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REFINEMENTS AND GENERALIZATIONS OF CERTAIN INEQUALITIES INVOLVING TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

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Abstract. Inequalities involving trigonometric and hyperbolic functions are established. Obtained results provide refinements or generalizations of inequalities proved by Adamović and Mitrinović, Cusa, Huygens, Lazarević and others. Some of the results included in this paper are obtained with the aid of the Schwab-Borchardt mean.

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1. Introduction

In this section we give a brief overview of known results which pertain to the main results of this paper.

In recent years the following two-sided inequality for trigonometric functions

$$(\cos\varphi)^{1/3} < \frac{\sin\varphi}{\varphi} < \frac{2+\cos\varphi}{3} \tag{1}$$

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($0 < |\varphi| < \dfrac{\pi}{2}$) has attracted attention of several researchers. The left inequality in (1) have been proven by Adamović and Mitrinović (see [8]), while the second one is due to Cusa and Huygens (see [5, 23]).

The counterpart of (1) for hyperbolic functions reads as follows

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3} \tag{2}$$

($x \neq 0$). The left inequality in (2) has been obtained by Lazarević (see, e.g. [8]) while the right one has been established in [21].

For a recent development in theory of inequalities related to (1) and (2) the interested reader is referred to [10], [12], [11], [15], [21], [22] and the references therein.

The following inequality

$$\left(\frac{\sin\varphi}{\varphi}\right)^2 + \frac{\tan\varphi}{\varphi} > 2 \tag{3}$$

($0 < |\varphi| < \frac{\pi}{2}$) is due to Wilker [24]. Several Wilker type inequalities appear in mathematical literature. For more details see [11, 14, 21, 25, 26] and the references therein. A hyperbolic counterpart of Wilker's inequality

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} > 2 \tag{4}$$

($x \neq 0$) has been established by L. Zhu [27]. See also [28] and [21]).

Another inequality which recently has been studied extensively is due to Huygens [5]

$$2\frac{\sin\varphi}{\varphi} + \frac{\tan\varphi}{\varphi} > 3 \tag{5}$$

($0 < |\varphi| < \frac{\pi}{2}$). Huygens inequality for the hyperbolic functions

$$2\frac{\sinh x}{x} + \frac{\tanh x}{x} > 3 \tag{6}$$

($x \neq 0$) was established by Neuman and Sándor in [21].

For generalizations and refinements of inequalities (3) - (6) the interested reader is referred to [25], [14], [16], [21] and the references therein.

This paper is a continuation of author's investigations reported in [11, 14, 15, 17, 22]. In Section 2 we give definitions of some bivariate means which will be used in the sequel.

Refinements of inequalities (1) and (2) are established in Section 3. In Section 4 we give generalizations of the Wilker and Huygens type inequalities.

2. Definitions of Some Means of Two Variables

In what follows the letters x and y will stand for the positive and unequal numbers. The power mean of order t, the logarithmic and the identric means of x and y will be denoted by A_t, L, and I, respectively. Recall that (see [2])

$$A_t \equiv A_t(x,y) = \begin{cases} (\frac{x^t+y^t}{2})^{1/t} & \text{if } \ t \neq 0, \\ \sqrt{xy} & \text{if } \ t = 0, \end{cases} \tag{7}$$

$$\begin{aligned} L \equiv L(x,y) &= \frac{x-y}{\ln x - \ln y}, \\ I \equiv I(x,y) &= e^{-1}\Big(\frac{x^x}{y^y}\Big)^{\frac{1}{x-y}}. \end{aligned} \tag{8}$$

The Schwab-Borchardt mean $SB(x,y) \equiv SB$ is defined as follows (see [1]), [3])

$$SB(x,y) = \begin{cases} \dfrac{\sqrt{y^2-x^2}}{\cos^{-1}(x/y)} & \text{if } \ x < y, \\ \dfrac{\sqrt{x^2-y^2}}{\cosh^{-1}(x/y)} & \text{if } \ y < x, \\ x & \text{if } \ x = y. \end{cases} \tag{9}$$

Mean SB is non-symmetric, homogeneous of degree 1 and strictly increasing in each variable.

