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ON GENERATING SEMIGROUPS FROM A FUNCTION

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Abstract. A semigroup is a well studied structure so is a function. This paper describes a way in generating a semigroup from a function on a finite set, using its directed graph. Further this paper also studies the properties of the semigroup generated.

Keywords: directed graphs; functions; semigroups.

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1. Introduction

Let α be a function on a set X. Recent studies have shown when a function forms an inner translation (also known called Cayley functions) of a semigroup. In 1972 Zupnik characterised all Cayley functions algebraically(in powers of β) [7]. In 2016, Araoujo et all characterised the Cayley functions using functional digraphs[1]. He also described a process(a problem half solved) on generating a semigroup from Cayley functions by constructing a graph with Cayley functions as the vertices and identifying the maximal cliques of the common semigroup graph G_S(two Cayley functions are edge connected if appear on the same semigroup). There are many ways to construct a semigroup from graphs, for example Endomorphism semigroup,

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graph semigroup, commutative graph semigroup, inverse graph semigroup, path semigroup, etc. There is no instance of a semigroup from a function. This paper describes a way in generating a semigroup from a single function on a finite set, using directed graphs, studies the properties of the semigroup generated.

In the sequel α will denote a function mapping a non-empty finite set X onto itself. For any positive integer n, α^n denotes the n^{th} iterate of α. By α^0 we mean the identity function on S, so $\alpha^0(x) = x$. Let X be a finite set, then $T(X)$ denotes the set of all transformations (functions) from X to X.

2. Directed Graphs

A directed graph is an ordered pair $D = (S, \rho)$ where the elements of S are called vertices and A is a set of ordered pairs of vertices (binary relation), called arrows, directed edges.

Any pair $(a,b) \in \rho$ is called an arc of D, which we will write as $a \to b$. A vertex a is called is called an initial vertex in D if there is no b in ρ such that $b \to a$; it is called a terminal vertex in D if there is no $b \in S$ such that $a \to b$

Definition: A digraph D is called a functional digraph if there is $\alpha : S \longrightarrow S$ such that for all $x,y \in S$, $x \to y$ is an arc in D if and only if $\alpha(x) = y$.

Such a functional digraph is denoted as D_α as there is one and only one function that represents a functional di-graph.

Let D be a digraph and if there exists pairwise distinct vertices $..., x_1, x_0, x_1, ...$ of D such that $x_0 \longrightarrow x_1 \longrightarrow .... \longrightarrow x_{k-1} \longrightarrow x_0$ then the graph is said to have a cycle of length k denoted as $(x_0 x_1 ... x_{k-1})$. D has a chain of length m, denoted $[x_0 x_1 ... x_m]$ if there are pairwise distinct vertices such that $x_0 \longrightarrow x_1 \longrightarrow .... \longrightarrow x_m$ in D. Similarly D has a right ray [left ray or double ray] denoted $[x_0 x_1 x_2 ...\rangle$; $[\langle ... x_2 x_1 x_0], \langle ... x_1 x_0 x_1 ... \rangle]$ if there exist pairwise distinct vertices such that $x_0 \to x_1 \to x_2 \to ....$ $[(...... \to x_2 \to x_1 \to x_0)$ or $(.... \to x_{-1} \to x_0 \to x_1 \to .....)]$

Let S be a non-empty set, then $T(S)$ denoted the set of all functions on S, $\alpha \in T(S)$ and D_α the directed graph that represents α, then a right ray $[x_0 x_1 x_2 ....\rangle$ in D_α is called a maximal right ray if x_0 is an initial vertex of D_α.

Definition: Let D_α be a functional digraph, where $\alpha \in T(S)$

- A left ray $L = \langle ...x_2x_1x_0]$ in D_α is called an infinite branch of a cycle C[double ray W] in D_α if x_0 lies on $C[W]$ and x_1 does not lie on $C[W]$. We will refer to any such L as an infinite branch in D_α.
- A chain $P = [x_0x_1...x_m]$ of length $m \geq 1$ in D_α is called a finite branch of a cycle C [double ray W, maximal right ray R, infinite branch L] in D_α if x_0 is an initial vertex of D_α, x_m lies on $C[W,R,L]$ and $x_m \geq 1$ does not lie on $C[W,R,L]$. If x_m lies on an infinite branch $L = \langle ...y_2y_1y_0]$, we also require that $x_m \neq y_0$.

The components of D_α correspond to the connected components of the underlying undirected graph of D_α where the component containing x is defined as follows

Definition: Let $\alpha \in T(S)$, $x \in S$. The subgraph of D_α induced by the set

$$\{y \in S : \alpha^k(y) = \alpha^m(x) \text{for some integers } k,m \geq 0\}$$

is called the component of D_α containing x.

