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GABLER INEQUALITY FOR FUNCTIONS WITH NONDECREASING INCREMENTS OF CONVEX TYPE

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Abstract. In this article we presents S.Gabler inequality [1] for function with nondecreasing increments of convex type and established results for Jensen, Mercer's and Neizgoda's results along with their refinements.

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1. Introduction and Preliminaries

Throughout this article we consider that $\acute{V}$ is an interval in a κ-dimensional vector space $\mathbb{R}^\kappa$ and $\rho_1, \cdots, \rho_n$ are wights such that $\rho_{\acute{i}} = \sum_{j=1}^{i} \rho_j, \acute{i} \in \{1, \cdots, n\}$ with $\rho_n = \sum_{j=1}^{n} \rho_j$ and $H \subset \mathbb{R}$. We write function with nondecreasing increments as FWNDI and continuous function with nondecreasing increments as CFWNDI.

H.D. Brunk [5] investigated an interesting class of multivariate real- valued function defined as follow:

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Definition 1.1. A function f of real valued is said to have nondecreasing increment on a κ-dimensional rectangle $\acute{V} \subset \mathbb{R}^{\kappa}$ if

$$f(\varepsilon + h) - f(\varepsilon) \leq f(\tau + h) - f(\tau),$$

whenever $0 \leq h \in \mathbb{R}^{\kappa}, \varepsilon \leq \tau, \tau + h \in \acute{V}$. We defined partial order on $\mathbb{R}^{\kappa}$ by

$$(\xi_1, \cdots, \xi_k) \leq (y_1, \cdots, y_k),$$

if and only if

$$\xi_{\acute{i}} \leq y_{\acute{i}}, \acute{i} \in \{1, \cdots, k\}.$$

In [2] author established the weighted Jensen inequality for the FWNDI as follow:

Proposition 1.2. *Let $f : \acute{V} \to \mathbb{R}$ be a CFWNDI defined on $\acute{V}$ and $\xi^{(i)} \in \acute{V}, \acute{i} \in \{1, \ldots, n\}$ with condition $\xi^{(1)} \leq \cdots \leq \xi^{(n)}$ or $\xi^{(1)} \geq \cdots \geq \xi^{(n)}$, ρ_n be a nonnegative n-tuple such that $\rho_n > 0$ then*

$$f\left(\frac{1}{\rho_n}\sum_{\acute{i}=1}^{n} \rho_{\acute{i}} \xi^{(\acute{i})}\right) \leq \frac{1}{\rho_n}\sum_{\acute{i}=1}^{n} \rho_{\acute{i}} f\left(\xi^{(\acute{i})}\right). \tag{1.1}$$

Remark 1.3. Let ρ be a real n-tuple such that

$$\rho_1 > 0, \rho_{\acute{i}} \leq 0, \acute{i} \in \{2, \ldots, n\}, \rho_n > 0.$$

And f be defined as proposition (1.2), and

$$\frac{1}{\rho_n}\sum_{\acute{i}=1}^{n} \rho_n \xi^{(\acute{i})} \in \acute{V},$$

then the reverse inequality (1.1) is valid.

In [2] a variant of Proposition (1.2) was established as follow:

Proposition 1.4. *By assuming the assumption of Proposition* (1.2) *we have*

$$f\left(\mathbf{m} + \mathbf{n} - \frac{\mathbf{1}}{\rho_{\mathbf{n}}}\sum_{\acute{\mathbf{i}}=\mathbf{1}}^{\mathbf{n}} \rho_{\acute{\mathbf{i}}} \xi^{(\acute{\mathbf{i}})}\right) \leq f(m) + f(n) - \frac{1}{\rho_n}\sum_{\acute{i}=1}^{n} \rho_{\acute{i}} f\left(\xi^{(\acute{i})}\right), \tag{1.2}$$

where $\mathbf{m} = (\mathbf{m_1}, \ldots, \mathbf{m_k})$ *and* $\mathbf{n} = (\mathbf{n_1}, \ldots, \mathbf{n_k})$ *are two specially chosen k-tuple related to* $\acute{V}$ *and* $m \leq \xi^{(\acute{i})} \leq n$*. If* ρ *is a real -tuple such that*

$$\rho_1 > 0, \rho_{\acute{i}} \leq 0, \acute{i} \in \{2, \cdots, n\}, \rho_n > 0,$$

and if

$$\frac{1}{\rho_n} \sum_{\acute{i}=1}^{n} \rho_n \xi^{(\acute{i})} \in \acute{V}.$$

