x_1/x_1)}{log(sin^{-1}x_2/x_2)}.x_2^2$, which is absurd.

Therefore , $\frac{log(sin^{-1}x_1/x_1)}{x_1^2} < \frac{log(sin^{-1}x_2/x_2)}{x_2^2}$ for $x_1 < x_2$ in $(0,1)$. Hence $\frac{log(sin^{-1}x/x)}{x^2}$ is strictly increasing in $(0,1)$.

Let, $g(x) = \frac{log(sin^{-1}x/x)}{x^2}$.

Consequently , $b = g(0+) = 1/6$ by l'Hôpital's rule and $a = g(1-) = log(sin^{-1}1) = log(\pi/2) \approx 0.451583$. □

Theorem 2. *If* $x \in (0,1)$ *then*

$$e^{-x^2/3} < \frac{tan^{-1}x}{x} < e^{-bx^2} \tag{2.2}$$

with the best possible constants $1/3$ *and* $b \approx 0.241564$.

Proof. Let $e^{-ax^2} < \frac{tan^{-1}x}{x} < e^{-bx^2}$,which implies that , $b < \frac{log(x/tan^{-1}x)}{x^2} < a$.

$$\text{Then, } f(x) = \frac{x}{tan^{-1}x} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = x$ and $f_2(x) = tan^{-1}x$, with $f_1(0) = f_2(0) = 0$. By differentiation we get

$$\frac{f_1'(x)}{f_2'(x)} = 1 + x^2, \text{ which is strictly increasing in } (0,1).$$

By Lemma 1, $f(x)$ is also strictly increasing in $(0,1)$. Therefore, for $x_1 < x_2$ in $(0,1)$ we have

$$\frac{x_1}{tan^{-1}x_1} < \frac{x_2}{tan^{-1}x_2}\ . \text{ Again } tan^{-1}x < x \text{ in } (0,1),$$

giving us $1 < \frac{x_1}{tan^{-1}x_1} < \frac{x_2}{tan^{-1}x_2} < \frac{1}{tan^{-1}(1)} \approx 1.273240$, which implies that

$$0 < log(x_1/tan^{-1}x_1) < log(x_2/tan^{-1}x_2) < 0.241565.$$

$$\text{Now, } \frac{log(x_1/tan^{-1}x_1)}{log(x_2/tan^{-1}x_2)} < 1 \text{ and } x_1^2 < x_2^2.$$

We claim that, $\frac{log(x_1/tan^{-1}x_1)}{x_1^2} > \frac{log(x_2/tan^{-1}x_2)}{x_2^2}$.

$$\text{For if, } \frac{log(x_1/tan^{-1}x_1)}{x_1^2} \leqslant \frac{log(x_2/tan^{-1}x_2)}{x_2^2} \text{ then}$$

$\frac{1}{x_1^2} \leqslant \frac{log(x_2/tan^{-1}x_2)}{log(x_1/tan^{-1}x_1)}.\frac{1}{x_2^2}$ which is nonsense. Therefore, $\frac{log(x_1/tan^{-1}x_1)}{x_1^2} > \frac{log(x_2/tan^{-1}x_2)}{x_2^2}$ for $x_1 < x_2$ in $(0,1)$. Thus, $\frac{log(x/tan^{-1}x)}{x^2}$ is strictly decreasing in $(0,1)$.

$$\text{Let, } g(x) = \frac{log(x/tan^{-1}x)}{x^2}.$$

Consequently, $a = g(0+) = 1/3$ by l'Hôpital's rule and $b = g(1-) = log(1/tan^{-1}1) = log(4/\pi) \approx 0.241564$. $\square$

Remark 1. Actually, we can see that $f(x) = \frac{x}{tan^{-1}x}$ is strictly decreasing in $(-\infty,0)$ and strictly increasing in $(0,\infty)$. Hence $g(x) = \frac{log(x/tan^{-1}x)}{x^2}$ is strictly increasing in $(-\infty,0)$ and strictly decreasing in $(0,\infty)$. Consequently, $a = g(0-) = 1/3, b = g(-\infty+) = 0$, by l'Hôpital's rule in $(-\infty,0)$ and $a = g(0+) = 1/3, b = g(\infty-) = 0$ by l'Hôpital's rule in $(0,\infty)$. Thus, for $x \in (-\infty,\infty)$

$$e^{-x^2/3} < \frac{tan^{-1}x}{x} < 1.$$

We now improve the bounds of (1.5) using natural exponential function.

Theorem 3. *If* $x \in (0,1)$ *then*

$$e^{-ax^2} < \frac{x}{tanx} < e^{-x^2/3} \tag{2.3}$$

with the best possible constants $a \approx 0.443023$ *and* $1/3$.