Simple bounds for the Schwab - Borchardt mean have been obtained in [18]

$$(xy^2)^{1/3} < SB(x,y) < \frac{x+2y}{3}. \tag{10}$$

For the later use let us record the following inequality [18]:

$$x < SB(y,x) < SB(x,y) < y \tag{11}$$

which holds true provided $x < y$. More inequalities for the Schwab - Borchardt mean can be found in [13] and [19].

We will also use the Heronian mean of order $\omega > 0$ of x and y:

$$H_\omega(x,y) \equiv H_\omega = \frac{2A + \omega G}{2+\omega} \tag{12}$$

(see [6]). It is clear that the function $\omega \to H_\omega$ is strictly decreasing on its domain.

3. Refinements of Inequalities (1) and (2)

The goal of this section is to establish several inequalities which provide refinements and extensions of inequalities (1) and (2).

We will use the following chain of inequalities

$$(xy^2)^{1/3} < (ySB(y,x))^{1/2} < SB(x,y) < \frac{1}{2}(y + SB(y,x)) < \frac{x+2y}{3}. \tag{13}$$

The first inequality in (13) is obtained by application of the left inequality in (10) to the second member of (13) while the second one is established in [19, Theorem 3.1]. The third inequality in (13) is established in [4] while the last one is obtained by application of the right inequality in (10) to the fourth member of (13).

The first result of this section is contained in the following.

Theorem 1. *Let* $0 < |\varphi| < \frac{\pi}{2}$. *Then*

$$\begin{aligned}
&(\cos\varphi)^{1/3} < (\cos\varphi\frac{\sin\varphi}{\varphi})^{1/4} < \Big(\frac{\sin\varphi}{\tanh^{-1}(\sin\varphi)}\Big)^{1/2} \\
&< \Big[\frac{1}{2}\Big(\cos\varphi + \frac{\sin\varphi}{\varphi}\Big)\Big]^{1/2} < \Big(\frac{1+2\cos\varphi}{3}\Big)^{1/2} \\
&< \Big(\frac{1+\cos\varphi}{2}\Big)^{2/3} < \frac{\sin\varphi}{\varphi}.
\end{aligned} \tag{14}$$

Proof. The first four inequalities in (14) can be obtained using (13) with $x = 1$, $y = \cos\varphi$ and two formulas

$$SB(\cos\varphi, 1) = \frac{\sin\varphi}{\varphi}, SB(1,\cos\varphi) = \frac{\sin\varphi}{\tanh^{-1}(\sin\varphi)} \tag{15}$$

which follow easily from (7). The fifth inequality in (14) is derived in [17] while the last one has been established in [21]. $\square$

Corollary 2. *Let $x \neq 0$. Then the following inequalities*

$$
\begin{aligned}
(\cosh x)^{1/3} &< \Big(\frac{\sinh x}{\sin^{-1}(\tanh\, x)}\Big)^{1/2} < \frac{\sinh x}{x} \\
&< \frac{1}{2}\Big(1 + \frac{\sinh x}{\sin^{-1}(\tanh\, x)}\Big) < \frac{2+\cosh x}{3} \\
&< \Big(\frac{1+\cosh x}{2}\Big)^{4/3}(\cosh x)^{-1/3} \\
&< \Big(\frac{\tanh\, x}{\sin^{-1}(\tanh\, x)}\Big)^{2}\cosh x
\end{aligned} \tag{16}
$$

are valid.

Proof. In order to establish inequalities (16) we substitute in (14) $\sin\varphi = \tanh\, x, \cos\varphi =$ sech x, and $\varphi = \sin^{-1}(\tanh\, x)$. Next raising each member of the resulting chain of inequalities to the power of 2 and finally multiplying all terms by $\cosh x$, we obtain, after a little algebra, the desired result. □

We close this section with another chain of inequalities involving hyperbolic functions. They provide a refinement and extensions of the second inequality in (2).

Theorem 3. *Let $x \neq 0$. Then*

$$
\begin{aligned}
\frac{\sinh x}{x} < \Big(\cosh\frac{x}{3}\Big)^3 < \frac{2\cosh x + \lambda}{2+\lambda} &< \Big(\cosh\frac{2x}{3}\Big)^{3/2} \\
&< \exp(x\coth\, x - 1) < \frac{2\cosh x + \omega}{2+\omega} \\
&< \cosh x,
\end{aligned} \tag{17}
$$

provided $1 \leq \lambda \leq 4$ and $0 < \omega \leq e - 2$.