Then the inequality (1.2) *remains valid.*

In [2] Gabler established the double index function to define sequentially convex function, a special case of convex functions.

Definition 1.5. For $\xi = \left(\xi_{\acute{i}_1}, \cdots, \xi_{\acute{i}_r}\right) \in H^n$, a real valued function $f : H \to \mathbb{R}$ then

$$F_{\kappa,n} = F_{\kappa,n}(\xi) = \binom{n}{\kappa}^{-1} \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} f\left(\frac{\xi_{\acute{i}_1} + \cdots + \xi_{\acute{i}_\kappa}}{\kappa}\right) \tag{1.3}$$

Gabler also established the inequality for the convex function f of the type (1.3) as follow:

$$f_{\kappa,n}(\xi) \geq f_{\kappa+1,n}(\xi), \quad k \in \{1, \ldots, n-1\}. \tag{1.4}$$

S. Gabler then defined sequentially convex functions as follows:

Definition 1.6. Let $f_{\kappa,n}$ be defined as in(1.4) and $f : H \to \mathbb{R}$, then f is said to be sequentially convex if $\{f_{\kappa,n}\}$ is a convex sequence in k for all $n > 2$ and all $\xi_1, \ldots, \xi_n \in H$.

Through the proof of [1] it is interesting to notice that (1.4) is valid for mid-convex functions can be seen in [4, 8].

These results along with the following interpolating inequality for jensen inequality can be seen as in [4, 6].

$$f\left(\frac{1}{\rho_n} \sum_{\acute{i}=1}^{n} \rho_{\acute{i}} \xi_{\acute{i}}\right) = f_{n,n} \leq \cdots \leq f_{\kappa+1,n} \leq f_{\kappa,n} \leq \cdots \leq f_{1,n} = \frac{1}{\rho_n} \sum_{\acute{i}=1}^{n} \rho_{\acute{i}} f\left(\xi_{\acute{i}}\right) \tag{1.5}$$

In [7] J. Pe*čarič* deduced the weighted version of (1.5) as follow:

Definition 1.7. For $\xi = \left(\xi_{\acute{i}_1}, \cdots, \xi_{\acute{i}_r}\right) \in H^n$ and $f : H \to \mathbb{R}$ ba a real valued function then

$$f_{\kappa,n} = \frac{1}{\binom{n-1}{\kappa-1}\rho_n} \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}\right) f\left(\frac{\rho_{\acute{i}_1}\xi_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k+1}\xi_{\acute{i}_k}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}}\right)$$

where $\rho_{\acute{i}}$ are nonnegative weights for $\acute{i} \in \{1, \ldots, n\}$ and $\rho_n = \sum_{\acute{i}=1}^{n} \rho_{\acute{i}}$.

In [11], author established the double index function in term of Jensen-Mercer inequality satisfying Gabler inequality as follow:

Proposition 1.8. *Let ρ be a positive real numbers with $\rho_n > 0$ and $\xi_{\acute{i}} \in [e, f], [e, f] \subset H$. If f is a convex function on $[e, f]$ and we define,*

(1.6)
$$f_{\kappa,n} = f_{\kappa,n} = \binom{n-1}{\kappa-1}^{-1} \rho_n \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}\right) f\left(e + f - \frac{\rho_{\acute{i}_1}\xi_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}\xi_{\acute{i}_\kappa}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}}\right),$$

then the inequality (1.4) *is valid.*

In [10] the author established the double index function in terms of Neizgoda's inequality satisfying Gabler inequality as follow:

