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## GLOBAL ANALYSIS FOR A TWO-STRAIN AVIAN INFLUENZA MODEL WITH DISTRIBUTED DELAY AND ENVIRONMENTAL TRANSMISSION

YAN-XIA DANG<sup>1</sup>, JUAN WANG<sup>2</sup>, MAIA MARTCHEVA<sup>3</sup>, XUE-ZHI LI<sup>4,\*</sup>

<sup>1</sup>Department of Public Education, Zhumadian Vocational and Technical College, Zhumadian 463000, China

<sup>2</sup>Department of Mathematics, Xinyang Normal University, Xinyang 464000, China

<sup>3</sup>Department of Mathematics, University of Florida, Gainesville, FL 32611–8105, USA

<sup>4</sup>Department of Mathematics and Physics, Anyang Institute of Technology, Anyang 455000, China

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**Abstract.** A two-strain avian influenza model with distributed delay and environmental spread in humans is investigated. The model describes well the transmission of avian influenza between poultry and humans. In this study, we introduce the behavior of both high pathogenic avian influenza (HPAI) as strain two and low pathogenic avian influenza (LPAI) as strain one in a domestic poultry population. We also include the distribution of the strain two through the contaminated environment. We compute the strain reproduction numbers  $\mathcal{R}_1$ ,  $\mathcal{R}_2$  and the invasion  $\hat{\mathcal{R}}_1$ ,  $\hat{\mathcal{R}}_2$ . We find that besides the disease-free equilibrium, there exist a dominance equilibrium for each strain and many coexistence equilibrium of both strain one and strain two if  $\mathcal{R}_1 = \mathcal{R}_2$ . Using a Lyapunov functional, we are able to establish global stability of the disease-free equilibrium if  $\max\{\mathcal{R}_1, \mathcal{R}_2\} < 1$ . If  $\mathcal{R}_i$ , the reproduction number of strain  $i$  is larger than one, then a single-strain equilibrium, corresponding to strain  $i$  exists. This single-strain equilibrium is locally stable whenever  $\hat{\mathcal{R}}_i > 1$ . Using a Lyapunov functional, we establish that the corresponding single-strain equilibrium  $\varepsilon_i$  is globally stable. When  $\mathcal{R}_1 = \mathcal{R}_2 > 1$  and  $\hat{\mathcal{R}}_1 = \hat{\mathcal{R}}_2 = 1$ , there are perhaps many coexistence equilibria of both strain one and strain two. Environmental transmission to humans may explain why avian influenza A (H7N9) virus has appear in humans in different places in China in 2013 and 2014.

**Keywords:** avian influenza; reproduction number; distributed delay; global stability; Lyapunov function.

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\*Corresponding author

E-mail address: xzli66@126.com

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## 1. Introduction

Avian Influenza (AI) virus chiefly infects birds, both wild and domestic. Avian influenza viruses of H5 and H7 subtypes have high pathogenic (HPAI) and low pathogenic (LPAI) form. Both forms infect poultry. Poultry infected with LPAI (strain one) show mild symptoms and recover. However, HPAI (strain two) is generally extremely virulent to poultry, with mortality rate 90%-100%. HPAI often kills chickens within two days of onset of symptoms. Highly pathogenic (strain two) H5N1 avian influenza have shown ability to transmit to humans and poses a big threat to public health since it may mutate to a pandemic human H5N1 influenza strain [1].

Human infections with a new strain of the avian influenza A (H7N9) virus were first reported in China in March in 2013. Most of these infections are believed to result from exposure to infected poultry or contaminated environment, as H7N9 viruses have also been found in poultry in China. While some mild illnesses in human H7N9 cases have been seen, most patients have had severe respiratory illness, with about one-third resulting in death.

In two successive and increasing waves, this virus has moved across China and crossed the Chinese border into Hong Kong, Taiwan and Malaysia. According to CDC it is possible that the virus can appear in the US.

The persistence and the pandemic threat of avian influenza as well as the very publicized cholera outbreak in Haiti have increased the awareness of diseases which transmit both directly and environmentally. Many recent articles have been devoted to indirectly transmitted diseases [2,3,4,5,6]. In this article, we investigate a two-strain avian influenza model describing the transmission of avian influenza between poultry and humans, including both direct and environmental transmission to humans.

Our model was inspired by the model introduced in [7]. Compared to the model in [7], our model includes distributed delay and the transmission between poultry and humans. Furthermore, we consider the environmental transmission. Our results are focused on the number of the equilibria and their local as well as global stabilities. While the authors in [7] mainly discuss the coexistence of pathogen strains caused by culling in a influenza model.

This paper is structured as follows. In section 2, we introduce a two-strain avian influenza model with distributed delay and environmental transmission. In section 3, we discuss the equilibria and establish their local stabilities. In section 4, we establish global stability of the disease-free equilibrium. In section 5, we use Lyapunov functional to derive the global stability of the single-strain equilibrium. In section 6, we summarize our results.

## 2. The two-strain avian influenza model

As in the introduction, we assume the pathogen exists through two strains. LPAI is strain one and HPAI is strain two. The two-strain model divides poultry under consideration into the following groups: susceptible poultry, denoted by  $S_v(t)$ , infected poultry with a strain  $i$ , denoted by  $I_{v_i}(t)$  ( $i = 1, 2$ ), and recovered poultry from strain one, denoted by  $R_v(t)$ . If we let  $N_v(t)$  be the total number of poultry at time  $t$ , we have  $N_v(t) = S_v(t) + I_{v_1}(t) + I_{v_2}(t) + R_v(t)$ . Let  $N_h(t)$  be the total number of humans at time  $t$ .  $N_h(t)$  is composed of the number of susceptible human individuals  $S_h(t)$ , the number of infective human individuals  $I_h(t)$ , and the number of recovered or immune human individuals  $R_h(t)$ . Thus,  $N_h(t) = S_h(t) + I_h(t) + R_h(t)$ . Let  $E(t)$  be the number of virus of strain two in the contaminated environment.

Because the dynamics of the virus in the human population is subjected to a significant influence from the incubation period of the pathogen within humans and in reality the incubation period is not a number but an interval during which the maturation of the parasite occurs in different individuals, we incorporate distributed delay in the humans to account for the delays. Let  $\tau$  be the incubation period of the parasite in humans. Here, we assume that  $\tau$  is distributed

parameter (see [8,9]). The model takes the form

$$\left\{ \begin{array}{l} \frac{dS_v}{dt} = \Lambda_v - \beta_{v_1} S_v I_{v_1} - \beta_{v_2} S_v I_{v_2} - \mu_v S_v, \\ \frac{dI_{v_1}}{dt} = \beta_{v_1} S_v I_{v_1} - (\mu_v + r_v) I_{v_1}, \\ \frac{dI_{v_2}}{dt} = \beta_{v_2} S_v I_{v_2} - (\mu_v + \alpha_v) I_{v_2}, \\ \frac{dR_v}{dt} = r_v I_{v_1} - \mu_v R_v, \\ \frac{dE}{dt} = \delta I_{v_2} - \gamma E, \\ \frac{dS_h}{dt} = \Lambda_h - \beta_{h_1} S_h \int_0^\tau f_1(s) I_{v_2}(t-s) ds - \beta_{h_2} S_h \int_0^\tau f_2(s) E(t-s) ds - \mu_h S_h, \\ \frac{dI_h}{dt} = \beta_{h_1} S_h \int_0^\tau f_1(s) I_{v_2}(t-s) ds + \beta_{h_2} S_h \int_0^\tau f_2(s) E(t-s) ds - (\mu_h + \alpha_h + r_h) I_h, \\ \frac{dR_h}{dt} = r_h I_h - \mu_h R_h(t). \end{array} \right. \quad (2.1)$$

In model (2.1),  $\Lambda_h$  and  $\Lambda_v$  are the birth/recruitment rate of humans and poultry,  $\beta_{v_i}$  is the transmission coefficient of strain  $i$  among poultry,  $i$  ( $i = 1, 2$ ). Similarly,  $\beta_{h_1}$  is the transmission coefficient of strain two from poultry to humans.  $\beta_{h_2}$  is the transmission rate to humans from the environmental contamination.  $\mu_h, \mu_v$  are the natural death rates of humans and poultry, respectively.  $r_v, r_h$  are the recovery rates of poultry and humans.  $\alpha_v, \alpha_h$  are the disease-induced death rates. The kernel functions  $f_1(\tau), f_2(\tau)$  expresses the distributed infectivity toward susceptible individuals during the infectious period of the surviving infectious poultry or the avian influenza virus in the environment. The term

$$\beta_{h_1} S_h(t) \int_0^\tau f_1(s) I_{v_2}(t-s) ds + \beta_{h_2} S_h(t) \int_0^\tau f_2(s) E(t-s) ds$$

gives the incidence of new cases of infection for humans at time  $t$ .

