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COMMON FIXED POINT THEOREM IN DISLOCATED METRIC SPACE

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Abstract: The aim of this paper is to prove a unique common fixed point theorem in dislocated metric space using the concept of weakly compatible mappings and E.A property.

Keywords: Dislocated metric space, E.A property, weakly compatible mappings, unique common fixed point.

AMS Mathematics subject classification (2010): 54H25, 47H10.

1. INTRODUCTION

The notion of dislocated metric space introduced by P. Hitzler and A.K Seda in 2000.Since then many others proved some useful fixed point results for dislocated metric space. G. Jungck and B.E.Rhoads initiated the concept of weakly compatible .The purpose of this paper is to prove a common fixed point for four self maps with the concept of weakly compatible maps and E.A property.

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2. DEFINITIONS AND PRELIMINARIES

Definition 2.1: Let X be a nonempty set. A function $d: X \times X \to [0\ \infty)$satisfying the following properties

(A1) $d(x,y) = d(y,x)$

(A2) $d(x,y) = d(y,x) = 0$ then $x = y$.

(A3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called dislocated or d-metric on X.

Definition 2.2: Let (X,G) is a d-metric space. A sequence $\{x_n\}$ in X is said be convergent if there is $x_0 \in X$ such that for each $\varepsilon > 0$ there is a natural number m such that $d\left(x_{n,} x_{0,}\right) < \varepsilon$ for all $n \geq m$.

Definition 2.3: Let (X,G) is a d-metric space. A sequence $\{x_n\}$ in X is said be Cauchy sequence for each $\varepsilon > 0$ there is a natural number $n_0$such that $d\left(x_{m,} x_{n,}\right) < \varepsilon$ for all $m, n \geq n_0$.

Definition 2.4: A d-metric space (X, d) is said to be complete if every Cauchy sequence in it is convergent in X with respect to d.

Definition 2.5: Suppose A and S are self maps of a d-metric space(X, d). The pair (A,S) is said to be commute if $ASx = SAx$ for all $x \in X$.

Definition 2.6: Suppose A and S are self maps of a d-metric space(X, d). The pair (A,S) is said to be weakly compatible pair if A and S are commute at their coincidence point. That is $ASx = SAx$ whenever $Ax = Sx$ for all $x \in X$.

Definition 2.7: Two self maps A and S of a d-metric space (X, d) we say that A and S satisfy the property (E.A) if there exists a sequencer $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = z$ for some $z \in X$.

In [5], Panthi proved the following Theorem.

2.8. Theorem: Let (X, d) be complete dislocated metric space. Let A, B, S and T be maps from X into itself satisfying following conditions

(2.8.1) $T(X) \subseteq A(X)$ and $S(X) \subseteq B(X)$

(2.8.2) $d(Tx, Sy) \leq \alpha[d(Bx, Tx) + d(Ay, Sy)] + \beta d(Bx, Ay) + \gamma d(Bx, Sy) + \delta d(Tx, Ay)$

where for all $x, y \in X$ and $\alpha, \beta, \gamma, \delta \geq 0$ and $0 \leq 2\alpha + \beta + 2\gamma + 2\delta < 1$

(2.8.3) (T, B) and (S, A) are compatible

(2.8.4) one of A, B, S and T are continuous.

Then A, B, S and T have a unique common fixed point.

We now generalize Theorem 2.8 as follows

3. MAIN RESULT

3.1Theorem: Let (X, d) be dislocated metric space. Let A, B, S and T be maps from X into itself satisfying following conditions

(3.1.1) $T(X) \subseteq A(X)$ and $S(X) \subseteq B(X)$

(3.1.2) the pair (S, A) (or) (T, B) satisfy E.A property

(3.1.3) $d(Tx, Sy) \leq \alpha[d(Bx, Tx) + d(Ay, Sy)] + \beta d(Bx, Ay) + \gamma d(Bx, Sy) + \delta d(Tx, Ay)$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $0 \leq 2\alpha + \beta + 2\gamma + 2\delta < 1$

(3.1.4) the pairs (T, B) and (S, A) are weakly compatible

(3.1.5) $B(X)$ or $A(X)$ is closed

then A, B, S and T have a unique common fixed point .

Proof: Suppose the pair (S, A) is satisfy E.A property then there exists a sequence $\{x_n\}$ in

X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = z$ for some $z \in X$.

Since $S(X) \subseteq B(X)$ so there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n = z \text{ as } n \to \infty$.

Now we shall show that $\lim_{n\to\infty} Ty_n == z$

Put $x = y_n$ and $y = x_n$ in the inequality (3.1.3), we get

$$d(Ty_n, Sx_n) \le \alpha[d(By_n, Ty_n) + d(Ax_n, Sx_n)] + \beta d(By_n, Ax_n)$$

$$+\gamma d(By_n, Sx_n) + \delta d(Ty_n, Ax_n)$$

$$d(Ty_n, Sx_n) \le (\alpha + \delta)[d(Sx_n, Ty_n)$$

$$d(Ty_n, z) \le (\alpha + \delta)[d(z, Ty_n) \text{ as } n \to \infty$$

implies $\lim_{n\to\infty} Ty_n == z$

Let $\lim_{n\to\infty} Ty_n == z$ then the sequences Ty_n Sx_n, By_n and Ay_n are converges to z as $n \to \infty$

Suppose B(X) is closed subspace of X then $z = Bu$ for some $u \in X$

Now we show that $Tu = z$

Put $x = u$ and $y = x_n$ in the equality (3.1.3), we get

$$d(Tu, Sx_n) \le \alpha[d(Bu, Tu) + d(Ax_n, Sx_n)] + \beta d(Bu, Ax_n) + \gamma d(Bu, Sx_n) + \delta d(Tu, Ax_n)$$

$$d(Tu, z) \le (\alpha + \delta)[d(z, Tu) \text{ as } n \to \infty$$

$[1 - (\alpha + \delta)]d(Tu, z) \le 0$ implies $Tu = z$ this implies $Tu = Bu = z$

Since $T(X) \subseteq A(X)$ so there exists a point v in X such that $Tu = Av = z$

Now we claim that $Sv = z$.