Proposition 1.9. *Let $f : [e, f] \to \mathbb{R}$ be a continuous convex function on $[e, f]$ and $\xi_{\acute{i}j} \in [e, f]$, $j \in \{1, \ldots, m\}, \acute{i} \in \{1, \ldots, n\}$ is a real $n \times m$ matrix. Suppose that a_j be a n-tuple, $a_j \in [e, f]$ and* $\boldsymbol{w}$ *be a real n-tuple such that $0 \leq W_{\acute{i}} \leq W_n, \quad W_n > 0$ where $W_{\acute{i}} = \sum_{j=1}^{n} w_j, \acute{i} \in \{1, \ldots, n\}$ and $W_n = \sum_{j=1}^{n} w_j$. If a majorizes each row of $\xi_{\acute{i}j}$ that is,*

$$\xi_{\kappa_1} = \left(\xi_{\acute{i}_1 1}, \ldots, \xi_{\acute{i}_1 m}\right) \prec (a_1, \ldots, a_m) = a \quad \forall \acute{i}_1 \in H_n, \tag{1.7}$$

if we define

$$f_{\kappa,n} = f_{\kappa,n}(\xi, a, w) = \frac{1}{\binom{n-1}{\kappa-1}W_n} \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_k \leq n} \left(w_{\acute{i}_1} + \cdots + w_{\acute{i}_k}\right) \times \tag{1.8}$$

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \frac{w_{\acute{i}_1}\xi_{\acute{i}_1} + \cdots + w_{\acute{i}_k}\xi_{\acute{i}_k}}{w_{\acute{i}_1} + \cdots + w_{\acute{i}_k}}\right), \tag{1.9}$$

then the inequality (1.4) *valid.*

In [9] J. Pecaric established the following results as follow:

Proposition 1.10. *For* $\xi_{\acute{i}} \in H, \acute{i} = 1, \cdots, n$, *let* $f : H \to \mathbb{R}$ *be a mid-convex function then*

$$f\left(\frac{1}{n}\sum_{j=1}^{n} y_j\right) \leq \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{y_1 + \cdots + \hat{y}_j + y_n}{n-1}\right), \tag{1.10}$$

for all $j = 1, \cdots, n$, *where* $\hat{y}_j$ *means that* y_j *is omitted.*

In present paper we establish S.Gabler inequality for Jensen inequality, Mercer and Niezdoga inequality for continuous function with nondecreasing increments of convex type.

2. Main Results

Theorem 2.1. *Let* $f : \acute{V} \to \mathbb{R}$ *be a CFWNDI of convex type defined on* $\acute{V}$ *and* $\xi^{(\acute{i})} \in \acute{V}, \acute{i} \in \{1, \ldots, n\}$ *with the condition* $\xi^{(1)} \leq \cdots \leq \xi^{(n)}$ *or* $\xi^{(1)} \geq \cdots \geq \xi^{(n)}$, *let*

$$f_{\kappa,n} = \binom{n}{\kappa}^{-1} \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} f\left(\frac{\xi^{(\acute{i}_1)} + \cdots + \xi^{(\acute{i}_\kappa)}}{\kappa}\right)$$

then

$$f_{\kappa,n} \geq f_{\kappa+1,n} \tag{2.1}$$

holds, for $k \in \{1, \cdots, n-1\}$.

Proof. We have

$$f_{\kappa+1,n} = \binom{n}{\kappa+1}^{-1} \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} f\left(\frac{\xi^{(\acute{i}_1)} + \cdots + \xi^{(\acute{i}_{\kappa+1})}}{\kappa+1}\right)$$

According to proposition (1.10)

$$\leq \binom{n}{\kappa+1}^{-1} \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} \frac{1}{\kappa+1} \sum_{j=1}^{\kappa+1} f\left(\frac{\xi^{(\acute{i}_1)} + \cdots + \xi^{(\acute{i}_{\kappa+1})} - \xi^{(\acute{i}j)}}{\kappa+1}\right)$$

$$= \binom{n}{\kappa}^{-1} \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} f\left(\frac{\xi^{(\acute{i}_1)} + \cdots + \xi^{(\acute{i}_\kappa)}}{\kappa}\right)$$