Proof. Let $e^{-ax^2} < \frac{x}{tanx} < e^{-bx^2}$, which implies that, $b < \frac{log(tanx/x)}{x^2} < a$.

$$\text{Then } f(x) = \frac{log(tanx/x)}{x^2} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = log(tanx/x)$ and $f_2(x) = x^2$, with $f_1(0+) = log1 = 0 = f_2(0)$. Differentiation gives

$$\frac{f_1'(x)}{f_2'(x)} = \frac{x.sec^2x - tanx}{2x^2.tanx} = \frac{f_3(x)}{f_4(x)}$$

where $f_3(x) = x.sec^2x - tanx$ and $f_4(x) = 2x^2.tanx$, with $f_3(0) = f_4(0) = 0$. Again by differentiation we have

$$\frac{f_3'(x)}{f_4'(x)} = \frac{sec^2x.tanx}{2tanx + x.sec^2x} = \frac{1}{2cos^2x + x/tanx}.$$

Clearly, cos^2x is strictly decreasing in $(0,1)$. And $x/tanx$ is also strictly decreasing in $(0,1)$. For if, $\frac{x_1}{tanx_1} \leqslant \frac{x_2}{tanx_2}$; $x_1 < x_2 \in (0,1)$ then $x_1 \leqslant \frac{tanx_1}{tanx_2}.x_2$ which is absurd , as $0 < \frac{tanx_1}{tanx_2} < 1$. Therefore, $\frac{f_3'(x)}{f_4'(x)} = \frac{1}{2cos^2x + x/tanx}$ is strictly increasing in $(0,1)$. By Lemma 1, $f(x)$ is strictly increasing in $(0,1)$. Consequently, $a = f(1-) = log(tan1) \approx 0.443023$ and $b = f(0+) = 1/3$ by l'Hôpital's rule. □

Note. There is no strict comparison between lower bounds $\frac{\pi^2 - 4x^2}{\pi^2}$ and e^{-ax^2}, where $a \approx 0.443023$ of $\frac{x}{tanx}$.

Remark 2. As a corollary , Theorem 2 and Theorem 3 give us

$$\frac{x}{tan^{-1}x} < \frac{tanx}{x};\ x \in (0,1). \tag{2.4}$$

which has already been proved in [9] and [13].

Theorem 4. *If* $x \in (0,1)$ *then*

$$e^{-x^2/6} < \frac{sinh^{-1}x}{x} < e^{-bx^2} \tag{2.5}$$

with the best possible constants $1/6$ *and* $b \approx 0.126274$.

Proof. Let $e^{-ax^2} < \frac{sinh^{-1}x}{x} < e^{-bx^2}$, which implies that, $b < \frac{log(x/sinh^{-1}x)}{x^2} < a$.

$$\text{Then } f(x) = \frac{x}{sinh^{-1}x} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = x$ and $f_2(x) = sinh^{-1}x$, with $f_1(0) = f_2(0) = 0$. By differentiation we get $\frac{f_1'(x)}{f_2'(x)} = \frac{1}{(1/\sqrt{1+x^2})} = \sqrt{1+x^2}$, which is clearly strictly increasing in $(0,1)$. By Lemma 1, $f(x)$ is also strictly increasing in $(0,1)$. Therefore for $x_1 < x_2$ in $(0,1)$ we have $\frac{x_1}{sinh^{-1}x_1} < \frac{x_2}{sinh^{-1}x_2}$.

Moreover, $sinh^{-1}x < x$ in $(0,1)$, giving us $1 < \frac{x_1}{sinh^{-1}x_1} < \frac{x_2}{sinh^{-1}x_2} < \frac{1}{sinh^{-1}1} \approx 1.134593$, which implies that

$$0 < log(x_1/sinh^{-1}x_1) < log(x_2/sinh^{-1}x_2) < 0.126274.$$

Now, $\frac{log(x_2/sinh^{-1}x_2)}{log(x_1/sinh^{-1}x_1)} > 1$ and $\frac{1}{x_2^2} < \frac{1}{x_1^2}$. We claim that, $\frac{log(x_1/sinh^{-1}x_1)}{x_1^2} > \frac{log(x_2/sinh^{-1}x_2)}{x_2^2}$.

For if, $\frac{log(x_1/sinh^{-1}x_1)}{x_1^2} \leqslant \frac{log(x_2/sinh^{-1}x_2)}{x_2^2}$ then $\frac{1}{x_1^2} \leqslant \frac{log(x_2/sinh^{-1}x_2)}{log(x_1/sinh^{-1}x_1)} . \frac{1}{x_2^2}$ which is nonsense. Therefore, $\frac{log(x/sinh^{-1}x)}{x^2}$ is strictly decreasing in $(0,1)$.

$$\text{Let } g(x) = \frac{log(x/sinh^{-1}x)}{x^2}.$$

Consequently, $a = g(0+) = 1/6$ by l'Hôpital's rule and $b = g(1-) = log(1/sinh^{-1}1) \approx 0.126274$. □

Theorem 5. *If* $x \in (0,1)$ *then*

$$e^{-x^2/3} < \frac{tanhx}{x} < e^{-bx^2} \tag{2.6}$$

with the best possible constants $1/3$ *and* $b \approx 0.272342$.