To understand the model, notice that susceptible poultry are recruited at a rate  $\Lambda_v$ . Susceptible poultry can become infected with strain  $i$  ( $i = 1, 2$ ) through a direct contact with an infected poultry with strain  $i$ . The infected poultry with strain one  $I_{v_1}$  can recover with recovery rate  $r_v$ , while the infected poultry with strain 2 may die with disease-induced death rate  $\alpha_v$ . In the same time, susceptible human individuals are recruited at a rate  $\Lambda_h$ . Susceptible individuals can become infected with strain two either through a direct contact with infected poultry infected with strain two or through coming into contact with viral particles of strain two that are in the

environment. Furthermore, it is assumed that a susceptible poultry, who has been exposed, may remain exposed for some period before becoming infectious and may have variable infectivity toward humans. As a consequence, the force of infection on susceptible human individuals through direct or indirect contact is given by the integral over all the incubation periods of the parasite in the poultry. Infected humans have a recovery rate  $r_h$  and move to the recovered class  $R_h(t)$ . Others infected humans may die with disease-induced death rate  $\alpha_h$ . Infected poultry with strain two shed the virus into the environment at a rate  $\delta$ . All viral particles shed by poultry infected with strain two are given by  $\delta I_{v_2}$ . We notice that the equations for the recovered poultry and recovered humans are decoupled from the system and the analysis of system (2.1) is equivalent to the analysis of the system.

$$\left\{ \begin{array}{l} \frac{dS_v}{dt} = \Lambda_v - \beta_{v_1} S_v I_{v_1} - \beta_{v_2} S_v I_{v_2} - \mu_v S_v, \\ \frac{dI_{v_1}}{dt} = \beta_{v_1} S_v I_{v_1} - (\mu_v + r_v) I_{v_1}, \\ \frac{dI_{v_2}}{dt} = \beta_{v_2} S_v I_{v_2} - (\mu_v + \alpha_v) I_{v_2}, \\ \frac{dE}{dt} = \delta I_{v_2} - \gamma E, \\ \frac{dS_h}{dt} = \Lambda_h - \beta_{h_1} S_h \int_0^\tau f_1(s) I_{v_2}(t-s) ds - \beta_{h_2} S_h \int_0^\tau f_2(s) E(t-s) ds - \mu_h S_h, \\ \frac{dI_h}{dt} = \beta_{h_1} S_h \int_0^\tau f_1(s) I_{v_2}(t-s) ds + \beta_{h_2} S_h \int_0^\tau f_2(s) E(t-s) ds - (\mu_h + \alpha_h + r_h) I_h. \end{array} \right. \quad (2.2)$$

In the remainder of this article we will focus on model (2.2). Model (2.2) is equipped with the following initial conditions:

$$\left\{ \begin{array}{l} S_v(0) = S_{v_0}, I_{v_1}(0) = I_{v_{1_0}}, I_{v_2}(\theta) = \psi_{v_2}(\theta), \\ S_h(0) = S_{h_0}, I_h(0) = I_{h_0}, E(\theta) = \psi_E(\theta), \theta \in [-\tau, 0]. \end{array} \right. \quad (2.3)$$

All parameters in model (2.2) are non-negative. We define the following space of functions

$$X = \mathbf{R}^+ \times \mathbf{R}^+ \times \prod_{i=1}^2 \left( C([-\tau, 0], \mathbf{R}^+) \right) \times \mathbf{R}^+ \times \mathbf{R}^+.$$

where the Banach space  $C([-\tau, 0], \mathbf{R}^+)$  of continuous functions mapping the interval  $[-\tau, 0]$  into  $\mathbf{R}^+$  is equipped with the sup-norm  $\|\psi\| = \sup_{-\tau \leq \theta \leq 0} |\psi(\theta)|$ . By the standard theory of functional differential equations [10], it can be verified that (2.2) with initial conditions (2.3) has a unique solution  $(S_v(t), I_{v_1}(t), I_{v_2}(t), E(t), S_h(t), I_h(t))$  which remains non-negative for all

$t \geq 0$ . Moreover, we can show the solutions of system (2.2) are ultimately uniformly bounded in  $X$ . In fact, it follows that the total poultry population size  $N_v(t) = S_v(t) + I_{v_1}(t) + I_{v_2}(t)$  satisfies

$$\frac{d}{dt} \left( S_v(t) + I_{v_1}(t) + I_{v_2}(t) \right) \leq \Lambda_v - \mu_v \left( S_v(t) + I_{v_1}(t) + I_{v_2}(t) \right).$$

Hence,

$$\limsup_{t \rightarrow \infty} \left( S_v(t) + I_{v_1}(t) + I_{v_2}(t) \right) = \frac{\Lambda_v}{\mu_v}.$$

Similarly, the total human population size  $N_h(t) = S_h(t) + I_h(t)$  satisfies

$$\frac{d}{dt} \left( S_h(t) + I_h(t) \right) \leq \Lambda_h - \mu_h \left( S_h(t) + I_h(t) \right),$$

so we have

$$\limsup_{t \rightarrow \infty} \left( S_h(t) + I_h(t) \right) \leq \frac{\Lambda_h}{\mu_h}.$$

The free virus in the environment can be bounded as follows:

$$E' \leq \delta \frac{\Lambda_v}{\mu_v} - \gamma E.$$

Hence

$$\limsup_{t \rightarrow \infty} E(t) \leq \frac{\delta \frac{\Lambda_v}{\mu_v}}{\gamma} = \frac{\delta \Lambda_v}{\gamma \mu_v}.$$

Therefore, the following set is positively invariant for system (2.2)

$$\Omega = \left\{ (S_v, I_{v_1}, I_{v_2}, E, S_h, I_h) \in X_+ : S_v + I_{v_1} + I_{v_2} \leq \frac{\Lambda_v}{\mu_v}, S_h + I_h \leq \frac{\Lambda_h}{\mu_h}, E \leq \frac{\delta \Lambda_v}{\gamma \mu_v} \right\}.$$

All positive semi-orbits in  $\Omega$  are precompact in  $X$ , and thus have non-empty  $\omega$ -limit sets. We have the following result.

**Lemma 2.1** *All positive semi-orbits in  $\Omega$  have non-empty  $\omega$ -limit sets.*

Furthermore, we impose the following assumptions:

### Assumptions 1:

- (1) It is assumed that  $f_1(s)$  and  $f_2(s)$  are continuous on  $[0, \tau]$ ;
- (2)  $f_1(s)$  and  $f_2(s)$  satisfy

$$\int_0^\tau f_1(s) ds = a_1, \quad \int_0^\tau f_2(s) ds = a_2;$$

- (3)  $f_1(s) \geq 0, f_2(s) \geq 0$  for  $0 \leq s \leq \tau$ .

The reproduction number of strain one and two are given by the following expressions

$$\mathcal{R}_1 = \frac{\beta_{v_1} \Lambda_v}{\mu_v (\mu_v + r_v)}, \quad \mathcal{R}_2 = \frac{\beta_{v_2} \Lambda_v}{\mu_v (\mu_v + \alpha_v)}. \quad (2.4)$$

respectively. The system has a reproduction number defined as

$$\mathcal{R}_0 = \max\{\mathcal{R}_1, \mathcal{R}_2\}.$$

We also introduce the invasion numbers of strain one and strain two. The invasion number of strain one (two) at the equilibrium of strain two (one) is given by

$$\hat{\mathcal{R}}_1 = \frac{\mathcal{R}_1}{\mathcal{R}_2}, \quad \hat{\mathcal{R}}_2 = \frac{\mathcal{R}_2}{\mathcal{R}_1}.$$

In the next section we compute explicit expressions for the equilibria and establish their local stability.

