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A TWO SPECIES AMENSALISM MODEL WITH NON-MONOTONIC FUNCTIONAL RESPONSE

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Abstract. A two species amensalism model with non-monotonic functional response takes the form

$$\begin{aligned}\frac{dx_1}{dt} &= x_1\Big(a_1 - b_1x_1 - \frac{c_1x_2}{d_1+x_2^2}\Big),\\ \frac{dx_2}{dt} &= x_2(a_2 - b_2x_2),\end{aligned}$$

is proposed and studied, where $a_i, b_i, i = 1,2$ c_1 and d_1 are all positive constants. If $a_1b_2^2d_1 + a_1a_2^2 - a_2b_2c_1 > 0$, then the system admits a unique globally stable positive equilibrium, which means that two species could coexistent in a stable state, and if $a_1b_2^2d_1 + a_1a_2^2 - a_2b_2c_1 < 0$, then the first species will be driven to extinction, and second species will be convergence to $x_2^* = \frac{r_2}{a_{22}}$.

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1. Introduction

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The aim of this paper is to investigate the dynamic behaviors of the following two species commensal symbiosis model with non-monotonic functional response

$$\begin{aligned} \frac{dx_1}{dt} &= x_1\Big(a_1 - b_1x_1 - \frac{c_1x_2}{d_1 + x_2^2}\Big), \\ \frac{dx_2}{dt} &= x_2(a_2 - b_2x_2), \end{aligned} \tag{1.1}$$

where $a_i, b_i, i = 1,2$, c_1 and d_1 are all positive constants. Here we make the following assumption:

(1) Two species obey the Logistic type growing;

(2) The commensal of the second species to the first one obey the non-monotonic functional response, i.e., $\frac{x_2}{1+x_2^2}$, which is a humped function and declines with the high densities of the second species after the hump.

During the past decade, many scholars investigated the dynamic behaviors of the amensalism model or commensalism model, some essential progress has been made on this direction(see [1]-[22]).

Sun[1] first time proposed the following two species amensalism model model:

$$\begin{aligned} \frac{dx}{dt} &= r_1x\Big(\frac{k_1 - x - \alpha y}{k_1}\Big), \\ \frac{dy}{dt} &= r_2y\Big(\frac{k_2 - y}{k_2}\Big). \end{aligned} \tag{1.2}$$

He investigated the local stability of all equilibria of the system.

Zhu and Chen [2] studied the qualitative property of the following two species amensalism model model:

$$\begin{aligned} \frac{dx}{dt} &= x\Big(r_1 - a_{11}x - a_{12}y\Big), \\ \frac{dy}{dt} &= y\Big(r_2 - a_{22}y\Big). \end{aligned} \tag{1.3}$$

They showed by vector field that system (1.3) may exist a positive equilibrium and it is globally stable, or the system has no positive equilibrium and one of the boundary equilibria is globally stable.

Han et al[5] argued that the non-autonomous case is more suitable, and they proposed the following two species amensalism model:

$$\begin{aligned} \frac{dx}{dt} &= x\Big(a_1(t) - b_1(t)x - c_1(t)y\Big), \\ \frac{dy}{dt} &= y\Big(a_2(t) - b_2(t)y\Big). \end{aligned} \tag{1.4}$$

By using the coincidence degree theory, a set of easily verified sufficient conditions which guarantee the global existence of positive periodic solutions of the system (1.4) was obtained.

It bring to our attention that all of the above works are based on the traditional Lotka-Volterra model, which suppose that the influence of the second species to the first one is linearize. This may not be suitable since the amensalism between two species become infinity as the density of the species become infinity. Generally speaking, there is a saturating state between two species, now, if we describe this saturation phenomenon by using the non-monotonic type functional response([23, 24]), under this assumption, then one could formulate the system (1.1).

The aim of this paper is to investigate the local and global stability property of the possible equilibria of system (1.1). We arrange the paper as follows: In the next section, we will investigate the existence and local stability property of the equilibria of system (1.1). In Section 3, we will investigate the global stability property of the positive equilibrium; In Section 4, an example together with its numeric simulation is presented to show the feasibility of our main results.

2. Local stability of the equilibria

The equilibria of system (1.1) is determined by the system

$$x_1\left(a_1 - b_1 x_1 - \frac{c_1 x_2}{d_1 + x_2^2}\right) = 0,$$

$$x_2(a_2 - b_2 x_2) = 0.$$

Hence, system (1.1) admits four possible equilibria, $E_0(0,0)$, $E_1\big(\frac{a_1}{b_1},0\big)$, $E_2\big(0,\frac{a_2}{b_2}\big)$ and $E_3\big(x^*,y^*\big)$, where

$$x^* = \frac{a_1 b_2^2 d_1 + a_1 a_2^2 - a_2 b_2 c_1}{b_1\big(b_2^2 d_1 + a_2^2\big)},\ y^* = \frac{a_2}{b_2}.$$

Obviously, E_3 is positive equilibrium if and only if $a_1 b_2^2 d_1 + a_1 a_2^2 - a_2 b_2 c_1 > 0$.

