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OPTIMAL BOUNDS FOR NEUMAN-SÁNDOR MEAN IN TERMS OF ONE-PARAMETER CENTROIDAL MEAN

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Abstract. In the paper, we find the best possible parameters $\alpha,\beta \in (0,1)$ and $\lambda,\mu \in (1/2,1)$ such that the double inequalities

$$\sqrt{\alpha E^2(a,b)+(1-\alpha)A^2(a,b)} < NS(a,b) < \sqrt{\beta E^2(a,b)+(1-\beta)A^2(a,b)},$$

$$E[\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a] < NS(a,b) < E[\mu a+(1-\mu)b, \mu b+(1-\mu)a]$$

holds for all $a,b>0$ with $a\neq b$, here $NS(a,b)=(a-b)/[2sinh^{-1}\ ((a-b)/(a+b))]$, $A(a,b)=(a+b)/2$ and $E(a,b)=2(a^2+ab+b^2)/[3(a+b)]$ are Neuman-Sándor, arithmetic and centroidal means of two positive real numbers a and b, respectively.

Keywords: Neuman-Sándor mean; arithmetic mean; centroidal mean.

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1. Introduction

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Let $a,b>0$ with $a\neq b$. Then Neuman-Sándor mean $NS(a,b)$ [1,2],arithmetic mean $A(a,b)$ and centroidal $E(a,b)$ are respectively defined by

$$NS(a,b)=\frac{a-b}{2sinh^{-1}[(a-b)/(a+b)]}, \tag{1.1}$$

$$A(a,b)=\frac{a+b}{2},E(a,b)=\frac{2(a^2+ab+b^2)}{3(a+b)}. \tag{1.2}$$

Recently, Neuman-Sándor mean have been the subject of intensive research. In particular, many remarkable inequalities and properties for these means can be found in the literature [7–17].

Let $T(a,b)=(a-b)/[2tan^{-1}((a-b)/(a+b))]$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$ and $C(a,b)=(a^2+b^2)/(a+b)$ be the second Seiffert, quadratic and contra-harmonic means of two positive real numbers a and b, respectively. Then it is well known that the inequalities

$$A(a,b)<NS(a,b)<T(a,b)<E(a,b)<Q(a,b)<C(a,b)$$

hold for all $a,b>0$ with $a\neq b$.

Neuman and Sándor [1,2] proved that the inequalities

$$A(a,b)<NS(a,b)<\frac{A(a,b)}{\log\left(1+\sqrt{2}\right)},\frac{\pi}{4}T(a,b)<NS(a,b)<T(a,b),$$

$$A(a,b)T(a,b)<NS^2(a,b)<\frac{1}{2}\left[A^2(a,b)+T^2(a,b)\right]$$

hold for all $a,b>0$ with $a\neq b$.

Neuman [3] proved that the double inequalities

$$\alpha Q(a,b)+(1-\alpha)A(a,b)<NS(a,b)<\beta Q(a,b)+(1-\beta)A(a,b)$$

$$\lambda C(a,b)+(1-\lambda)A(a,b)<NS(a,b)<\mu C(a,b)+(1-\mu)A(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq 1-\log(1+\sqrt{2})/[(\sqrt{2}-1)\log(1+\sqrt{2})]$, $\beta\geq 1/3,\lambda\leq 1-\log(1+\sqrt{2})/\log(1+\sqrt{2})$ and $\mu\geq 1/6$.

Qian and Chu [4] found the greatest value α, λ and the least value β, μ such that the double inequalities

$$E^{\alpha}(a,b)A^{1-\alpha}(a,b) < NS(a,b) < E^{\beta}(a,b)A^{1-\beta}(a,b),$$

$$\lambda E(a,b) + (1-\lambda)A(a,b) < NS(a,b) < \mu E(a,b) + (1-\mu)A(a,b)$$

for all $a,b > 0$ with $a \neq b$.

For $\alpha, \beta, \lambda, \mu \in (1/2, 1)$, Jiang and Qi [5,6] proved that the double inequalities

$$Q(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < NS(a,b) < Q(\beta a + (1-\beta)b, \beta b + (1-\beta)a),$$

$$C(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) < NS(a,b) < C(\mu a + (1-\mu)b, \mu b + (1-\mu)a),$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 1/2 + \sqrt{1/[\log(1+\sqrt{2})]^2 - 1}/2$, $\beta \geq 1/2 + \sqrt{3}/6$, $\lambda \leq 1/2 + \sqrt{1/\log(1+\sqrt{2}) - 1}/2$,$\mu \geq 1/2 + \sqrt{6}/12$.