### 3. Equilibria and their local stability

In the positively invariant region

$$\Omega = \left\{ (S_v, I_{v_1}, I_{v_2}, E, S_h, I_h) \in X_+ : S_v + I_{v_1} + I_{v_2} \leq \frac{\Lambda_v}{\mu_v}, S_h + I_h \leq \frac{\Lambda_h}{\mu_h}, E \leq \frac{\delta \Lambda_v}{\gamma \mu_v} \right\},$$

system (2.2) always has a unique disease-free equilibrium  $\mathcal{E}_0$ , which is given by

$$\mathcal{E}_0 = \left( \frac{\Lambda_v}{\mu_v}, 0, 0, 0, \frac{\Lambda_h}{\mu_h}, 0 \right).$$

In addition, for each  $i$  there is a corresponding single-strain equilibrium  $\mathcal{E}_i$  given by

$$\mathcal{E}_1 = (S_{v_1}^*, I_{v_1}^*, 0, 0, S_{h_1}^*, 0), \quad \mathcal{E}_2 = (S_{v_2}^*, 0, I_{v_2}^*, E^*, S_{h_2}^*, I_h^*),$$

where  $S_{v_1}^*, I_{v_1}^*, S_{h_1}^*$  and  $S_{v_2}^*, I_{v_2}^*, E^*, S_{h_2}^*, I_h^*$  satisfy

$$\begin{cases} \Lambda_v - \beta_{v_1} S_{v_1}^* I_{v_1}^* - \mu_v S_{v_1}^* = 0, \\ \beta_{v_1} S_{v_1}^* I_{v_1}^* - (\mu_v + r_v) I_{v_1}^* = 0, \\ \Lambda_h - \mu_h S_{h_1}^* = 0, \end{cases} \quad (3.1)$$

and

$$\begin{cases} \Lambda_v - \beta_{v_2} S_{v_2}^* I_{v_2}^* - \mu_v S_{v_2}^* = 0, \\ \beta_{v_2} S_{v_2}^* I_{v_2}^* - (\mu_v + \alpha_v) I_{v_2}^* = 0, \\ \delta I_{v_2}^* - \gamma E^* = 0, \\ \Lambda_h - \beta_{h_1} S_{h_2}^* I_{v_2}^* a_1 - \beta_{h_2} S_{h_2}^* E^* a_2 - \mu_h S_{h_2}^* = 0, \\ \beta_{h_1} S_{h_2}^* I_{v_2}^* a_1 + \beta_{h_2} S_{h_2}^* E^* a_2 - (\mu_h + \alpha_h + r_h) I_h^* = 0. \end{cases} \quad (3.2)$$

The non-zero components of the equilibrium  $\mathcal{E}_j$  are given by

$$\begin{aligned} S_{v_1}^* &= \frac{\mu_v + r_v}{\beta_{v_1}}, & I_{v_1}^* &= \frac{\Lambda_v}{\mu_v + r_v} \left(1 - \frac{1}{\mathcal{R}_1}\right), & S_{h_1}^* &= \frac{\Lambda_h}{\mu_h}, \\ S_{v_2}^* &= \frac{\mu_v + \alpha_v}{\beta_{v_2}}, & I_{v_2}^* &= \frac{\Lambda_v}{\mu_v + \alpha_v} \left(1 - \frac{1}{\mathcal{R}_2}\right), & E^* &= \frac{\delta}{\gamma} I_{v_2}^*, \\ S_{h_2}^* &= \frac{\Lambda_h}{\beta_{h_1} I_{v_2}^* a_1 + \beta_{h_2} E^* a_2 + \mu_h}, & I_h^* &= \frac{\beta_{h_1} I_{v_2}^* a_1 + \beta_{h_2} E^* a_2}{\mu_h + \alpha_h + r_h} S_{h_2}^*. \end{aligned}$$

The endemic equilibrium  $\mathcal{E}_i$  exists if and only if  $\mathcal{R}_i > 1$ . So we have the following results.

**Theorem 3.1** *The model (2.2) has a unique strain 1 dominance equilibrium  $\mathcal{E}_1 = (S_{v_1}^*, I_{v_1}^*, 0, 0, S_{h_1}^*, 0)$  if  $\mathcal{R}_1 > 1$ , and a unique strain 2 dominance equilibrium  $\mathcal{E}_2 = (S_{v_2}^*, 0, I_{v_2}^*, E^*, S_{h_2}^*, I_h^*)$  if  $\mathcal{R}_2 > 1$ .*

Concerning the coexistence equilibria, we have:

**Theorem 3.2** *If  $\mathcal{R}_1 = \mathcal{R}_2 > 1, \hat{\mathcal{R}}_1 = \hat{\mathcal{R}}_2 = 1$ , then there exist many coexistence equilibria*

*$(\bar{S}_v^*, \bar{I}_{v_1}^*, \bar{I}_{v_2}^*, \bar{E}^*, \bar{S}_h^*, \bar{I}_h^*)$ , where*

$$\bar{S}_v^* = S_{v_1}^* = S_{v_2}^*, \quad \bar{E}^* = \frac{\delta}{\gamma} \bar{I}_{v_2}^*, \quad \bar{S}_h^* = \frac{\Lambda_h}{\beta_{h_1} \bar{I}_{v_2}^* a_1 + \beta_{h_2} \bar{E}^* a_2 + \mu_h}, \quad \bar{I}_h^* = \frac{\beta_{h_1} \bar{I}_{v_2}^* a_1 + \beta_{h_2} \bar{E}^* a_2}{\mu_h + \alpha_h + r_h} \bar{S}_h^*,$$

where  $\bar{I}_{v_1}^*$  and  $\bar{I}_{v_2}^*$  satisfy the following equation:

$$\beta_{v_1} \bar{I}_{v_1}^* + \beta_{v_2} \bar{I}_{v_2}^* = \beta_{v_1} I_{v_1}^* = \beta_{v_2} I_{v_2}^*$$



**Proof.** We assume that  $(\bar{S}_v^*, \bar{I}_{v_1}^*, \bar{I}_{v_2}^*, \bar{E}^*, \bar{S}_h^*, \bar{I}_h^*)$  is an equilibrium of the system (2.2), then it must satisfy the following system:

$$\begin{cases} \Lambda_v - \beta_{v_1} \bar{S}_v^* \bar{I}_{v_1}^* - \beta_{v_2} \bar{S}_v^* \bar{I}_{v_2}^* - \mu_v \bar{S}_v^* = 0, \\ \beta_{v_1} \bar{S}_v^* \bar{I}_{v_1}^* - (\mu_v + r_v) \bar{I}_{v_1}^* = 0, \\ \beta_{v_2} \bar{S}_v^* \bar{I}_{v_2}^* - (\mu_v + \alpha_v) \bar{I}_{v_2}^* = 0, \\ \delta \bar{I}_{v_2}^* - \gamma \bar{E}^* = 0, \\ \Lambda_h - \beta_{h_1} \bar{S}_h^* \bar{I}_{v_2}^* a_1 - \beta_{h_2} \bar{S}_h^* \bar{E}^* a_2 - \mu_h \bar{S}_h^* = 0, \\ \beta_{h_1} \bar{S}_h^* \bar{I}_{v_2}^* a_1 + \beta_{h_2} \bar{S}_h^* \bar{E}^* a_2 - (\mu_h + \alpha_h + r_h) \bar{I}_h^* = 0. \end{cases} \quad (3.3)$$

By the second and third equation of (3.3), we obtain

$$\bar{S}_v^* = \frac{\mu_v + r_v}{\beta_{v_1}} = \frac{\mu_v + \alpha_v}{\beta_{v_2}}$$

Noticing that

$$S_{v_1}^* = \frac{\mu_v + r_v}{\beta_{v_1}}, \quad S_{v_2}^* = \frac{\mu_v + \alpha_v}{\beta_{v_2}}$$

So we have

$$\bar{S}_v^* = S_{v_1}^* = S_{v_2}^*$$

From the first equation of (3.3), we have

$$\Lambda_v - \mu_v \bar{S}_v^* = \beta_{v_1} \bar{S}_v^* \bar{I}_{v_1}^* + \beta_{v_2} \bar{S}_v^* \bar{I}_{v_2}^*.$$

Using the the first equation of (3.1), satisfied by equilibrium  $\mathcal{E}_1$ , we have the relation

$$\Lambda_v - \mu_v \bar{S}_v^* = \Lambda_v - \mu_v S_{v_1}^* = \beta_{v_1} S_{v_1}^* I_{v_1}^* = \beta_{v_1} \bar{S}_v^* I_{v_1}^*.$$