Concerned with the local stability property of the above four equilibria, we have

Theorem 2.1. *$E_0(0,0)$ and $E_1(\frac{a_1}{b_1},0)$ are unstable; $E_2\left(0,\frac{a_2}{b_2}\right)$ is unstable if $a_1b_2^2d_1+a_1a_2^2-a_2b_2c_1>0$ and locally stable if $a_1b_2^2d_1+a_1a_2^2-a_2b_2c_1<0$ holds; The positive equilibrium $E_3\left(x^*,y^*\right)$ if exist, it is locally stable.*

Proof. The Jacobian matrix of the system (1.8) is calculated as

$$J(x,y)=\begin{pmatrix} a_1-2b_1x_1-\dfrac{c_1x_2}{1+x_2^2} & -\dfrac{c_1x_1(-x_2^2+d_1)}{(x_2^2+d_1)^2} \\ 0 & -2b_2x_2+a_2 \end{pmatrix}. \tag{2.3}$$

Then the Jacobian matrix of the system (1.1) about the equilibrium $E_0(0,0)$ and $E_1(\frac{a_1}{b_1},0)$ are given by

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \tag{2.4}$$

and

$$\begin{pmatrix} -a_1 & -\dfrac{c_1a_1}{b_1d_1} \\ 0 & a_2 \end{pmatrix} \tag{2.5}$$

respectively. One could easily see that all of the above two matrix has at least one positive eigenvalues, which means that $E_0(0,0)$ and $E_1(\frac{a_1}{b_1},0)$ are all unstable.

The Jacobian matrix of the system (1.1) about the equilibrium $E_2(0,\frac{a_2}{b_2})$ are

$$\begin{pmatrix} \dfrac{a_1b_2^2d_1+a_1a_2^2-a_2b_2c_1}{b_2^2d_1+a_2^2} & 0 \\ 0 & -a_2 \end{pmatrix} \tag{2.6}$$

The corresponding eigenvalue are $\lambda_1=\dfrac{a_1b_2^2d_1+a_1a_2^2-a_2b_2c_1}{b_2^2d_1+a_2^2}$, $\lambda_2=-a_2<0$. Obviously, if $a_1b_2^2d_1+a_1a_2^2-a_2b_2c_1<0$, then $\lambda_1<0$, in this case, $E_2(0,\frac{a_2}{b_2})$ is locally stable; And $E_2(0,\frac{a_2}{b_2})$ is unstable if $a_1b_2^2d_1+a_1a_2^2-a_2b_2c_1>0$.

The Jacobian matrix about the equilibrium E_3 is given by

$$\begin{pmatrix} -\dfrac{a_1b_2^2d_1+a_1a_2^2-a_2b_2c_1}{b_2^2d_1+a_2^2} & A \\ 0 & -a_2 \end{pmatrix}, \tag{2.7}$$

where

$$A = \frac{c_1 b_2^2 (a_1 b_2^2 d_1 + a_1 a_2^2 - a_2 b_2 c_1)(-b_2^2 d_1 + a_2^2)}{b_1 (b_2^2 d_1 + a_2^2)^3}. \tag{2.8}$$

The eigenvalues of the above matrix are $\lambda_1 = -\dfrac{a_1 b_2^2 d_1 + a_1 a_2^2 - a_2 b_2 c_1}{b_2^2 d_1 + a_2^2}, \lambda_2 = -r_2$. To ensure the existence of the positive equilibrium, one should make the assumption $a_1 b_2^2 d_1 + a_1 a_2^2 - a_2 b_2 c_1 > 0$. In this case, $\lambda_1 < 0, \lambda_2 < 0$, which implies $E_3(x^*, y^*)$ is locally stable.

This ends the proof of Theorem 2.1.

3. Global stability

Previously we had showed that the boundary equilibrium E_2 and the positive equilibrium E_3 are all possible locally stable. One of the interesting issue is to find out the conditions to ensure the global stability of those two equilibria.

Lemma 3.1.[25] System

$$\frac{dy}{dt} = y(a - by)$$

has a unique globally attractive positive equilibrium $y^* = \frac{a}{b}$.