Proof. In order to establish the first inequality in (17) we utilize Lin's inequality [7]: $L < A_{1/3}$ (see also [9]). Letting

$$(x, y) := (e^x, e^{-x}) \tag{18}$$

we obtain, using (7) and (8), $L = \sinh x/x$ and $A_{1/3} = (\cosh(x/3))^3$. Hence the desired follows. For the proof of the second inequality in (17) we put $x := 3x$ in the second and third members of the inequality in question and write it as $2\cosh 3x-(2+\lambda)(\cosh x)^3+\lambda > 0$. Using the triple argument formula $\cosh 3x = 4\cosh^3 x - 3\cosh x$ we write the last inequality as $(6-\lambda)c^3-6c+\lambda>0$, where we have used $c=\cosh x$ for the sake of brevity. A straightforward algebra allow us to write the last inequality as

$$(c-1)[(6-\lambda)c^2+(6-\lambda)c-\lambda]>0.$$

It is easy to verify that the quadratic polynomial inside the square brackets is a strictly increasing function if $c>1$ and λ is the same as assumed. For the proof of the remaining inequalities in (17) we shall utilize the following chain of inequalities for means [20]:

$$H_1 < A_{2/3} < I < H_\omega < A. \tag{19}$$

Using (12), (7), and (8), with x and y as defined in (18), we obtain the assertion. □

If $\lambda=4$, then the third member of (17) becomes the third member of (2).

4. Wilker and Huygens Type Inequalities

In this section we shall establish inequalities which have similar structures as the inequalities (3) - (6).

Throughout the sequel the letters u and v will stand for two positive numbers which satisfy the following conditions

$$\min(u,v)<1<\max(u,v), \tag{20}$$

$$1<u^\alpha v^\beta, \tag{21}$$

and

$$1<\frac{\alpha}{\alpha+\beta}\frac{1}{u}+\frac{\beta}{\alpha+\beta}\frac{1}{v}, \tag{22}$$

where the last two inequalities must be satisfied for some positive numbers α and β.

In the proofs of the main results of this section we will utilize the following result [16]:

Theorem A. Let $\lambda>0$ and $\mu>0$. If $u<1<v$, then

$$1<\frac{\lambda}{\lambda+\mu}u^p+\frac{\mu}{\lambda+\mu}v^q \tag{23}$$

if either

$$q > 0 \qquad \text{and} \qquad p \le q\frac{\alpha\mu}{\beta\lambda} \tag{24}$$

or if

$$p \le q \le -1 \qquad \text{and} \qquad \beta\lambda \ge \alpha\mu. \tag{25}$$

If $v < 1 < u$, then the inequality (23) holds true if either

$$p > 0 \qquad \text{and} \qquad q \le p\frac{\beta\lambda}{\alpha\mu} \tag{26}$$

or if

$$q \le p \le -1 \qquad \text{and} \qquad \alpha\mu \ge \beta\lambda. \tag{27}$$

Before we will formulate and prove the main results of this section let us define

$$u = \frac{SB(x,y)}{SB(y,x)}, \tag{28}$$

$$v = \frac{SB(x,y)}{y}. \tag{29}$$

It follows from the second and the third inequalities in (11) that $1 < uv$ and $1 < \frac{1}{2}(\frac{1}{u}+\frac{1}{v})$, respectively. Thus the conditions (21) and (22) are satisfied with $\alpha = \beta = 1$. Also, using (11), with x interchanged with y, we conclude that

$$u < 1 < v \tag{30}$$

provided $x > y$. Similarly, if $x < y$, then we use (11), to obtain

$$v < 1 < u. \tag{31}$$

Thus if u and v are defined as in (28) and (29), then they satisfy conditions (20) - (22).

We are in a position to prove the following.