The following two propositions is due to [4] shows the character of functional digraphs.

Proposition: Let D_α be a functional digraph. Then for every component A of D_α , exactly one of the following three conditions holds:

(1) A has a unique cycle but not a double ray or right ray;
(2) A has a double ray but not a cycle ; or
(3) A has a maximal right ray but not a cycle or double ray.

Proposition: Let D_α be a functional digraph. Then for every component A of D_α :

(1) if A has a (unique) cycle C, then A is the join of C and its branches;
(2) if A has a double ray W, then A is the join of W and its branches;
(3) if A has a maximal right ray R but not a double ray , then A is the join of R and its (finite) branches(type *rro*).

As we are dealing only with finite functions every component will be a join of a cycle C and its branches.

3. Constructing a semigroup from a function

Now consider a function $\alpha \in T(X)$, let $A_1,.....A_t$ be the connected components of D_α now fix a relation $\leq$ on $A_1,.....A_t$ such that $\leq$ is a total ordering on $A_1,.....A_t$. Without loss of generality, let $A_1 \leq ..... \leq A_t$ (otherwise renumbering as required). Now for each A_i fix a $a_i \in A_i$ such that a_i is the initial vertex of the longest branch of the cycle (if the cycle has no branches then choose any one vertex in the cycle), if there two or more branches of the same length choose any one.

For $x, y \in X$ define $N(x,y) = \{(r,q) : \alpha^r(x) = \alpha^q(y)\}$. If $N(x,y)$ is empty then x, y are in different connected components. If $N(x,y)$ is non-empty then x, y are in same connected component and in this case we define $\delta_{xy} = \psi_{xy} - \tau_{xy}$ where $\tau_{xy} = min\{q : (r,q) \in N(x,y)\}$ and $\psi_{xy} = min\{r : \alpha^r(x) = \alpha^{\tau_{xy}}(y)\}$. If you take two vertices x, y in the directed graph D_α, then τ_{xy} is the minimum distance from y to the path travelled by x as α is composed again and again which is zero if y is on the path, that is $\tau_{xy} = 0$ if there is an integer n such that $\alpha^n(x) = y$, again if $y' = \alpha^{\tau_{xy}}(y)$ then ψ_{xy} is the minimum distance from x to y'.

The following lemma is required to prove the associativity of the semigroup constructed.

Lemma 1: Let X be a finite set and $\alpha \in T(X)$. Let δ_{xy} be as defined above, then $\delta_{x\alpha^l(y)} = \delta_{xy} + l$

Proof. The proof is by induction, when $l = 1$ we have to prove that

$$\delta_{x\alpha(y)} = \delta_{xy} + 1$$

but $\delta_{xy} = \psi_{xy} - \tau_{xy}$ and $\delta_{x\alpha(y)} = \psi_{x\alpha(y)} - \tau_{x\alpha(y)}$ by definition, also $\tau_{xy} = min\{q : (r,q) \in N(x,y)\}$ and $\psi_{xy} = min\{r : \alpha^r(x) = \alpha^{\tau_{xy}}(y)\}$

$$\text{if } \tau_{xy} = 0, \qquad \tau_{x\alpha(y)} = 0 \text{ and } \psi_{x\alpha(y)} = \psi_{xy} + 1$$

$$\text{if } \tau_{xy} \geq 0, \qquad \tau_{x\alpha(y)} = \tau_{xy} + 1 \text{ and } \psi_{x\alpha(y)} = \psi_{xy}$$

so either way

$$\delta_{x\alpha(y)} = \delta_{xy} + 1 \tag{1}$$

Now assume that $\delta_{x\alpha^l(y)} = \delta_{xy} + l$ to prove $\delta_{x\alpha^{l+1}(y)} = \delta_{xy} + l + 1$,

$$\begin{aligned} \delta_{x\alpha^{l+1}(y)} &= \delta_{x\alpha^l(y)} + 1 \text{ from (1)} \\ &= \delta_{xy} + l + 1 \text{ from assumption} \end{aligned}$$

Hence proved

□

Now for two elements $x, y \in X$, $\alpha \in T(X)$ define

$$x * y = \begin{cases} \alpha_{|A_i}^{\delta_{a_i x}+1}(y) & \text{if } x \text{ and } y \text{ are in the same connected component } A_i. \\ x & \text{if } x \in A_i, y \in A_j \text{ and } A_i \geq A_j \\ y & \text{if } x \in A_i, y \in A_j \text{ and } A_i \leq A_j. \end{cases} \tag{2}$$