Theorem 3.1. *Assume that $a_1 b_2^2 d_1 + a_1 a_2^2 - a_2 b_2 c_1 < 0$, then $E_2\big(0, \frac{a_2}{b_2}\big)$ is globally stable.*

Proof. $a_1 b_2^2 d_1 + a_1 a_2^2 - a_2 b_2 c_1 < 0$ is equivalent to $a_1 - \dfrac{c_1 \frac{a_2}{b_2}}{d_1 + (\frac{a_2}{b_2})^2} < 0$, hence, for $\varepsilon > 0$ enough small, without loss of generality, we may assume $\varepsilon < \frac{1}{2}\frac{a_2}{b_2}$, the following inequality holds.

$$a_1 - \frac{c_1(\frac{a_2}{b_2} - \varepsilon)}{d_1 + (\frac{a_2}{b_2} + \varepsilon)^2} < -\varepsilon. \tag{3.1}$$

Noting that the second equation of (1.1) takes the form

$$\frac{dx_2}{dt} = x_2(a_2 - b_2 x_2). \tag{3.2}$$

It then follows from Lemma 3.1 that $\lim\limits_{t \to +\infty} x_2(t) = \frac{a_2}{b_2}$, hence, for above $\varepsilon > 0$, there exists a $T > 0$, such that for all $t \geq T$,

$$\frac{a_2}{b_2} - \varepsilon < x_2(t) < \frac{a_2}{b_2} + \varepsilon. \tag{3.3}$$

From the first equation of system (1.1), (3.1) and (3.3), we have

$$\begin{aligned} \frac{dx_1}{dt} &\leq x_1\Big(a_1 - b_1x_1 - \frac{c_1(\frac{a_2}{b_2}-\varepsilon)}{d_1+(\frac{a_2}{b_2}+\varepsilon)^2}\Big) \\ &< -\varepsilon x_1, \end{aligned} \tag{3.5}$$

Hence,

$$0 \leq x_1(t) < x_1(0)\exp\{-\varepsilon t\} \to 0 \text{ as } t \to +\infty. \tag{3.6}$$

That is,

$$\lim_{t\to+\infty} x_1(t) = 0.$$

This ends the proof of Theorem 3.1.

Remark 3.1. From the proof of Theorem 3.1, we only use the continuity of the function $\frac{x_2}{d_1+x_2^2}$, and did not use the fact the functional response is a humped function.

Remark 3.2. Theorem 3.1 shows that the local stability of $E_2(0,\frac{a_2}{b_2})$ is enough to ensure its global stability.

Theorem 3.2. *If the positive equilibrium $E_3\big(x^*,y^*\big)$ exist, it is globally stable.*

Proof. Firstly we proof that every solution of system (1.1) that starts in R_+^2 is uniformly bounded. It follows from the first equation of system (1.1) that

$$\frac{dx_1}{dt} \leq x_1(a_1 - b_1x_1),$$

and so

$$\limsup_{t\to+\infty} x_1(t) \leq \frac{a_1}{b_1}. \tag{3.7}$$

(3.7) together with (3.2)-(3.3) shows that there exists a $\varepsilon > 0$ such that for all $t > T$

$$x_1(t) < \frac{a_1}{b_1}+\varepsilon,\ x_2(t) < \frac{a_2}{b_2}+\varepsilon. \tag{3.8}$$

Let $D = \{(x,y) \in R_+^2 : x_1 < \frac{a_1}{b_1}+\varepsilon,\ x_2 < \frac{a_2}{b_2}+\varepsilon\}$. Then every solution of system (1.1) starts in R_+^2 is uniformly bounded on D. Also, from Theorem 2.1 there is a unique local stable positive equilibrium $E_3(x^*,y^*)$. To show that $E_3(x^*,y^*)$ is globally stable, it's enough to show that the system admits no limit cycle in the area D. To this end, let's consider the Dulac function $u(x,y) = x_1^{-1}x_2^{-1}$, then

$$\frac{\partial(uP)}{\partial x_1} + \frac{\partial(uQ)}{\partial x_2} = -b_1x_2^{-1} - b_2x_1^{-1} < 0,$$

where $P(x_1,x_2) = x_1\Big(a_1 - b_1x_1 - \dfrac{c_1x_2}{d_1+x_2^2}\Big), Q(x_1,x_2) = x_2(a_2 - b_2x_2)$. By Dulac Theorem[26], there is no closed orbit in area D. Consequently, $E_3(x^*,y^*)$ is globally asymptotically stable. This completes the proof of Theorem 3.2.

4. Numeric simulation

Now let us consider the following example.