Let $x \in [1/2, 1]$ and

$$J(x) = E[xa + (1-x)b, xb + (1-x)a]. \tag{1.3}$$

It is not difficult to directly verify that $J(x)$ is continuous and strictly increasing on $[1/2, 1]$ and to notice that

$$J(1/2) = A(a,b) < NS(a,b) < E(a,b) = J(1) \tag{1.4}$$

for all $a,b > 0$ with $a \neq b$.

Motivated by (1.3) and (1.4), it is natural to ask a question: what are the best possible parameters $\alpha, \beta \in (0,1)$ and $\lambda, \mu \in (1/2, 1)$ such that the double inequalities

$$\sqrt{\alpha E^2(a,b) + (1-\alpha)A^2(a,b)} < NS(a,b) < \sqrt{\beta E^2(a,b) + (1-\beta)A^2(a,b)},$$

$$E[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a] < NS(a,b) < E[\mu a + (1-\mu)b, \mu b + (1-\mu)a]$$

holds for all $a,b > 0$ with $a \neq b$? The main purpose of this paper is to answer there questions.

2. Lemmas

In order to prove our main results we need two Lemmas, which we present in this section.

Lemma 2.1. Let $p \in (0,1)$, $l_0 = \log\left(1+\sqrt{2}\right) = 0.8813\cdots$ and

$$f(x) = p^3x^{10} - p^2(18-13p)x^8 + 2p^2(23p+90)x^6 - 2p\left(5p^2+27p-486\right)x^4$$
$$+p(25p+9)(27-7p)x^2 + 125p^3 - 720p^2 + 1701p - 1458 \tag{2.1}$$

Then the following statements are true:

(1) If $p = 1/2$, then $f(x) > 0$ for all $x \in \left(1, \sqrt{2}\right)$;

(2) If $p = 9\left(1-l_0{}^2\right)/\left(7l_0{}^2\right) = 0.3693\cdots$, then there exists $\theta_0 \in \left(1, \sqrt{2}\right)$ such that $f(x) < 0$ for $x \in (1, \theta_0)$ and $f(x) > 0$ for $x \in \left(\theta_0, \sqrt{2}\right)$.

Proof For part (1), if $p = 1/2$, then (2.1) lead to

$$f(x) = \frac{1}{8}\left(x^2-1\right)\left(x^8 - 22x^6 + 384x^4 + 4154x^2 + 6175\right)$$
$$> \frac{1}{8}\left(x^2-1\right)\left(x^8 + 340x^4 + 4154x^2 + 6175\right) > 0 \tag{2.2}$$

$x \in \left(1, \sqrt{2}\right)$. Therefore, part (1) follows easily from (2.2).

For part (2), if $p = 9\left(1-l_0{}^2\right)/\left(7l_0{}^2\right) = 0.3693\cdots$. Then numerical computations lead to

$$5p^2 + 27p - 486 = -475.3443\cdots < 0, \tag{2.3}$$

$$f(1) = 2916p - 1458 = -380.8693\cdots < 0, \tag{2.4}$$

$$f\left(\sqrt{2}\right) = 243p^3 + 1440p^2 + 6075p - 1458 = 999.7924\cdots > 0 \tag{2.5}$$

It follows from (2.3) that

$$f'(x) = 2px\left[5p^2x^8 - 4p(18-13p)x^6 + 6p(23p+90)x^4\right.$$
$$\left.-4\left(5p^2+27p-486\right)x^2 + (25p+9)(27p-7p)\right]$$
$$> 2px\left[5p^2x^8 + 6p(11p+18)x^4 - 4\left(5p^2+27p-486\right)x^2\right.$$
$$\left.+(25p+9)(27p-7p)\right] > 0 \tag{2.6}$$

for $x \in (0,1)$.

Therefore, part (2) follows easily from (2.4),(2.5) and (2.6).

Lemma 2.2. Let $p \in [1/2, 1]$, and

$$g(x) = (1-2p)^4 x^3 + 4\left(4p^4 - 8p^3 + 9p^2 - 5p + 1\right)x^2$$
$$+4\left(-4p^4 + 8p^3 + 3p^2 - 7p + 2\right)x - 4\left(2p^2 - 2p - 1\right)^2 \tag{2.7}$$

Then the following statements are true:

(1) If $p = 1/2 + \sqrt{2}/4$, then $f(x) > 0$ for all $x \in (1, \sqrt{2})$;
(2) If $p = 1/2 + \sqrt{3\left[1/\log\left(1+\sqrt{2}\right) - 1\right]}/2 = 0.8177\cdots$, then there exists $\xi_0 \in \left(1, \sqrt{2}\right)$ such that $g(x) < 0$ for $x \in (1, \xi_0)$ and $g(x) > 0$ for $x \in \left(\xi_0, \sqrt{2}\right)$.