Then we obtain

$$\beta_{v_1} \bar{S}_v^* I_{v_1}^* = \beta_{v_1} \bar{S}_v^* \bar{I}_{v_1}^* + \beta_{v_2} \bar{S}_v^* \bar{I}_{v_2}^*.$$

We divide both sides by  $\bar{S}_v^*$ ,

$$\beta_{v_1} I_{v_1}^* = \beta_{v_1} \bar{I}_{v_1}^* + \beta_{v_2} \bar{I}_{v_2}^*.$$

Similarly, we have

$$\beta_{v_2} I_{v_2}^* = \beta_{v_1} \bar{I}_{v_1}^* + \beta_{v_2} \bar{I}_{v_2}^*.$$

Thus  $\bar{I}_{v_1}^*$  and  $\bar{I}_{v_2}^*$  satisfy the following equation:

$$\beta_{v_1}\bar{I}_{v_1}^* + \beta_{v_2}\bar{I}_{v_2}^* = \beta_{v_1}I_{v_1}^* = \beta_{v_2}I_{v_2}^*.$$

Similarly to  $E^*, S_{h_2}^*, I_h^*$ , we can get  $\bar{E}^*, \bar{S}_h^*$  and  $\bar{I}_h^*$ .

$$\bar{E}^* = \frac{\delta}{\gamma}\bar{I}_{v_2}^*, \quad \bar{S}_h^* = \frac{\Lambda_h}{\beta_{h_1}\bar{I}_{v_2}^*a_1 + \beta_{h_2}\bar{E}^*a_2 + \mu_h}, \quad \bar{I}_h^* = \frac{\beta_{h_1}\bar{I}_{v_2}^*a_1 + \beta_{h_2}\bar{E}^*a_2}{\mu_h + \alpha_h + r_h}\bar{S}_h^*.$$

Since there are many  $\bar{I}_{v_1}^*$  and  $\bar{I}_{v_2}^*$  that satisfy the equation  $\beta_{v_1}\bar{I}_{v_1}^* + \beta_{v_2}\bar{I}_{v_2}^* = \beta_{v_1}I_{v_1}^* = \beta_{v_2}I_{v_2}^*$ , the proof is complete.  $\square$

Now we are ready to establish the following result.

**Theorem 3.3** *If  $\mathcal{R}_0 = \max\{\mathcal{R}_1, \mathcal{R}_2\} < 1$ , then the disease-free equilibrium  $\mathcal{E}_0$  is locally asymptotically stable. If  $\mathcal{R}_0 > 1$ , then it is unstable.*

**Proof.** In order to investigate the local stability of the model, let us first linearize system (2.2) at  $\mathcal{E}_0$ . Let  $S_v(t) = \Lambda_v/\mu_v + x_v(t), I_{v_1}(t) = y_{v_1}(t), I_{v_2}(t) = y_{v_2}(t), E(t) = z(t), S_h(t) = \Lambda_h/\mu_h + x_h(t), I_h(t) = y_h(t)$ . Thus, we obtain the following linearized system

$$\left\{ \begin{array}{l} \frac{dx_v}{dt} = -\beta_{v_1}\frac{\Lambda_v}{\mu_v}y_{v_1} - \beta_{v_2}\frac{\Lambda_v}{\mu_v}y_{v_2} - \mu_v x_v, \\ \frac{dy_{v_1}}{dt} = \beta_{v_1}\frac{\Lambda_v}{\mu_v}y_{v_1} - (\mu_v + r_v)y_{v_1}, \\ \frac{dy_{v_2}}{dt} = \beta_{v_2}\frac{\Lambda_v}{\mu_v}y_{v_2} - (\mu_v + \alpha_v)y_{v_2}, \\ \frac{dz}{dt} = \delta y_{v_2} - \gamma z, \\ \frac{dx_h}{dt} = -\beta_{h_1}\frac{\Lambda_h}{\mu_h}\int_0^\tau f_1(s)y_{v_2}(t-s)ds - \beta_{h_2}\frac{\Lambda_h}{\mu_h}\int_0^\tau f_2(s)z(t-s)ds - \mu_h x_h, \\ \frac{dy_h}{dt} = \beta_{h_1}\frac{\Lambda_h}{\mu_h}\int_0^\tau f_1(s)y_{v_2}(t-s)ds + \beta_{h_2}\frac{\Lambda_h}{\mu_h}\int_0^\tau f_2(s)z(t-s)ds - (\mu_h + \alpha_h + r_h)y_h. \end{array} \right. \quad (3.4)$$

To study system (3.4), we notice that the system for  $x_h$  and  $y_h$  is decoupled from the equations for  $x_v, y_{v_1}, y_{v_2}$  and  $z$ . Hence, the equation for  $x_h$  and  $y_h$  are independent from the first to the fourth equation.

We investigate solutions of the form

$$x_v(t) = x_v e^{\lambda t}, \quad y_{v_1}(t) = y_{v_1} e^{\lambda t}, \quad y_{v_2}(t) = y_{v_2} e^{\lambda t}, \quad z(t) = z e^{\lambda t}.$$

This leads to solving the following set of equations.

$$\begin{cases} \lambda x_v = -\beta_{v_1} \frac{\Lambda_v}{\mu_v} y_{v_1} - \beta_{v_2} \frac{\Lambda_v}{\mu_v} y_{v_2} - \mu_v x_v, \\ \lambda y_{v_1} = \beta_{v_1} \frac{\Lambda_v}{\mu_v} y_{v_1} - (\mu_v + r_v) y_{v_1}, \\ \lambda y_{v_2} = \beta_{v_2} \frac{\Lambda_v}{\mu_v} y_{v_2} - (\mu_v + \alpha_v) y_{v_2}, \\ \lambda z = \delta y_{v_2} - \gamma z. \end{cases} \quad (3.5)$$

System (3.5) is a linear system. Thus, looking for eigenvalues in the model is equivalent to find the characteristic roots which are determined by the following equation:

$$\begin{aligned} & \begin{vmatrix} \lambda + \mu_v & \beta_{v_1} \frac{\Lambda_v}{\mu_v} & \beta_{v_2} \frac{\Lambda_v}{\mu_v} & 0 \\ 0 & \lambda + \mu_v + r_v - \beta_{v_1} \frac{\Lambda_v}{\mu_v} & 0 & 0 \\ 0 & 0 & \lambda + \mu_v + \alpha_v - \beta_{v_2} \frac{\Lambda_v}{\mu_v} & 0 \\ 0 & 0 & -\delta & \lambda + \gamma \end{vmatrix} \\ = & \begin{vmatrix} \lambda + \mu_v & \beta_{v_1} \frac{\Lambda_v}{\mu_v} & \beta_{v_2} \frac{\Lambda_v}{\mu_v} & 0 \\ 0 & \lambda - (\mu_v + r_v)(\mathcal{R}_1 - 1) & 0 & 0 \\ 0 & 0 & \lambda - (\mu_v + \alpha_v)(\mathcal{R}_2 - 1) & 0 \\ 0 & 0 & -\delta & \lambda + \gamma \end{vmatrix} = 0. \end{aligned} \quad (3.6)$$

It is easy to obtain that the eigenvalues of system (2.2) are

$$\lambda_1 = (\mu_v + r_v)(\mathcal{R}_1 - 1), \quad \lambda_2 = (\mu_v + \alpha_v)(\mathcal{R}_2 - 1), \quad \lambda_3 = -\mu_v, \quad \lambda_4 = -\gamma.$$

Note that if  $\mathcal{R}_0 = \max\{\mathcal{R}_1, \mathcal{R}_2\} < 1$ , then, all the four eigenvalues  $\lambda_1, \lambda_2, \lambda_3, \lambda_4 < 0$  are negative real numbers. Therefore, the stability of  $\mathcal{E}_0$  depends on the eigenvalues of the following system

$$\begin{cases} \frac{dx_h}{dt} = -\mu_h x_h, \\ \frac{dy_h}{dt} = -(\mu_h + \alpha_h + r_h) y_h. \end{cases}$$