Example 4.1. Consider the following system

$$\begin{aligned}\frac{dx_1}{dt} &= x_1\Big(1 - 2x_1 - \frac{5x_2}{1+x_2^2}\Big),\\ \frac{dx_2}{dt} &= x_2(1-2x_2).\end{aligned} \tag{4.1}$$

In this system, corresponding to system (1.1), we take $a_1 = 1, b_1 = 2, c_1 = 5, d_1 = 1, a_2 = 1, b_2 = 2$. By simple computation, $a_1b_2^2d_1 + a_1a_2^2 - a_2b_2c_1 = -5 < 0$. From Theorem 3.1, the boundary equilibrium $\left(0, \frac{1}{2}\right)$ is globally asymptotically stable. Numeric simulation (Fig.1) also support this assertion.

Example 4.2. Consider the following system

$$\begin{aligned}\frac{dx_1}{dt} &= x_1\Big(1 - x_1 - \frac{x_2}{3+x_2^2}\Big),\\ \frac{dx_2}{dt} &= x_2(1-x_2).\end{aligned} \tag{4.2}$$

In this system, corresponding to system (1.1), we take $a_1 = 1, b_1 = 1, c_1 = 1, d_1 = 3, a_2 = 1, b_2 = 1$. By simple computation, $a_1b_2^2d_1 + a_1a_2^2 - a_2b_2c_1 = 3 > 0$. From Theorem 3.2, the positive equilibrium $\left(\frac{3}{8}, \frac{1}{2}\right)$ is globally asymptotically stable. Numeric simulation (Fig.2) also support this assertion.

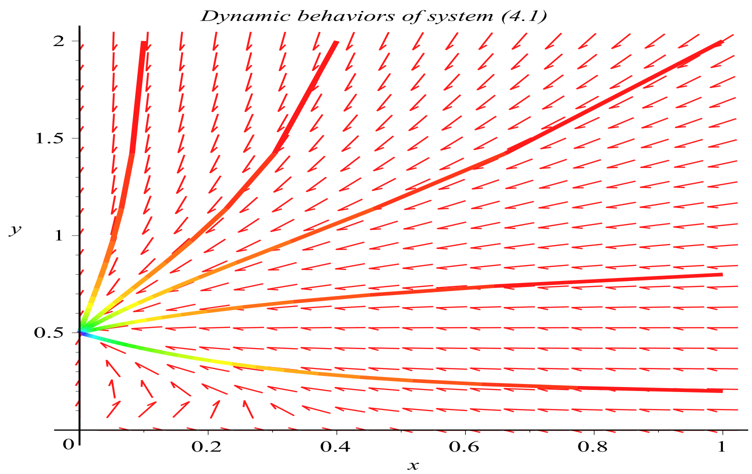

FIGURE 1. Numeric simulations of system (4.1) with the initial conditions $(x(0),y(0)) = (0.4,2),(1,0.8),(1,0.2),(1,2)$ and $(0.1,2)$, respectively.

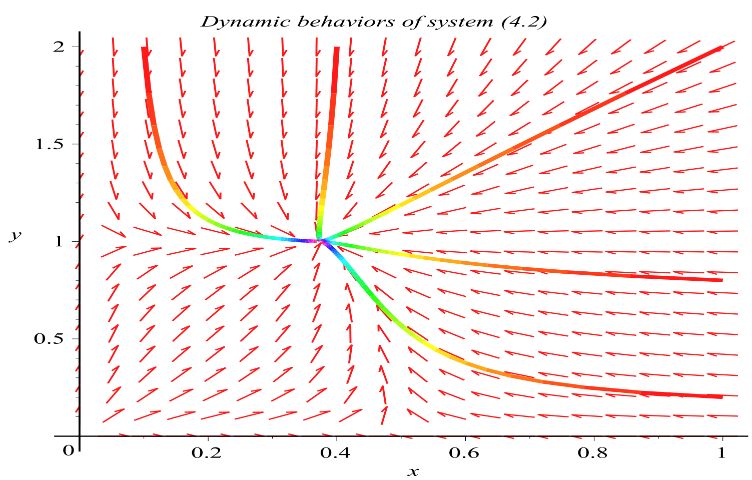

FIGURE 2. Numeric simulations of system (4.2) with the initial conditions $(x(0),y(0)) = (0.4,2),(1,0.8),(1,0.2),(1,2)$ and $(0.1,2)$, respectively.

5. Conclusion

We propose a two species amensalism model with non-monotonic functional response. Our study shows that the dynamic behaviors of the system depending on the sign of $a_1b_2^2d_1 + a_1a_2^2 - a_2b_2c_1$, if it is positive, then the system admits a unique globally stable positive equilibrium, and if it is negative, then the boundary equilibrium $E_2(0, \frac{a_2}{b_2})$ is globally stable, in this case, the first species will be driven to extinction.

Conflict of Interests

The authors declare that there is no conflict of interests.

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