To show that $(X, *)$ forms a semigroup we need to show that its an associative binary operation $(x*y)*z = x*(y*z) \forall x, y, z \in X$. For this we have five cases,

Case 1: x, y, z are in 5 different components say A_i A_j A_k respectively, then

$$x*(y*z) = (x*y)*z = x \text{ if } A_i \geq A_j, A_i \geq A_k$$

$$x*(y*z) = (x*y)*z = y \text{ if } A_j \geq A_i, A_j \geq A_k$$

$$x*(y*z) = (x*y)*z = z \text{ if } A_k \geq A_i, A_k \geq A_j$$

Case 2: x, y are in one component say A_i and z in A_j then

$$x*(y*z) = (x*y)*z = x*y \qquad \text{if } A_i \geq A_j$$

$$x*(y*z) = (x*y)*z = z \qquad \text{if } A_i \leq A_j$$

Case 3: x, z are in one component say A_i and y in A_j

$$x*(y*z) = (x*y)*z = x*z \qquad \text{if } A_i \geq A_j$$

$$x*(y*z) = (x*y)*z = y \qquad \text{if } A_i \leq A_j$$

Case 4: y, z are in one component say A_i and x in A_j

$$x*(y*z) = (x*y)*z = x \qquad \text{if } A_i \geq A_j$$

$$x*(y*z) = (x*y)*z = y*z \qquad \text{if } A_i \leq A_j$$

Case 5: x, y, z are in one component say A_i,

Let $k = \delta_{a_i y} + 1$ and $l = \delta_{a_i x} + 1$ then

$$x*(y*z) = x*\alpha^{\delta_{a_i y}+1}(z) \quad = \alpha^{\delta_{a_i x}+1}(\alpha^{\delta_{a_i y}+1}(z)) = \alpha^{k+l}(z)$$

$$(x*y)*z = \alpha^{\delta_{a_i x}+1}(y)*z \quad = \alpha^{\delta_{a_i(\alpha^{\delta_{a_i x}+1}(y))}+1}(z) = \alpha^{\delta_{a_i(\alpha^{l}(y))}+1}(z) \quad = \alpha^{\delta_{a_i y}+l+1}(z) \text{by lemma 1}$$

$$= \alpha^{k+l}(z)$$

Hence $*$ is a associative operation on X, and so $(X,*)$ is a semigroup, and this summaries to the following theorem,

Theorem 1: Let X be a finite set and $\alpha \in T(X)$ then there exist a semigroup operation on X derived from the function α.

So now one can construct a semigroup from any function α on a finite set which I would rather call a functional semigroup and denote by S_α the semigroup constructed from a function in the above manner. S_α changes with a change in the total order though there is a similarity in them begin that the principal sub-semigroups are isomorphic.

Theorem 2: Let X be a finite set and $\alpha \in T(X)$, S_α^1 S_α^2 be two semigroups constructed from α as described in (2), with two different total ordering but same fixed points, then a principal sub-semigroup of S_α^1 is isomorphic to a principal sub-semigroup of S_α^2.

Proof. Let $x \in X$, be a vertex of A_i a connected component of α then $x * x = \alpha^{\delta_{a_i x}+1}(x)$ in both S^1_α and S^2_α, invariant of change in the total order, hence the theorem. □

S_α changes with a change in the choice of the fixed points a_i as well, though there is a similarity in them being that the sub-semigroups generated by a_i, and b_i, are isomorphic.

Theorem 3: Let X be a finite set and $\alpha \in T(X)$, Suppose that α has t connected components and a connected component (say A_k)such that the length of the longest branch is s and there is at-least two branches of length s, let the initial vertex say of these branches be a_k, b_k,... Let S^1_α S^2_α be two semigroups constructed from α one with fixed points $a_1, ...a_K, ..., a_t$ and the other with fixed points $a_1, ..b_k, ...a_t$with two different choice of fixed points then a principal sub-semigroup generated by a_k S^1_α is isomorphic to a principal sub-semigroup generated by b_k of S^2_α. Also the principal sub-semigroup a_i is a commutative sub-semigroup.

Proof. Let a_i be a fixed point as described, then
$a_i * a_i = \alpha^{\delta_{a_i a_i}+1}(a_i) = \alpha(a_i)$,
$a_i * \alpha(a_i) = \alpha(\alpha(a_i)) = \alpha^2(a_i)$
$\alpha(a_i) * a_i = \alpha^{\delta_{a_i \alpha(a_i)}+1}(a_i) = \alpha^2(a_i)$ by lemma 1
and hence we get the sub-semigroup generated by $a_i = \{\alpha^n(a_i); n \in \mathrm{N}\}$ Also, $\alpha^n(a_i) * \alpha^m(a_i) = \alpha^{\delta_{a_i \alpha^n(a_i)}+1}(\alpha^m(a_i)) = \alpha^{\delta_{a_i a_i}+n+1}(\alpha^m(a_i))$ by lemma 1 $= \alpha^{m+n+1}(a_i) = \alpha^m(a_i) * \alpha^n(a_i)$, hence the commutativity is proved.