Proof For part (1), if $p = 1/2 + \sqrt{2}/4$, then (2.7) becomes

$$g(x) = \frac{1}{4}(x-1)\left(x^2 + 8x + 25\right) \tag{2.8}$$

Therefore, part (1) follows easily from (2.8).

For part (2), if $p = 1/2 + \sqrt{3\left[1/\log\left(1+\sqrt{2}\right) - 1\right]}/2 = 0.8177\cdots$. Then numerical computations lead to

$$4p^4 - 8p^3 + 9p^2 - 5p + 1 = 0.3425\cdots > 0, \tag{2.9}$$

$$-4p^4 + 8p^3 + 3p^2 - 7p + 2 = 0.8677\cdots > 0, \tag{2.10}$$

$$8p^2 - 8p + 1 = -0.1924\cdots < 0, \tag{2.11}$$

$$8p^4 - 16p^3 + 6\left(4+\sqrt{2}\right)p^2 - 2\left(8+3\sqrt{2}\right)p + 8 - 3\sqrt{2} = 0.2854\cdots > 0, \tag{2.12}$$

$$g(1) = 9\left(8p^2 - 8p + 1\right), \tag{2.13}$$

$$g\left(\sqrt{2}\right) = 2\left(1+\sqrt{2}\right)\left[8p^4 - 16p^3 + 6\left(4+\sqrt{2}\right)p^2 - 2\left(8+3\sqrt{2}\right)p + 8 - 3\sqrt{2}\right], \tag{2.14}$$

$$g'(x) = 3(1-2p)^4 x^2 + 8\left(4p^4 - 8p^3 + 9p^2 - 5p + 1\right)x$$
$$+4\left(-4p^4 + 8p^3 + 3p^2 - 7p + 2\right). \tag{2.15}$$

It follows from (2.8)-(2.15) that

$$g(1) < 0, g\left(\sqrt{2}\right) > 0, \tag{2.16}$$

and

$$g'(x) > 0 \tag{2.17}$$

for $x \in \left(1, \sqrt{2}\right)$.

Therefore, part (2) follows easily from (2.16) and (2.17).

3. Main results

Theorem 3.1. For $\alpha, \beta \in (0,1)$,the double inequality

$$\sqrt{\alpha E^2(a,b) + (1-\alpha)A^2(a,b)} < NS(a,b) < \sqrt{\beta E^2(a,b) + (1-\beta)A^2(a,b)}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 9\left(1 - {l_0}^2\right) / \left(7{l_0}^2\right) = 0.3693\cdots$, $\beta \geq 1/2$.

Proof Since $A(a,b)$, $E(a,b)$ and $NS(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$,$x = \sqrt{1+v^2} \in \left(1, \sqrt{2}\right)$ and $p \in (0,1)$. Then from (1.1) and (1.2) lead to

$$\begin{aligned} & NS\left(a,b\right) - \sqrt{pE^2\left(a,b\right) + (1-p)A^2\left(a,b\right)} \\ &= A\left(a,b\right)\left[\frac{v}{sinh^{-1}\left(v\right)} - \sqrt{p\left(\frac{1}{3}v^2+1\right)^2 + 1 - p}\right] \end{aligned}$$

$$= \frac{A\left(a,b\right)\left[p\left(x^2/3+2/3\right)^2 + 1 - p\right]}{sinh^{-1}\left(\sqrt{x^2-1}\right)\left[\sqrt{x^2-1} + sinh^{-1}\left(\sqrt{x^2-1}\right)\sqrt{p(x^2/3+2/3)^2+1-p}\right]}F\left(x\right), \quad (3.1)$$

where

$$F\left(x\right) = \frac{x^2-1}{p(x^2/3+2/3)^2+1-p} - \left[sinh^{-1}\left(\sqrt{x^2-1}\right)\right]^2 \quad (3.2)$$