□

Theorem 2.2. *let $f = \acute{V} \to \mathbb{R}$ be a continuous function with nondecreasing increments of convex type defined on u and $\xi^{i} \in \acute{V}, \acute{i} \in \{1,\dots,n\}$ satisfy the condition $\xi^1 \leq \cdots \leq \xi^n$ or $\xi^1 \geq \cdots \geq \xi^n$, with $\rho_{\acute{i}}$ is a nonnegative n tuple such that $\rho_n > 0$. Let*

$$f_{\kappa,n} = \binom{n-1}{\kappa-1}^{-1} \rho_n \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}\right) f\left(\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_k}\xi^{(\acute{i}_\kappa)}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}}\right),$$

then

$$f_{\kappa,n}(\xi, p) \geq f_{\kappa+1,n}(\xi, p), \tag{2.2}$$

holds for $k = 1, \cdots, n-1$.

Proof. We have

$$\begin{aligned}
&\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k+1}\right) f\left(\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_k}\xi^{(\acute{i}_{\kappa+1})}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k+1}}\right) \\
&= \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k+1}\right) f\left(\frac{\sum_{j=1}^{\kappa+1}\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k+1} - \rho_{\acute{i}j}\right)\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_k+1}\xi^{(\acute{i}_{\kappa+1})} - \xi^{(\acute{i}j)}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{ij}}}{\left(\sum_{j=1}^{\kappa+1}\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}j}\right)}\right) \\
&\leq \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k+1}\right) \frac{\sum_{j=1}^{\kappa+1}\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k+1} - \rho_{\acute{i}j}\right) f\left(\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_k+1}\xi^{(\acute{i}_{\kappa+1})} - \xi^{(\acute{i}j)}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}j}}\right)}{\sum_{j=1}^{\kappa+1}\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}j}\right)} \\
&= \frac{1}{\kappa}\sum_{j=1}^{\kappa+1}\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k+1} - \rho_{\acute{i}j}\right) f\left(\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_k+1}\xi^{(\acute{i}_{\kappa+1})} - \xi^{(\acute{i}j)}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}j}}\right)
\end{aligned}$$

therefore

$$\begin{aligned}
f_{\kappa+1,n} &= \binom{n-1}{\kappa}^{-1} \rho_n \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}}\right) f\left(\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_k}\xi^{(\acute{i}_{\kappa+1})}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}}}\right) \\
&\leq \frac{1}{\kappa\binom{n-1}{\kappa}\rho_n} \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} \sum_{j=1}^{\kappa+1} \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k+1} - \rho_{\acute{i}j}\right) f\left(\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_k+1}\xi^{(\acute{i}_{\kappa+1})} - \xi^{(\acute{i}j)}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}j}}\right) \\
&= \frac{1}{\kappa\binom{n-1}{\kappa}\rho_n} \sum_{1 \leq \acute{i}_1 < \cdots < \acute{i}_\kappa \leq n} \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}\right) f\left(\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_k+1}\xi^{(\acute{i}_k)}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}}\right) \\
&= f_{\kappa,n}
\end{aligned}$$

□

Theorem 2.3. *Let $f = \acute{V} \to \mathbb{R}$ be a CFWNI of convex type and $\xi^{(\acute{i})} \in \acute{V}, \acute{i} \in \{1,\dots,n\}$ with the condition $\xi^{(1)} \leq \cdots \leq \xi^{(n)}$ or $\xi^{(1)} \geq \cdots \geq \xi^{(n)}$, with $\rho_{\acute{i}}$ is a positive n-tuple such that $\rho_n > 0$. Let*

(2.3)

$$f_{\kappa,n} = f_{\kappa,n}(\xi,\rho) = \frac{1}{\binom{n-1}{\kappa-1}\rho_n} \sum_{1\leq \acute{i}_1 < \cdots < \acute{i}_k \leq n} \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}\right) f\left(e + f - \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_k}\xi^{(\acute{i}_k)}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_k}}\right)$$

then

$$f_{\kappa,n}(\xi,\rho) \geq f_{\kappa+1,n}(\xi,\rho) \tag{2.4}$$

for $k \in \{1,\cdots,n-1\}$ holds.