To prove that $S_{a_k} = \{\alpha^n(a_k); n \in \mathrm{N}\}$ is isomorphic to $S_{b_k} = \{\alpha^n(b_k); n \in \mathrm{N}\}$, Let $\phi : S_{a_k} \longrightarrow S_{b_k}$ be such that $\phi(\alpha^n(a_k)) = \alpha^n(b_k)$, then for $x = \alpha^n(a_k), y = \alpha^m(a_k) \in S_{a_k}$

$$\begin{aligned} \phi(x*y) = &\quad \phi(\alpha^n(a_k)*\alpha^m(a_k)) = &\quad \phi(\alpha^{m+n+1}(a_k)) = &\quad \alpha^{m+n+1}(b_k) \\ \phi(x)*\phi(y) = &\quad \phi(\alpha^n(a_k))*\phi(\alpha^m(a_k)) = &\quad \alpha^n(b_k)*\alpha^m(b_k) = &\quad \alpha^{m+n+1}(b_k) \end{aligned}$$

Hence ϕ is an isomorphism

□

Hence if the directed graph D_α has at-most one branch in each connected component then the semigroup generated from α is commutative. Moreover if α is a permutation then the semigroup generated is a union of groups. Further for a function α, if D_α has t connected components then S_α has t maximal-subgroups. Precisely the vertices of a cycle in D_α form a group.

Now comparing the number of semigroup on a finite set and the number of function on a finite set one can see that [3]

n	functions	semigroups	commutative semigroup
2	4	4	3
3	27	18	12
4	256	126	58
5	3,125	1,160	325
6	46,656	15,973	2,143
7	823,543	836,021	1,7291
8	16,777,216	1,843,120,128	221,805

So now one concludes that most of the semigroups constructed from a function in a manner described above must be isomorphic.

Theorem 4: Let X be a finite set and α and β two functions on X, then for any S_α there is a ordering on the directed graph D_β and a choice of vertices b_i such that S_α and S_β are isomorphic if and only if the directed graphs D_α and D_β are digraph isomorphic.

Proof. Suppose that D_α and D_β are digraph isomorphic with isomorphism ϕ_D and S_α is constructed from α such that $A_1 \leq ..... \leq A_t$ and for each A_i, $a_i \in A_i$ is the chosen fixed vertex, then let $b_i = \phi_D(a_i)$ be the fixed vertices and B_i be the connected component such that $b_i \in B_i$. Then the semigroup constructed with the order $B_1 \leq ..... \leq B_t$ and b_i the fixed vertices is isomorphic to S_α as when $a, b \in A_i$, $\phi_D(a*b) = \phi_D(\alpha^{\delta_{a_i a}+1}(b)) = \beta^{\delta_{\phi_D(a_i)\phi_D(a)}+1}(\phi_D(b))($ by digraph isomorphism $) =$ $\phi_D(a)*\phi_D(b)$, when a, b belong to different components its trivial .

Conversely suppose that S_α and S_β are isomorphic and that ϕ_S is the semigroup-isomorphism, then ϕ_S is also the digraph isomorphism for when $\alpha(a) = b$ then by semigroup isomorphism $\phi_S(a_i)*\phi_S(a) = \phi_S(a_i*a)$

$$\phi_S(a_i*a) = \phi_S(\alpha(a)) = \phi_S(b)$$

$$\phi_S((a_i))*\phi_S(a) = \beta^{\delta_{b_i\phi_S(a_i)}+1}(\phi_S(a)) = \qquad \beta^{\delta_{b_ib_i}+1}(\phi_S(a)) = \beta(\phi_s(a))$$

which shows that if there is an edge from a to b in D_α ($\alpha(a) = b$) then $\beta(\phi_S(a)) = \phi_S(b)$ so there is an edge from $\phi_S(a)$ to $\phi_S(b)$ and vice-versa. Hence D_α and D_β are digraph isomorphic. $\square$

Remark: If in equation (2) $\alpha_{|A_i}^{\delta_{a_I x}+1}(y)$ is replaced by $\alpha_{|A_i}^{\delta_{a_I x}}(y)$ then the isomorphism in theorem 4 is not obtained.

This construction could be extended to the case when X is countably infinite

Conflict of Interests

The authors declare that there is no conflict of interests.

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