It is easy to obtain that the eigenvalues are  $\lambda_5 = -\mu_h$ ,  $\lambda_6 = -(\mu_h + \alpha_h + r_h) < 0$ . Hence, all the eigenvalues of system (2.2) are negative. Thus, the disease free equilibrium is locally asymptotically stable for  $\max\{\mathcal{R}_1, \mathcal{R}_2\} < 1$ . However, when  $\max\{\mathcal{R}_1, \mathcal{R}_2\} > 1$ , we have  $\lambda_1$  or  $\lambda_2 > 0$ . Hence, the disease-free equilibrium is unstable for  $\max\{\mathcal{R}_1, \mathcal{R}_2\} > 1$ .  $\square$

Now we turn to the local stability of the endemic equilibrium  $\mathcal{E}_i$  for a fixed  $i$ . We assume that strain  $j$  cannot invade the equilibrium of strain  $i$ , that is we assume  $\hat{\mathcal{R}}_j < 1$  for  $j \neq i$ . In this case we are able to show that the endemic equilibrium is locally stable. That is, the endemic equilibrium of strain  $i$  is locally stable if the other strain cannot invade it. The results on local stability of single-strain equilibrium  $\mathcal{E}_i$  are summarized below

**Theorem 3.4** *Assume  $\mathcal{R}_i > 1$ . If  $\hat{\mathcal{R}}_j < 1$ , then the endemic equilibrium  $\mathcal{E}_i$  is locally asymptotically stable. If  $\hat{\mathcal{R}}_j > 1$ , it is unstable.*

**Proof.** Similarly to the proof in Theorem 3.3, let us first linearize system (2.2) at  $\mathcal{E}_1$ . Let  $S_v(t) = S_{v_1}^* + x_v(t), I_{v_1}(t) = I_{v_1}^* + y_{v_1}(t), I_{v_2}(t) = y_{v_2}(t), E(t) = z(t), S_h(t) = S_{h_1}^* + x_h(t), I_h(t) = y_h(t)$ . Thus, we obtain the following linearized system

$$\left\{ \begin{array}{l} \frac{dx_v}{dt} = -\beta_{v_1} S_{v_1}^* y_{v_1} - \beta_{v_1} I_{v_1}^* x_v - \beta_{v_2} S_{v_1}^* y_{v_2} - \mu_v x_v, \\ \frac{dy_{v_1}}{dt} = \beta_{v_1} S_{v_1}^* y_{v_1} + \beta_{v_1} I_{v_1}^* x_v - (\mu_v + r_v) y_{v_1}, \\ \frac{dy_{v_2}}{dt} = \beta_{v_2} S_{v_1}^* y_{v_2} - (\mu_v + \alpha_v) y_{v_2}, \\ \frac{dz}{dt} = \delta y_{v_2} - \gamma z, \\ \frac{dx_h}{dt} = -\beta_{h_1} S_{h_1}^* \int_0^\tau f_1(s) y_{v_2}(t-s) ds - \beta_{h_2} S_{h_1}^* \int_0^\tau f_2(s) z(t-s) ds - \mu_h x_h, \\ \frac{dy_h}{dt} = \beta_{h_1} S_{h_1}^* \int_0^\tau f_1(s) y_{v_2}(t-s) ds + \beta_{h_2} S_{h_1}^* \int_0^\tau f_2(s) z(t-s) ds - (\mu_h + \alpha_h + r_h) y_h. \end{array} \right. \quad (3.7)$$

We get the following characteristic equation:

$$= \begin{vmatrix} \lambda + \beta_{v_1} I_{v_1}^* + \mu_v & \beta_{v_1} S_{v_1}^* & \beta_{v_2} S_{v_1}^* & 0 \\ -\beta_{v_1} I_{v_1}^* & \lambda & 0 & 0 \\ 0 & 0 & \lambda - \beta_{v_2} S_{v_1}^* + (\mu_v + \alpha_v) & 0 \\ 0 & 0 & -\delta & \lambda + \gamma \end{vmatrix} \quad (3.8)$$

$$= \begin{vmatrix} \lambda + \beta_{v_1} I_{v_1}^* + \mu_v & \beta_{v_1} S_{v_1}^* & \beta_{v_2} S_{v_1}^* & 0 \\ -\beta_{v_1} I_{v_1}^* & \lambda & 0 & 0 \\ 0 & 0 & \lambda - (\mu_v + \alpha_v)(\hat{\mathcal{R}}_2 - 1) & 0 \\ 0 & 0 & -\delta & \lambda + \gamma \end{vmatrix} = 0.$$

It is easy to obtain that  $\lambda_1 = (\mu_v + \alpha_v)(\hat{\mathcal{R}}_2 - 1)$ ,  $\lambda_2 = -\gamma$  are two negative real characteristic roots of system (2.2) for  $\hat{\mathcal{R}}_2 < 1$ . The other two characteristic roots are determined by the following equation:

$$\lambda^2 + (\beta_{v_1} I_{v_1}^* + \mu_v)\lambda + \beta_{v_1}^2 S_{v_1}^* I_{v_1}^* = 0. \quad (3.9)$$

Since  $\beta_{v_1} I_{v_1}^* + \mu_v > 0$ ,  $\beta_{v_1}^2 S_{v_1}^* I_{v_1}^* > 0$ , from the Routh-Hurwitz criterion, the eigenvalues  $\lambda_3, \lambda_4$  from equation (3.9) have negative real parts or are negative. Furthermore,  $\lambda_5 = -\mu_h$ ,  $\lambda_6 = -(\mu_h + \alpha_h + r_h) < 0$ . Hence all eigenvalues have negative real part. This proves that when  $\hat{\mathcal{R}}_2 < 1$ , the dominance equilibrium  $\mathcal{E}_1$  is locally asymptotically stable. Moreover, if  $\hat{\mathcal{R}}_2 > 1$ , we have  $\lambda_1 = (\mu_v + \alpha_v)(\hat{\mathcal{R}}_2 - 1) > 0$ . Then the equilibrium  $\mathcal{E}_1$  is unstable for  $\hat{\mathcal{R}}_2 > 1$ .

By a similar argument as above, we can also analyze the stability of the dominance equilibrium  $\mathcal{E}_2$ . The characteristic equation at  $\mathcal{E}_2$  is as follows:

$$= \begin{vmatrix} \lambda + \beta_{v_2} I_{v_2}^* + \mu_v & \beta_{v_1} S_{v_2}^* & \beta_{v_2} S_{v_2}^* & 0 \\ 0 & \lambda - \beta_{v_1} S_{v_2}^* + (\mu_v + r_v) & 0 & 0 \\ -\beta_{v_2} I_{v_2}^* & 0 & \lambda & 0 \\ 0 & 0 & -\delta & \lambda + \gamma \end{vmatrix} \quad (3.10)$$

$$= \begin{vmatrix} \lambda + \beta_{v_2} I_{v_2}^* + \mu_v & \beta_{v_1} S_{v_2}^* & \beta_{v_2} S_{v_2}^* & 0 \\ 0 & \lambda - (\mu_v + r_v)(\hat{\mathcal{R}}_1 - 1) & 0 & 0 \\ -\beta_{v_2} I_{v_2}^* & 0 & \lambda & 0 \\ 0 & 0 & -\delta & \lambda + \gamma \end{vmatrix} = 0.$$

It is easy to see that system (2.2) has two negative real number eigenvalues  $\lambda_1 = (\mu_v + r_v)(\hat{\mathcal{R}}_1 - 1) < 0$ ,  $\lambda_2 = -\gamma$  for  $\hat{\mathcal{R}}_1 < 1$ . The others two characteristic roots are determined by the following equation:

$$\lambda^2 + (\beta_{v_2} I_{v_2}^* + \mu_v)\lambda + \beta_{v_2}^2 S_{v_2}^* I_{v_2}^* = 0. \quad (3.11)$$

It is evident that the equation (3.11) have two eigenvalues with negative real parts. Then  $\lambda_5 = -\mu_h$ ,  $\lambda_6 = -(\mu_h + \alpha_h + r_h) < 0$ . Hence System (2.2) has six eigenvalues with negative real part. Therefore, when  $\hat{\mathcal{R}}_1 < 1$ , the dominance equilibrium  $\mathcal{E}_2$  is locally asymptotically stable. Otherwise, the equilibrium  $\mathcal{E}_2$  is unstable for  $\hat{\mathcal{R}}_1 > 1$ .  $\square$

## 4. Global stability of the disease-free equilibrium

In the previous section we established the local stability of the equilibria, that is, given the conditions on the parameters, if the initial conditions are close enough to the equilibrium, the solution will converge to that equilibrium. In this section our objective is to extend these results to global results. That is, given the conditions on the parameters, convergence to the equilibrium occurs independent of the initial conditions.