Proof. By using the definition of convex functions and rearrangements we have

$$\begin{aligned}
&\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}}\right) f\left(e + f - \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_{\kappa+1}}\xi^{(\acute{i}_{\kappa+1})}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}}}\right) \\
&= \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}}\right) f\left[\frac{1}{\sum_{j=1}^{\kappa+1}\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}_j}\right)} \times \right. \\
&\left. \sum_{j=1}^{\kappa+1}\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}_j}\right)\left(e + f - \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_{\kappa+1}\xi^{(\acute{i}_{\kappa+1})}} - \rho_{\acute{i}_j}\xi^{(\acute{i}_j)}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}_j}}\right)\right] \\
&\leq \left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}}\right)\left[\frac{1}{\sum_{j=1}^{\kappa+1}\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}_j}\right)} \times \right. \\
&\left. \sum_{j=1}^{\kappa+1}\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}_j}\right) f\left(e + f - \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_{\kappa+1}\xi^{(\acute{i}_{\kappa+1})}} - \rho_{\acute{i}_j\xi^{(\acute{i}_j)}}}{\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}_j}}\right)\right] \\
&= \frac{1}{\kappa}\sum_{j=1}^{\kappa+1}\left(\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}_j}\right) f\left(e + f - \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)} + \cdots + \rho_{\acute{i}_{\kappa+1}}\xi^{(\acute{i}_{\kappa+1})}}{-}\rho_{\acute{i}_j}\xi^{(\acute{i}_j)}\rho_{\acute{i}_1} + \cdots + \rho_{\acute{i}_{\kappa+1}} - \rho_{\acute{i}_j}\right)
\end{aligned}$$

in order to use above result we consider

$$f_{\kappa+1,n}=\frac{1}{\binom{n-1}{\kappa}\rho_n}\sum_{1\le i_1<\cdots<i_{\kappa+1}\le n}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}\right)f\left(e+f-\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_{\kappa+1}}\xi^{(\acute{i}_{\kappa+1})}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}}\right)$$

$$\le\frac{1}{\kappa\binom{n-1}{\kappa}\rho_n}\sum_{1\le i_1<\cdots<i_{\kappa+1}\le n}\sum_{j=1}^{\kappa+1}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_j}\right)\times$$

$$f\left(e+f-\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_{\kappa+1}}\xi^{(\acute{i}_{\kappa+1})}-\rho_{\acute{i}_j}\xi^{(\acute{i}_j)}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_j}}\right)$$

$$=\frac{1}{\binom{n-1}{\kappa-1}\rho_n}\sum_{1\le\acute{i}_1<\cdots<\acute{i}_k\le n}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_k}\right)f\left(e+f-\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_k\xi^{(\acute{i}_k)}}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_k}}\right)=f_{\kappa,n}$$

which concludes our proof. ∎

Remark 2.4. Similar to (1.5), the following refinement for (2.3) can be defined.

$$f\left(e+f-\frac{1}{\rho_n}\sum_{\acute{i}=1}^{n}\rho_{\acute{i}}\xi_{\acute{i}}\right)=f_{n,n}\le\cdots\le\quad f_{\kappa+1,n}\le$$

$$f_{\kappa,n}\le\cdots\le f_{1,n}\le f(e)+f(f)-\frac{1}{\rho_n}\sum_{\acute{i}=1}^{n}\rho_{\acute{i}}f(\xi_{\acute{i}}).$$

Now, we establish Gabler inequality for Neizgoda's inequality for functions with nondecreasing increments for convex type.