Theorem 4. *Let $\lambda > 0$, $\mu > 0$, and let $x \ne 0$. Then*

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{\sin^{-1}(\tanh\, x)}{x}\Big)^p + \frac{\mu}{\lambda+\mu}\Big(\frac{\sinh x}{x}\Big)^q \tag{32}$$

if either

$$q > 0 \qquad and \qquad p \le q\frac{\mu}{\lambda} \tag{33}$$

or if

$$p \le q \le -1 \qquad and \qquad \lambda \ge \mu. \tag{34}$$

Proof. Let $x = \sqrt{1+t^2}$ $(t \neq 0)$ and let $y = 1$. Clearly $x > y$. Using [13, (2.10), (2.11)] we have

$$SB(x,y) = \frac{t}{\sinh^{-1} t} \qquad \text{and} \qquad SB(y,x) = \frac{t}{\tan^{-1} t}.$$

Making use of (28) and (29) we obtain

$$u = \frac{\tan^{-1} t}{\sinh^{-1} t} \qquad \text{and} \qquad v = \frac{t}{\tan^{-1} t}.$$

Since $x > y$, u and v satisfy inequalities (30). To complete the proof it suffices to use Theorem A to obtain

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{\tan^{-1} t}{\sinh^{-1} t}\Big)^p + \frac{\mu}{\lambda+\mu}\Big(\frac{t}{\sinh^{-1} t}\Big)^q.$$

Now we let $t = \sinh x$ and utilize an elementary identity $\tan^{-1}(\sinh x) = \sin^{-1}(\tanh x)$ to obtain the desired result. □

Corollary 5. *Let* $\lambda \ge \mu > 0$ *and let* $x \neq 0$. *Then*

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{x}{\tanh x}\Big)^r + \frac{\mu}{\lambda+\mu}\Big(\frac{x}{\sinh x}\Big)^s \tag{35}$$

provided

$$r \ge s \ge 1. \tag{36}$$

Proof. We utilize inequality (32) assuming that p, q, λ and μ satisfy conditions (34). Letting $p = -r$ and $q = -s$ we obtain

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{x}{\sinh^{-1}(\tanh x)}\Big)^r + \frac{\mu}{\lambda+\mu}\Big(\frac{x}{\sinh x}\Big)^s, \tag{37}$$

where r and s must to satisfy (36). Taking into account that the inequality

$$1 < \frac{x}{\sin^{-1}(\tanh x)} < \frac{x}{\tanh x}$$

holds true for all $x \neq 0$ we see that (37) implies inequality (35). □

Our next result reads as follows.

Theorem 6. *Let $\lambda > 0$, $\mu > 0$, and let $0 < |\varphi| < \frac{\pi}{2}$. Then*

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{\sinh^{-1}(\tan\varphi)}{\varphi}\Big)^p + \frac{\mu}{\lambda+\mu}\Big(\frac{\sin\varphi}{\varphi}\Big)^q \tag{38}$$

if either

$$p > 0 \qquad \text{and} \qquad q \le p\frac{\lambda}{\mu} \tag{39}$$

or if

$$q \le p \le -1 \qquad \text{and} \qquad \mu \ge \lambda. \tag{40}$$

Proof. Let $x = 1$ and $y = \sqrt{1+t^2}$ $(t \neq 0)$. Making use of [13, (2.11)] and [13, (2.12)] we get

$$SB(x,y) = \frac{t}{\tan^{-1} t} \qquad \text{and} \qquad SB(y,x) = \frac{t}{\sinh^{-1} t}.$$

This in conjunction with (28) and (29) yields

$$u = \frac{\sinh^{-1} t}{\tan^{-1} t} \qquad \text{and} \qquad v = \frac{t}{\sqrt{1+t^2}\tan^{-1} t}.$$

Since $x < y$, $v < 1 < u$. Letting above $t = \tan\varphi$ and substituting resulting expressions for u and v into (23), we obtain inequality (38). Conditions (39) and (40) follow from (26) and (27), respectively, because $\alpha = \beta = 1$. □

We close this section with the following.

Corollary 7. *Let $\lambda, \mu, \varphi, p, q$ satisfy the same conditions as in Theorem 6. Then*

$$1 < \frac{\lambda}{\lambda+\mu}\Big(\frac{\tan\varphi}{\varphi}\Big)^p + \frac{\mu}{\lambda+\mu}\Big(\frac{\sin\varphi}{\varphi}\Big)^q. \tag{41}$$

Proof. To obtain the desired result we use Theorem 6 together with the inequality

$$1 < \frac{\sinh^{-1}(\tan\varphi)}{\varphi} < \frac{\tan\varphi}{\varphi}$$

which is a consequence of the well known inequality $t < \sinh t$ $(t \neq 0)$. □

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