As a first step, we establish the global stability of the disease-free equilibrium.

**Theorem 4.1** *If  $\mathcal{R}_0 = \max\{\mathcal{R}_1, \mathcal{R}_2\} < 1$ , the disease-free equilibrium  $\mathcal{E}_0$  is globally asymptotically stable.*

**Proof.** For the global stability analysis of the disease-free equilibrium  $\mathcal{E}_0$ , we will use a Lyapunov function with Lasalle Invariance Principle. Let us consider the function  $V_0 = I_{v_1} + I_{v_2}$ . Note that the derivative of it along the solutions of the system (2.2) is

$$\begin{aligned} \frac{dV_0}{dt} &= [\beta_{v_1} S_v(t) - (\mu_v + r_v)] I_{v_1}(t) + [\beta_{v_2} S_v(t) - (\mu_v + \alpha_v)] I_{v_2}(t) \\ &\leq [\beta_{v_1} \frac{\Lambda_v}{\mu_v} - (\mu_v + r_v)] I_{v_1}(t) + [\beta_{v_2} \frac{\Lambda_v}{\mu_v} - (\mu_v + \alpha_v)] I_{v_2}(t) \\ &= (\mu_v + r_v)(\mathcal{R}_1 - 1) I_{v_1}(t) + (\mu_v + \alpha_v)(\mathcal{R}_2 - 1) I_{v_2}(t) \leq 0, \end{aligned}$$

since  $\mathcal{R}_0 = \max\{\mathcal{R}_1, \mathcal{R}_2\} < 1$ . Hence, by Lasalle Invariance Principle, for any solution  $(S_v, I_{v_1}, I_{v_2}, E, S_h, I_h) \in \Omega$ , the omega limit set of this solution is a subset of the largest invariant set in  $\{x \in \Omega | V'(x) = 0\}$ . It is easy to see that the largest invariant set in  $\{x \in \Omega | V'(x) = 0\}$  is the singleton set of  $\mathcal{E}_0$ . Then any solution in  $\Omega$  converges to the DFE when  $\max\{\mathcal{R}_1, \mathcal{R}_2\} < 1$ .  $\square$

## 5. Global stability of the single-strain equilibrium $\mathcal{E}_i$

From Theorem 3.4 we know that under the specified conditions the equilibrium  $\mathcal{E}_i$  is locally asymptotically stable. It remains to be established that  $\mathcal{E}_i$  is globally stable. We expect to show this result using a Lyapunov function. With  $g(x) = x - 1 - \ln x$ , we define the following Lyapunov functions.

$$V_1(t) = S_{v_1}^* g\left(\frac{S_v}{S_{v_1}^*}\right) + I_{v_1}^* g\left(\frac{I_{v_1}}{I_{v_1}^*}\right) + I_{v_2}, \quad V_2(t) = S_{v_2}^* g\left(\frac{S_v}{S_{v_2}^*}\right) + I_{v_2}^* g\left(\frac{I_{v_2}}{I_{v_2}^*}\right) + I_{v_1}.$$

We have to establish that  $V'(t) \leq 0$  along the solution curves of system (2.2). Before proof, let us make some preparations first. We denote

$$\varphi_{I_v}(t) = \beta_{h_1} \int_0^\tau f_1(s) I_{v_2}(t-s) ds, \quad \varphi_E(t) = \beta_{h_2} \int_0^\tau f_2(s) E(t-s) ds.$$

From the third equation, we have

$$E(t) = E_0 e^{-\gamma t} + \delta \int_0^t I_{v_2}(\sigma) e^{-\gamma(t-\sigma)} d\sigma. \quad (5.1)$$

Similarly, from the fourth and the fifth equation, we obtain

$$S_h(t) = S_{h_0} e^{-\int_0^t (\varphi_{I_v}(\sigma) + \varphi_E(\sigma) + \mu_h) d\sigma} + \Lambda_h \int_0^t e^{-\int_\sigma^t (\varphi_{I_v}(b) + \varphi_E(b) + \mu_h) db} d\sigma, \quad (5.2)$$

and

$$I_h(t) = I_{h_0} e^{-(\mu_h + \alpha_h + r_h)t} + \int_0^t S_h(\sigma) (\varphi_{I_v}(\sigma) + \varphi_E(\sigma)) e^{-(\mu_h + \alpha_h + r_h)(t-\sigma)} d\sigma. \quad (5.3)$$

The following Theorem summarizes the result.

**Theorem 5.1** *Assume  $\hat{\mathcal{R}}_2 < 1$ . Then equilibrium  $\mathcal{E}_1$  is globally asymptotically stable, that is, for any initial condition  $x^0 \in X$ , the solution of system (2.2) converges to  $\mathcal{E}_1$ .*

**Proof.** Calculating the derivative of the expressions of  $V_1(t)$  along the system (2.2), we obtain

$$\begin{aligned} \frac{dV_1(t)}{dt} &= S_{v_1}^* \left(1 - \frac{S_{v_1}^*}{S_v}\right) \frac{1}{S_{v_1}^*} [\Lambda_v - \beta_{v_1} S_v I_{v_1} - \beta_{v_2} S_v I_{v_2} - \mu_v S_v] \\ &\quad + I_{v_1}^* \left(1 - \frac{I_{v_1}^*}{I_{v_1}}\right) \frac{1}{I_{v_1}^*} [\beta_{v_1} S_v I_{v_1} - (\mu_v + r_v) I_{v_1}] + [\beta_{v_2} S_v I_{v_2} - (\mu_v + \alpha_v) I_{v_2}] \\ &= \left(1 - \frac{S_{v_1}^*}{S_v}\right) [\beta_{v_1} S_{v_1}^* I_{v_1}^* + \mu_v S_{v_1}^* - \beta_{v_1} S_v I_{v_1} - \beta_{v_2} S_v I_{v_2} - \mu_v S_v] \\ &\quad + \left(1 - \frac{I_{v_1}^*}{I_{v_1}}\right) [\beta_{v_1} S_v I_{v_1} - \beta_{v_1} S_{v_1}^* I_{v_1}] + [\beta_{v_2} S_v I_{v_2} - (\mu_v + \alpha_v) I_{v_2}] \\ &= -\frac{\mu_v (S_v - S_{v_1}^*)^2}{S_v} + \beta_{v_1} S_{v_1}^* I_{v_1}^* \left(1 - \frac{S_{v_1}^*}{S_v}\right) \left(1 - \frac{S_v I_{v_1}}{S_{v_1}^* I_{v_1}^*}\right) - \left(1 - \frac{S_{v_1}^*}{S_v}\right) \beta_{v_2} S_v I_{v_2} \\ &\quad + \beta_{v_1} S_{v_1}^* I_{v_1}^* \left(1 - \frac{I_{v_1}^*}{I_{v_1}}\right) \left(\frac{S_v I_{v_1}}{S_{v_1}^* I_{v_1}^*} - \frac{I_{v_1}}{I_{v_1}^*}\right) + [\beta_{v_2} S_v I_{v_2} - (\mu_v + \alpha_v) I_{v_2}] \\ &= -\frac{\mu_v (S_v - S_{v_1}^*)^2}{S_v} + \beta_{v_1} S_{v_1}^* I_{v_1}^* \left(1 - \frac{S_{v_1}^*}{S_v} - \frac{S_v I_{v_1}}{S_{v_1}^* I_{v_1}^*} + \frac{I_{v_1}}{I_{v_1}^*}\right) - (\beta_{v_2} S_v I_{v_2} - \beta_{v_2} S_{v_1}^* I_{v_2}) \\ &\quad + \beta_{v_1} S_{v_1}^* I_{v_1}^* \left(1 + \frac{S_v I_{v_1}}{S_{v_1}^* I_{v_1}^*} - \frac{I_{v_1}}{I_{v_1}^*} - \frac{S_v}{S_{v_1}^*}\right) + [\beta_{v_2} S_v I_{v_2} - (\mu_v + \alpha_v) I_{v_2}] \\ &= -\frac{\mu_v (S_v - S_{v_1}^*)^2}{S_v} + \beta_{v_1} S_{v_1}^* I_{v_1}^* \left(2 - \frac{S_{v_1}^*}{S_v} - \frac{S_v}{S_{v_1}^*}\right) + [\beta_{v_2} S_{v_1}^* - (\mu_v + \alpha_v)] I_{v_2} \end{aligned}$$

$$= -\frac{\mu_v(S_v - S_{v_1}^*)^2}{S_v} - \beta_{v_1} S_{v_1}^* I_{v_1}^* \left[ g\left(\frac{S_{v_1}^*}{S_v}\right) + g\left(\frac{S_v}{S_{v_1}^*}\right) \right] + (\mu_v + \alpha_v)(\hat{\mathcal{R}}_2 - 1)I_{v_2}. \quad (5.4)$$