Then simple computations lead to

$$F\left(1\right) = 0, \quad (3.3)$$

$$F\left(\sqrt{2}\right) = \frac{9}{7p+9} - {l_0}^2, \quad (3.4)$$

$$F'\left(x\right) = \frac{2}{\sqrt{x^2-1}}F_1\left(x\right), \quad (3.5)$$

where

$$F_1(x) = \frac{9x\sqrt{x^2-1}\left[9-p\left(x^2-1\right)^2\right]}{\left[p(x^2+2)^2+9(1-p)\right]^2} - sinh^{-1}\left(\sqrt{x^2-1}\right),$$

$$F_1(1) = 0, F_1\left(\sqrt{2}\right) = \frac{9\sqrt{2}(9-p)}{(7p+9)^2} - l_0 \tag{3.6}$$

$$F_1'(x) = -\frac{\sqrt{x^2-1}}{\left[p(x^2+2)^2+9(1-p)\right]^3} f(x) \tag{3.7}$$

where $f(x)$ is defined Lemma 2.1.

We divide the proof into four cases.

Case 1 $p = 1/2$. Then it follows from Lemma 2.1(1) and (3.7) that $F_1(x)$ is strictly decreasing on $\left(1, \sqrt{2}\right)$. Therefore,

$$NS(a,b) < \sqrt{\frac{E^2(a,b)+A^2(a,b)}{2}}$$

for all $a, b > 0$ with $a \neq b$ follows from (3.1)-(3.3),(3.5),(3.6) and the monotonicity of $F(x)$.

Case 2 $0 < p < 1/2$. Let $x > 0, x \to 0^+$, then it follows from (1.1) and (1.2) together with the Taylor expansion we get

$$\begin{aligned} & NS(1,1+x) - \sqrt{pE^2(1,1+x)+(1-p)A^2(1,1+x)} \\ = & \frac{x}{2sinh^{-1}\left(\frac{x}{2+x}\right)} - \sqrt{p\left[\frac{2(3+3x+x^2)}{3(2+x)}\right]^2 + (1-p)\left(\frac{2+x}{2}\right)^2} \\ = & \frac{(1-2p)}{24}x^2 + O\left(x^2\right). \end{aligned} \tag{3.8}$$

Equations (3.8) implies that there exists $0 < \delta_0 < 1$ such that

$$NS(1,1+x) > \sqrt{pE^2(1,1+x)+(1-p)A^2(1,1+x)}$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (0, \delta_0)$.

Case 3 $p = 9\left(1-{l_0}^2\right)/\left(7{l_0}^2\right) = 0.3693\cdots$. Then it follows from Lemma 2.1(2) and (3.7) that there exists $\theta_0 \in \left(1, \sqrt{2}\right)$ such that $F_1(x)$ is strictly increasing on $(1, \theta_0]$ and strictly decreasing on $\left[\theta_0, \sqrt{2}\right)$.

Equations (3.5) and (3.6) together with the piecewise monotonicity of $F_1(x)$ lead to the conclusion that there exists $\theta_0 \in \left(1, \sqrt{2}\right)$ such that $F(x)$ is strictly increasing on $(1, \theta_0]$ and strictly decreasing on $\left[\theta_0, \sqrt{2}\right)$, and

$$F\left(\sqrt{2}\right) = 0. \tag{3.9}$$

Therefore,

$$NS(a,b) > \sqrt{pE^2(a,b) + (1-p)A^2(a,b)}$$

for all $a, b > 0$ with $a \neq b$ follows from (3.1)-(3.3) and (3.9) together with the piecewise monotonicity of $F(x)$.

Case 4 $9\left(1 - {l_0}^2\right)/\left(7{l_0}^2\right) < p < 1$. Then (3.4) lead to

$$\lim_{x \to \sqrt{2}} F(x) < 0 \tag{3.10}$$

Equations (3.1), (3.2) and the inequality (3.10) imply that there exists $0 < \delta_1 < 1$ such that

$$NS(a,b) < \sqrt{pE^2(a,b) + (1-p)A^2(a,b)}$$

for all $a > b > 0$ with $(a-b)/(a+b) \in (1-\delta_1, 1)$.

Theorem 3.2. For $\lambda, \mu \in (1/2, 1)$, the double inequality

$$E[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a] < NS(a,b) < E[\mu a + (1-\mu)b, \mu b + (1-\mu)a]$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\lambda \leq 1/2 + \sqrt{3\left[1/\log\left(1+\sqrt{2}\right) - 1\right]}/2 = 0.8177\cdots$, $\mu \geq 1/2 + \sqrt{2}/4 = 0.8535\cdots$.