Proof. Let $e^{-ax^2} < \frac{tanhx}{x} < e^{-bx^2}$, which implies that, $b < \frac{log(x/tanhx)}{x^2} < a$.

$$\text{Then } f(x) = \frac{x}{tanhx} = \frac{f_1(x)}{f_2(x)}$$

where $f_1(x) = x$ and $f_2(x) = tanhx$, with $f_1(0) = f_2(0) = 0$. Differentiation gives $\frac{f_1'(x)}{f_2'(x)} = \frac{1}{sech^2x} = cosh^2x$, which is strictly increasing in $(0,1)$. By Lemma 1, $f(x)$ is strictly increasing in $(0,1)$.

Now, $g(x) = \frac{log(x/tanhx)}{x^2} = \frac{g_1(x)}{g_2(x)}$, where $g_1(x) = log(x/tanhx)$ and $g_2(x) = x^2$, with $g_1(0+) = g_2(0) = 0$. By differentiation we get

$$\frac{g_1'(x)}{g_2'(x)} = \frac{tanhx - x.sech^2x}{2x^2.tanhx} = \frac{g_3(x)}{g_4(x)}$$

where, $g_3(x) = tanhx - x.sech^2x$ and $g_4(x) = 2x^2.tanhx$, with $g_3(0) = g_4(0) = 0$. Again differentiating

$\frac{g_3'(x)}{g_4'(x)} = \frac{sech^2x.tanhx}{x.sech^2x + 2tanhx} = \frac{1}{x/tanhx + 4cosh^2x}$, which is decreasing as $x/tanhx$ and $cosh^2x$ are both increasing. By Lemma 1, $g(x)$ is decreasing in $(0,1)$. Consequently, $a = g(0+) = 1/3$ and $b = g(1-) = log(1/tanh1) \approx 0.272342$. □

Remark 3. Actually, $f(x) = \frac{x}{tanhx}$ is strictly decreasing in $(-\infty, 0)$ and strictly increasing in $(0, \infty)$. Hence, $g(x) = \frac{log(x/tanhx)}{x^2}$ is strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$. Consequently

$$e^{-x^2/3} < \frac{tanhx}{x} < 1;\ x \in (-\infty, \infty).$$

Corollary 1. *If* $x \in (0,1)$ *then*

$$\frac{x}{tanx} < \frac{tanhx}{x}. \tag{2.7}$$

Proof. The corollary is an immediate consequence of Theorem 3 and Theorem 5. □

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] Bhayo B. A. and Sándor, On certain old and new trigonometric and hyperbolic inequalities , J. Anal. Math. 41(2015), Article ID 3.

[2] Cătălin Barbu and Laurian-Ioan Pişcoran, Jordan type Inequalities using Monotony of Functions , J. Math. Inequal. 8 (1) (2014), 83-89.

[3] Chao-Ping Chen and József Sándor, Sharp Inequalities for Trigonometric and Hyperbolic Functions , J. Math. Inequal. 9 (1) (2015), 203-217.

[4] D. S. Mitrinovic, Analytic Inequalities , Springer-Verlag, New York, Berlin 1970.

[5] Edward Neuman and József Sándor, Optimal Inequalities For Hyperbolic And Trigonometric Functions, Bull. Math. Anal. Appl. 3 (3) (2011), 177-181.

[6] Edward Neuman, Inequalities Involving Inverse Circular and Inverse Hyperbolic Functions , Univ. Beograd. Publ. Elektrotehn. Fak. Ser.Mat. 18 (2007), 32-37.

[7] Feng Qi, Da-Wei Niu, and Bai-Ni Guo, Refinements, Generalizations, and Applications of Jordan's inequality and related problems , J. Inequal. Appl. 2009 (2009), Article ID 271923, 52 pages.

[8] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, Conformal Invarients, Inequalities and Quasiconformal maps , John Wiley and Sons, New York, 1997.

[9] József Sándor, On Certain Inequalities For Hyperbolic And Trigonometric Functions , J. Math. Inequal. 7 (3) (2013), 421-425.

[10] L. Zhu, J.Sun, Six new Redheffer-type inequalities for circular and hyperbolic functions, Comput. Math. Appl. 56 (2) (2008), 522-529.

[11] M. Becker and E. L. Stark, On a hierarchy of quolynomial Inequalities For tanx , Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602-633, 1978, 133-138.

[12] P. S. Bullen, A Directory of Inequalities , Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol.97 , Addison Wesley Longman Limited, Longman, Harlow 1998.

[13] R. Klén, M. Visuri and M. Vuorinen, On Jordan Type Inequalities for Hyperbolic Functions , J. Inequal. Appl. 2010 (2010), Art. ID 362548.

[14] Yogesh J. Bagul, Inequalities Involving Circular, Hyperbolic and Exponential Functions , J. Math. Inequal. 11 (3) (2017), 695-699.

[15] Yupei LV, Gendi Wang, Yuming Chu, A note on Jordan type inequalities for hyperbolic functions , Appl. Math. Lett. 25 (2012), 505-508.

[16] Zhen-Hang Yang, Yu-Ming Chu, Jordan Type Inequalities for Hyperbolic Functions and Their Applications , J. Funct. Spaces 2015 (2015), Article ID 370979.