Since  $g(x) \geq 0$  for  $x > 0$ ,  $\hat{\mathcal{R}}_2 < 1$ , we have  $dV_1(t)/dt \leq 0$ . Define:

$$\bar{\Omega} = \left\{ (S_v, I_{v_1}, I_{v_2}, E, S_h, I_h) \in \Omega \mid \frac{dV_1(t)}{dt} = 0 \right\}$$

We want to show that the largest invariant set in  $\bar{\Omega}$  is the singleton  $\mathcal{E}_1$ . In fact, from equation (5.4),  $dV_1(t)/dt = 0$ , and using the fact that  $1 - x + \ln x \leq 0$  for all  $x > 0$  with equality holding if  $x = 1$ , we have

$$S_v(t) = S_{v_1}^*, \quad I_{v_2}(t) = 0. \quad (5.5)$$

Using Equation (5.5), we obtain

$$0 = \frac{dS_v(t)}{dt} = \Lambda_v - \beta_{v_1} S_{v_1}^* I_{v_1}(t) - \mu_v S_{v_1}^*.$$

Rearranging gives

$$I_{v_1}(t) = \frac{\Lambda_v - \mu_v S_{v_1}^*}{\beta_{v_1} S_{v_1}^*}.$$

Using the fact that the equilibrium  $\varepsilon_1$  satisfies the relation

$$\Lambda_v - \beta_{v_1} S_{v_1}^* I_{v_1}^* - \mu_v S_{v_1}^* = 0.$$

we easily obtain

$$I_{v_1}(t) = I_{v_1}^*, \text{ for } t \geq 0.$$

From the equation (5.1), we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} E(t) &= \limsup_{t \rightarrow \infty} \left( E_0 e^{-\gamma t} + \delta \int_0^t I_{v_2}(\sigma) e^{-\gamma(t-\sigma)} d\sigma \right) \\ &= \delta \limsup_{t \rightarrow \infty} I_{v_2}(t) \limsup_{t \rightarrow \infty} \int_0^t e^{-\gamma(t-\sigma)} d\sigma = 0. \end{aligned}$$

Thus we have

$$\limsup_{t \rightarrow \infty} \varphi_{I_v}(t) = \limsup_{t \rightarrow \infty} \beta_{h_1} \int_0^\tau f_1(s) I_{v_2}(t-s) ds = \beta_{h_1} a_1 \limsup_{t \rightarrow \infty} I_{v_2}(t) = 0,$$

and

$$\limsup_{t \rightarrow \infty} \varphi_E(t) = \limsup_{t \rightarrow \infty} \beta_{h_2} \int_0^\tau f_2(s) E(t-s) ds = \beta_{h_2} a_2 \limsup_{t \rightarrow \infty} E(t) = 0,$$



From the equation (5.2), we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} S_h(t) &= \limsup_{t \rightarrow \infty} \left( S_{h_0} e^{-\int_0^t (\varphi_{I_v}(\sigma) + \varphi_E(\sigma) + \mu_h) d\sigma} + \Lambda_h \int_0^t e^{-\int_\sigma^t (\varphi_{I_v}(b) + \varphi_E(b) + \mu_h) db} d\sigma \right) \\ &= \Lambda_h \limsup_{t \rightarrow \infty} \int_0^t e^{-\mu_h(t-\sigma)} d\sigma = \frac{\Lambda_h}{\mu_h} = S_{h_1}^*. \end{aligned}$$

From the equation (5.3), we obtain

$$\begin{aligned} &\limsup_{t \rightarrow \infty} I_h(t) \\ &= \limsup_{t \rightarrow \infty} \left( I_{h_0} e^{-(\mu_h + \alpha_h + r_h)t} + \int_0^t S_h(\sigma) (\varphi_{I_v}(\sigma) + \varphi_E(\sigma)) e^{-(\mu_h + \alpha_h + r_h)(t-\sigma)} d\sigma \right) = 0. \end{aligned}$$

Hence, the largest invariant set in  $\bar{\Omega}$  is the singleton  $\mathcal{E}_1$ . By the LaSalle Invariance Principle and Theorem 3.4, we see that the equilibrium  $\mathcal{E}_1$  is globally asymptotically stable.  $\square$

Using the same proof as in Theorem 5.1, we have the following Theorem.

**Theorem 5.2** *Assume  $\hat{\mathcal{R}}_1 < 1$ . Then, equilibrium  $\mathcal{E}_2$  is globally asymptotically stable*

**Proof.** Calculating the derivative of the expressions of  $V_2(t)$  along the system (2.2), we obtain

$$\begin{aligned} \frac{dV_2(t)}{dt} &= S_{v_2}^* \left(1 - \frac{S_{v_2}^*}{S_v}\right) \frac{1}{S_{v_2}^*} [\Lambda_v - \beta_{v_1} S_v I_{v_1} - \beta_{v_2} S_v I_{v_2} - \mu_v S_v] \\ &\quad + I_{v_2}^* \left(1 - \frac{I_{v_2}^*}{I_v}\right) \frac{1}{I_{v_2}^*} [\beta_{v_2} S_v I_{v_2} - (\mu_v + \alpha_v) I_{v_2}] + [\beta_{v_1} S_v I_{v_1} - (\mu_v + r_v) I_{v_1}] \\ &= \left(1 - \frac{S_{v_2}^*}{S_v}\right) [\beta_{v_2} S_{v_2}^* I_{v_2}^* + \mu_v S_{v_2}^* - \beta_{v_1} S_v I_{v_1} - \beta_{v_2} S_v I_{v_2} - \mu_v S_v] \\ &\quad + \left(1 - \frac{I_{v_2}^*}{I_v}\right) [\beta_{v_2} S_v I_{v_2} - \beta_{v_2} S_{v_2}^* I_{v_2}^*] + [\beta_{v_1} S_v I_{v_1} - (\mu_v + r_v) I_{v_1}] \\ &= -\frac{\mu_v (S_v - S_{v_2}^*)^2}{S_v} + \beta_{v_2} S_{v_2}^* I_{v_2}^* \left(1 - \frac{S_{v_2}^*}{S_v}\right) \left(1 - \frac{S_v I_{v_2}}{S_{v_2}^* I_{v_2}^*}\right) - \left(1 - \frac{S_{v_2}^*}{S_v}\right) \beta_{v_1} S_v I_{v_1} \\ &\quad + \beta_{v_2} S_{v_2}^* I_{v_2}^* \left(1 - \frac{I_{v_2}^*}{I_v}\right) \left(\frac{S_v I_{v_2}}{S_{v_2}^* I_{v_2}^*} - \frac{I_{v_2}}{I_{v_2}^*}\right) + [\beta_{v_1} S_v I_{v_1} - (\mu_v + r_v) I_{v_1}] \\ &= -\frac{\mu_v (S_v - S_{v_2}^*)^2}{S_v} + \beta_{v_2} S_{v_2}^* I_{v_2}^* \left(1 - \frac{S_{v_2}^*}{S_v} - \frac{S_v I_{v_2}}{S_{v_2}^* I_{v_2}^*} + \frac{I_{v_2}}{I_{v_2}^*}\right) - (\beta_{v_1} S_v I_{v_1} - \beta_{v_1} S_{v_2}^* I_{v_1}) \\ &\quad + \beta_{v_2} S_{v_2}^* I_{v_2}^* \left(1 + \frac{S_v I_{v_2}}{S_{v_2}^* I_{v_2}^*} - \frac{I_{v_2}}{I_{v_2}^*} - \frac{S_v}{S_{v_2}^*}\right) + [\beta_{v_1} S_v I_{v_1} - (\mu_v + r_v) I_{v_1}] \\ &= -\frac{\mu_v (S_v - S_{v_2}^*)^2}{S_v} + \beta_{v_2} S_{v_2}^* I_{v_2}^* \left(2 - \frac{S_{v_2}^*}{S_v} - \frac{S_v}{S_{v_2}^*}\right) + [\beta_{v_1} S_{v_2}^* - (\mu_v + r_v)] I_{v_1} \\ &= -\frac{\mu_v (S_v - S_{v_2}^*)^2}{S_v} - \beta_{v_2} S_{v_2}^* I_{v_2}^* \left[g\left(\frac{S_{v_2}^*}{S_v}\right) + g\left(\frac{S_v}{S_{v_2}^*}\right)\right] + (\mu_v + r_v) (\hat{\mathcal{R}}_1 - 1) I_{v_1}. \end{aligned} \tag{5.6}$$