Proof Since both $E(a,b)$ and $NS(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$, $x = \sqrt{1+v^2} \in \left(1, \sqrt{2}\right)$ and $p \in (1/2, 1)$. Then from (1.1) and (1.2) lead to

$$\begin{aligned} &E[pa + (1-p)b, pb + (1-p)a] - NS(a,b) \\ &= A(a,b)\left[\frac{(1-2p)^2 v^2 + 3}{3} - \frac{v}{sinh^{-1}(v)}\right] \\ &= \frac{A(a,b)\left[(1-2p)^2\left(x^2-1\right) + 3\right]}{3sinh^{-1}\left(\sqrt{x^2-1}\right)} G(x), \end{aligned} \tag{3.11}$$

where

$$G(x) = sinh^{-1}\left(\sqrt{x^2-1}\right) - \frac{3\sqrt{x^2-1}}{(1-2p)^2x^2+3-(2p-1)^2}. \tag{3.12}$$

Then simple computations lead to

$$G(1) = 0, \tag{3.13}$$

$$G\left(\sqrt{2}\right) = \log\left(1+\sqrt{2}\right) - \frac{3}{4(p^2-p+1)}, \tag{3.14}$$

$$G'(x) = \frac{(x-1)}{\sqrt{x^2-1}\left[(1-2p)^2x^2+3-(1-2p)^2\right]^2} g(x) \tag{3.15}$$

where $g(x)$ is defined Lemma 2.2.

We divide the proof into four cases.

Case 1 $p = 1/2+\sqrt{2}/4$. Then from Lemma 2.2(1) and (3.15) lead to the conclusion that $G(x)$ is strictly increasing on $\left(1,\sqrt{2}\right)$. Therefore,

$$NS(a,b) < E\left[pa+(1-p)b, pb+(1-p)a\right]$$

for all $a,b>0$ with $a \neq b$ follows easily from (3.11) and (3.13) together with the monotonicity of $F(x)$.

Case 2 $1/2+\sqrt{2}/4 < p \leq 1$. let $q = (1-2p)^2$ and $v \to 0^+$,then $1/2 < q \leq 1$ and power series expansions lead to

$$\begin{aligned}\frac{(1-2p)^2v^2+3}{3} - \frac{v}{sinh^{-1}(v)} &= \frac{\left(qv^2+3\right)sinh^{-1}(v)-3v}{3sinh^{-1}(v)} \\ &= \frac{1}{3sinh^{-1}(v)}\left[\left(q-\frac{1}{2}\right)v^3+o\left(v^3\right)\right].\end{aligned} \tag{3.16}$$

Equations (3.11) and (3.16) imply that there exists small enough $0<\gamma_0<1$ such that

$$NS(a,b) > E\left[pa+(1-p)b, pb+(1-p)a\right]$$

for all $a>b>0$ with $(a-b)/(a+b) \in (0,\gamma_0)$.

Case 3 $p = 1/2+\sqrt{3\left[1/\log\left(1+\sqrt{2}\right)-1\right]}/2$. Then (3.14) and (3.15) together with Lemma 2.2(2) lead to the conclusion that there exists $\xi_0 \in \left(1,\sqrt{2}\right)$ such that $G(x)$ is strictly decreasing on $(1,\xi_0]$ and strictly increasing on $\left[\xi_0,\sqrt{2}\right)$, and

$$G\left(\sqrt{2}\right) = 0. \tag{3.17}$$

Therefore,

$$NS(a,b) > E\left[pa+(1-p)b, pb+(1-p)a\right]$$

for all $a,b>0$ with $a\neq b$ follows easily from (3.3) and (3.5) together with (3.17) and the piecewise monotonicity of $G(x)$.

Case 4 $1/2 \le p < 1/2+\sqrt{3\left[1/\log\left(1+\sqrt{2}\right)-1\right]}/2$. Then

$$\lim_{v\to 1^-}\left[\frac{(1-2p)^2v^2+3}{3}-\frac{v}{sinh^{-1}(v)}\right]=\frac{1}{3}\left[(1-2p)^2+3\right]-\frac{1}{\log\left(1+\sqrt{2}\right)}<0 \qquad (3.18)$$

Equations (3.3) and inequality (3.18) imply that there exists $0<\gamma_1<1$ such that

$$NS(a,b) < E\left[pa+(1-p)b, pb+(1-p)a\right]$$

for all $a>b>0$ with $(a-b)/(a+b)\in(1-\gamma_1,1)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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