Put $x = u, y = v$ in the inequality (3.1.3), we get

$d(Tu,Sv) \leq \alpha[d(Bu,Tu) + d(Av,Sv)] + \beta d(Bu,Av) + \gamma d(Bu,Sv) + \delta d(Tu,Av)$

$d(z,Sv) \leq \alpha[d(z,z) + d(z,Sv)] + \beta d(z,Sv) + \gamma d(z,Sv) + \delta d(z,z)$

$[1 - \alpha - \beta - \gamma]d(z,Sv) \leq 0$ implies $z = Sv$

Therefore $Av = Sv = z$

Since the pair (T,B) is weakly compatible therefore $TBu = BTu$ implies $Tz = Bz$.

Now we show that $Tz = z$

Put $x = z, y = v$

Put $x = z$ and $y = v$ in the inequality (3.1.3), we get

$d(Tz,Sv) \leq \alpha[d(Bz,Tz) + d(Av,Sv)] + \beta d(Bz,Av) + \gamma d(Bz,Sv) + \delta d(Tz,Av)$

$d(Tz,z) \leq \alpha[d(Tz,Tz) + d(z,z)] + \beta d(Tz,z) + \gamma d(Tz,z) + \delta d(Tz,z)$

$[1 - \beta - \gamma - \delta]d(Tz,z) \leq 0$ implies $Tz = z$ therefore $Tz = Bz = z$

The weakly compatibility of (S,A) implies $SAv = ASv$ implies $Sz = Az$

Now we shell show that $Sz = z$

Put $x = z, y = z$ in inequality (3.1.3), we get

$d(Tz,Sz) \leq \alpha[d(Bz,Tz) + d(Az,Sz)] + \beta d(Bz,Az) + \gamma d(Bz,Sz) + \delta d(Tz,Az)$

this implies

$d(z,Sz) \leq \alpha[d(z,z) + d(Sz,Sz)] + \beta d(z,Sz) + \gamma d(z,Sz) + \delta d(z,Sz)$

this gives

$[1 - \beta - \gamma - \delta]d(z,Sz) \leq 0$ implies $z = Sz$

Therefore $Az = Bz = Tz = Sz = z$ hence z is common fixed point of mappings A, B, S and T.

Now we prove the Uniqueness.

Let z^* be another common fixed point of mappings A,B,S and T then $Az^* = Bz^* = Tz^* = Sz^*$

Put $x = z$ and $y = z^*$ in the inequality (3.1.3), we get

$$d(Tz,Sz^*) \le \alpha[d(Bz,Tz) + d(Az^*,Sz^*)] + \beta d(Bz,Az^*) + \gamma d(Bz,Sz^*) + \delta d(Tz,Az^*)$$

implies

$$d(z,z^*) \le \alpha[d(z,z) + d(z^*,z^*)] + \beta d(z,z^*) + \gamma d(z,z^*) + \delta d(z,z^*)$$

and this gives

$[1 - \beta - \gamma - \delta]d(z,z^*) \le 0$ implies $z = z^*$

Hence z is unique common fixed point of mappings A, B, S and T.

Our main Theorem 3.1 can be validating by using following example.

3.2 Example: Let $X = [-1,1]$ be dislocated metric space $d(x,y) = |x - y|$. Define self maps A,B,S and T of X defined as

$$A(x) = \frac{x(x-1)}{2} \text{ if } -1 \le x \le 1, \quad Bx = \begin{cases} x^2 & \text{if } -1 \le x \le 0 \\ \frac{1}{2} & \text{if } 0 < x \le 1 \end{cases}$$

$$Sx = \begin{cases} -x & \text{if } -1 \le x \le 0 \\ \frac{1}{2} & \text{if } 0 \le x \le 1 \end{cases} \text{ and } Tx = \begin{cases} -x & \text{if } -1 \le x \le 0 \\ \frac{1}{3} & \text{if } 0 \le x \le 1 \end{cases}$$

Clearly $A(X) = B(X) = S(X) = T(X) = [0,1]$ so that $A(X) \subseteq T(X)$,and $B(X) \subseteq S(X)$,

Now consider the sequence $\{x_n\}$ where $x_n = \left\{\frac{1}{n}\right\}$ for $n \ge 1$.

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} A\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{n} - 1\right) = 0 \text{ , } \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} S\left(\frac{1}{n}\right) = 0$$

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} B\left(\frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{n^2}\right) = 0 \text{ and } \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} T\left(\frac{1}{n}\right) = 0$$

Therefore $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0$ and this implies that the pair (S,A) satisfies E.A property.

and also we observe that $BT(0) = TB(0)$ when $B(0) = T(0)$ and $AS(0) = SA(0)$ when $A(0) = S(0)$.

Hence the pairs (A,S) and (B,T) are weakly compatible.

3.3 Conclusion: From the example 3.2, Clearly the pairs (B,T) and (A,S) are weakly compatible and the pair (S,A) satisfy E.A. Property. Also none of the mappings are continuous and the rational inequality holds for the values $0 \le 2\alpha + \beta + 2\gamma + 2\delta < 1,\ \alpha, \beta, \gamma, \delta \ge 0$ .Clearly 0 is the unique common fixed point of A, B, S and T .

Conflict of Interests

The authors declare that there is no conflict of interests.

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