Theorem 2.5. *Under the assumptions of Proposition* (1.9)*, if we define*

$$f_{\kappa,n}=f_{\kappa,n}(\xi,a,\rho)=\frac{1}{\binom{n-1}{\kappa-1}\rho_n}\sum_{1\le\acute{i}_1<\cdots<\acute{i}_k\le n}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_k}\right)\times$$

$$f\left(\sum_{j=1}^{m}a_j-\sum_{j=1}^{m-1}\frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_k\xi^{(\acute{i}_k)}}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_k}}\right)\tag{2.5}$$

then the inequality (1.4) *holds.*

Proof. By using the definition of convex functions and rearrangements we have

$$\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}\right) f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1} \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_{\kappa+1}}\xi^{(\acute{i}_{\kappa+1})}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}}\right)$$

$$=\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}\right) f\left[\frac{1}{\sum_{l=1}^{\kappa+1}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_l}\right)}\times\right.$$

$$\left.\left[\sum_{l=1}^{\kappa+1}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_l}\right)\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1} \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_{\kappa+1}}\xi^{(\acute{i}_{\kappa+1})}-\rho_{\acute{i}_l}\xi^{(\acute{i}_l)}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_l}}\right)\right]\right]$$

$$\leq\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}\right)\left[\frac{1}{\sum_{l=1}^{\kappa+1}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_l}\right)}\times\right.$$

$$\left.\sum_{l=1}^{\kappa+1}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_l}\right) f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1} \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_{\kappa+1}}\xi^{(\acute{i}_{\kappa+1})}-\rho_{\acute{i}_l}\xi^{(\acute{i}_l)}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_l}}\right)\right]$$

$$=\frac{1}{\kappa}\sum_{l=1}^{\kappa+1}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_l}\right) f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1} \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_{\kappa+1}}\xi^{(\acute{i}_{\kappa+1})}-\rho_{\acute{i}_l}\xi^{(\acute{i}_l)}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_l}}\right)$$

in order to use the above result we consider

$$f_{\kappa+1,n}=\frac{1}{\binom{n-1}{\kappa}\rho_n}\sum_{1\leq i_1<\cdots<i_{\kappa+1}\leq n}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}\right)\times$$

$$f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1} \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_{\kappa+1}}\xi^{(\acute{i}_{\kappa+1})}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}}\right)$$

$$\leq\frac{1}{\kappa\binom{n-1}{\kappa}\rho_n}\sum_{1\leq i_1<\cdots<i_{\kappa+1}\leq n}\sum_{l=1}^{\kappa+1}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_l}\right)\times$$

$$f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1} \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_{\kappa+1}}\xi^{(\acute{i}_{\kappa+1})}-\rho_{\acute{i}_l}\xi_{\acute{i}_l}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_{\kappa+1}}-\rho_{\acute{i}_l}}\right)$$

$$=\frac{1}{\binom{n-1}{\kappa-1}\rho_n}\sum_{1\leq\acute{i}_1<\cdots<\acute{i}_k\leq n}\left(\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_k}\right) f\left(\sum_{j=1}^{m} a_j-\sum_{j=1}^{m-1} \frac{\rho_{\acute{i}_1}\xi^{(\acute{i}_1)}+\cdots+\rho_{\acute{i}_k}\xi^{(\acute{i}_k)}}{\rho_{\acute{i}_1}+\cdots+\rho_{\acute{i}_k}}\right)=f_{\kappa,n}$$

Which concludes our proof.

□

Remark 2.6. Similar to (1.5), the following refinement for (2.5) can be defined.

$$f\left(\sum_{j=1}^{m} a_j - \frac{1}{\rho_n}\sum_{j=1}^{m-1}\sum_{\acute{i}=1}^{n}\rho_{\acute{i}}\xi^{(\acute{i}j)}\right) = \quad f_{n,n} \leq \cdots \leq f_{\kappa+1,n} \leq$$

$$f_{\kappa,n} \leq \cdots \leq f_{1,n} \leq \sum_{j=1}^{m} f(a_j) - \frac{1}{\rho_n}\sum_{j=1}^{m-1}\sum_{\acute{i}=1}^{n}\rho_{\acute{i}} f(\xi^{(\acute{i}j)}).$$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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