Since  $g(x) \geq 0$  for  $x > 0$ ,  $\hat{\mathcal{R}}_1 < 1$ , we have  $dV_2(t)/dt \leq 0$ .

$$\bar{\Omega} = \left\{ (S_v, I_{v_1}, I_{v_2}, E, S_h, I_h) \in \Omega \mid \frac{dV_2(t)}{dt} = 0 \right\}.$$

We want to show that the largest invariant set in  $\bar{\Omega}$  is the singleton  $\mathcal{E}_2$ . In fact, from equation (5.6),  $dV_2(t)/dt = 0$ , and using the fact that  $1 - x + \ln x \leq 0$  for all  $x > 0$  with equality holding if  $x = 1$ , we have

$$S_v(t) = S_{v_2}^*, \quad I_{v_1}(t) = 0. \quad (5.7)$$

Using equation (5.7), we obtain

$$0 = \frac{dS_v(t)}{dt} = \Lambda_v - \beta_{v_2} S_{v_2}^* I_{v_2}(t) - \mu_v S_{v_2}^*$$

Rearranging gives

$$I_{v_2}(t) = \frac{\Lambda_v - \mu_v S_{v_2}^*}{\beta_{v_2} S_{v_2}^*}$$

Using the fact that the equilibrium  $\mathcal{E}_2$  satisfies the relation

$$\Lambda_v - \beta_{v_2} S_{v_2}^* I_{v_2}^* - \mu_v S_{v_2}^* = 0.$$

we easily obtain

$$I_{v_2}(t) = I_{v_2}^*, \text{ for } t \geq 0.$$

From the equation (5.1), we obtain

$$\begin{aligned} \limsup_{t \rightarrow \infty} E(t) &= \limsup_{t \rightarrow \infty} \left( E_0 e^{-\gamma t} + \delta \int_0^t I_{v_2}(\sigma) e^{-\gamma(t-\sigma)} d\sigma \right) \\ &= \delta I_{v_2}^* \limsup_{t \rightarrow \infty} \int_0^t e^{-\gamma(t-\sigma)} d\sigma = \frac{\delta I_{v_2}^*}{\gamma} = E^*. \end{aligned}$$

Thus we have

$$\limsup_{t \rightarrow \infty} \varphi_{I_v}(t) = \limsup_{t \rightarrow \infty} \beta_{h_1} \int_0^\tau f_1(s) I_{v_2}(t-s) ds = \beta_{h_1} a_1 I_{v_2}^*,$$

and

$$\limsup_{t \rightarrow \infty} \varphi_E(t) = \limsup_{t \rightarrow \infty} \beta_{h_2} \int_0^\tau f_2(s) E(t-s) ds = \beta_{h_2} a_2 E^*,$$

From the equation (5.2), we obtain

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} S_h(t) \\
&= \limsup_{t \rightarrow \infty} \left( S_{h_0} e^{-\int_0^t (\varphi_{I_v}(\sigma) + \varphi_E(\sigma) + \mu_h) d\sigma} + \Lambda_h \int_0^t e^{-\int_\sigma^t (\varphi_{I_v}(b) + \varphi_E(b) + \mu_h) db} d\sigma \right) \\
&= \Lambda_h \limsup_{t \rightarrow \infty} \int_0^t e^{-(\beta_{h_1} a_1 I_{v_2}^* + \beta_{h_2} a_2 E^* + \mu_h)(t-\sigma)} d\sigma = \frac{\Lambda_h}{\beta_{h_1} a_1 I_{v_2}^* + \beta_{h_2} a_2 E^* + \mu_h} = S_{h_2}^*.
\end{aligned}$$

From the equation (5.3), we obtain

$$\begin{aligned}
& \limsup_{t \rightarrow \infty} I_h(t) \\
&= \limsup_{t \rightarrow \infty} \left( I_{h_0} e^{-(\mu_h + \alpha_h + r_h)t} + \int_0^t S_h(\sigma) (\varphi_{I_v}(\sigma) + \varphi_E(\sigma)) e^{-(\mu_h + \alpha_h + r_h)(t-\sigma)} d\sigma \right) \\
&= S_{h_2}^* (\beta_{h_1} a_1 I_{v_2}^* + \beta_{h_2} a_2 E^*) \limsup_{t \rightarrow \infty} \int_0^t e^{-(\mu_h + \alpha_h + r_h)(t-\sigma)} d\sigma \\
&= \frac{(\beta_{h_1} a_1 I_{v_2}^* + \beta_{h_2} a_2 E^*)}{\mu_h + \alpha_h + r_h} S_{h_2}^* = I_h^*.
\end{aligned}$$

Hence, the largest invariant set in  $\bar{\Omega}$  is the singleton  $\mathcal{E}_2$ . By the LaSalle Invariance Principle and Theorem 3.4, we see that the equilibrium  $\mathcal{E}_2$  is globally asymptotically stable.  $\square$

## 6. Discussion

In this paper, we introduce a two-strain avian influenza model with distributed delay and environmental transmission between poultry and humans. We define the basic reproduction number  $\mathcal{R}_0$  of the disease as the maximum of the reproduction numbers of each strain. We show that if  $\mathcal{R}_0 < 1$  the disease-free equilibrium  $\mathcal{E}_0$  is locally and globally stable, that is the number of infected with each strain goes to zero. Furthermore, we show that if  $\mathcal{R}_0 > 1$ , then the disease persists. Moreover, the single-strain equilibrium  $\mathcal{E}_1$  (or  $\mathcal{E}_2$ ) is locally asymptotically stable if the invasion numbers  $\hat{\mathcal{R}}_2$  (or  $\hat{\mathcal{R}}_1$ ) is smaller than one. Furthermore, we show that the single-strain equilibrium is globally stable, that is the strain 1 persists in poultry (or the strain 2 persists in poultry, the environment and humans). The existence and lack of uniqueness of the coexistence equilibrium is verified analytically when the invasion numbers  $\hat{\mathcal{R}}_1 = \hat{\mathcal{R}}_2 = 1$  and the reproduction numbers of each strain  $\mathcal{R}_1 = \mathcal{R}_2 > 1$ .

From the perspective of public health, controlling avian influenza A (H7N9) virus may be performed by monitoring the reproduction number of strain 2  $\mathcal{R}_2$  and the invasion number

$\hat{\mathcal{R}}_1$ . If  $\mathcal{R}_2 < 1$ , then the single-strain equilibrium  $\mathcal{E}_2$  does not exist, and humans cannot be infected by HPAI strain. Reducing  $\mathcal{R}_2$  may be done by reducing the transmission rate  $\beta_{v_2}$  through vaccination or increasing the HPAI-generated disease-induced death rate  $\alpha_v$  through selective culling of infected poultry. Mass culling which decreases the poultry lifespan  $1/\mu_v$  is also an effective way to decrease the reproduction number, as long as mass culling is not only performed in response to an outbreak but is also performed as preventive measure. If  $\mathcal{R}_2 > 1, \hat{\mathcal{R}}_1 > 1$ , then the presence of LPAI in poultry will lead to elimination of HPAI in poultry. Thus, maintaining high levels of LPAI in poultry is a possible, although not very advisable, strategy to reduce HPAI. If  $\mathcal{R}_2 > 1, \hat{\mathcal{R}}_1 < 1$ , the single-strain equilibrium not only exists but also is locally asymptotically and global stable. Humans can be infected with strain 2. If  $\mathcal{R}_1 = \mathcal{R}_2 > 1, \hat{\mathcal{R}}_2 = \hat{\mathcal{R}}_2 = 1$ , then many coexistence equilibria exist.

### Conflict of Interests

The authors declare that there is no conflict of interests.

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