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STABILITY ANALYSIS OF A LOTKA-VOLTERRA TYPE PREDATOR-PREY SYSTEM WITH ALLEE EFFECT ON THE PREDATOR SPECIES

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Abstract. In this paper, we propose a Lotka-Volterra type predator-prey system with Allee effect on the predator species. Sufficient condition which ensure the existence of a unique globally asymptotically stable positive equilibrium is obtained. Numeric simulation s show that the system subject to an Allee effect takes much longer time to reach its stable steady-state solution, however, Allee effect has no influence on the final density of the predator and prey species. This result differs from that obtained for the predator prey system with Allee effect on the prey species.

Keywords: predator; prey; Allee effect; global stability.

2000 Mathematics Subject Classification: 34C25, 92D25, 34D20, 34D40.

1. Introduction

As was pointed out by Berryman[1], the dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. In the past decades, many

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scholars investigated the dynamic behaviors of the predator-prey system (see [2]-[23] and the reference cited therein). Recently, several scholars also investigated the dynamic behaviors of the predator-prey model involving the Allee effect, which reflects the fact that the population growth rate is reduced at low population size (see [2]-[9] and the references cited therein).

H*ü*seyin Merdan [2] investigated the influence of the Allee effect on the Lotka-Volterra type predator-prey system. To do so, the author proposed the following predator-prey without and with Allee effect system

$$\frac{dx}{dt} = rx(1-x) - axy,\ \ \frac{dy}{dt} = ay(x-y). \tag{1.1}$$

$$\frac{dx}{dt} = r\alpha(x)x(1-x) - axy,\ \ \frac{dy}{dt} = ay(x-y). \tag{1.2}$$

where $\alpha(x) = \frac{x}{\beta+x}$, β is positive constant. H*ü*seyin Merdan showed that the model (1.1) has three steady-state solutions: $A(0,0), B(1,0)$ and $C\big(x^*, y^*\big) = \big(\frac{r}{a+r}, \frac{r}{a+r}\big)$. The first two are locally unstable, while the third one is locally asymptotically stable. He also showed that if $r - a\beta > 0$ hold, then system (1.2) admits similar dynamic behaviors as that of (1.1). By carrying out a series of numeric simulations, the author found the following two phenomena. (1) The system subject to an Allee effect takes a longer time to reach its steady-state solution than the system (1.1); (2) The Allee effect reduces the population densities of both predator and prey at the steady-state.

It seems interesting to consider the influence of the Allee effect on the predator species, since generally speaking, the higher the hierarchy in the food chain, the more likely it is to become extinct. Thus, in this paper, we will consider the following model:

$$\frac{dx}{dt} = rx(1-x) - axy,\ \ \frac{dy}{dt} = a\alpha(y)y(x-y). \tag{1.3}$$

where $\alpha(y) = \frac{y}{\beta+y}$, β is positive constant, represents the Allee effect of the predator species, r, a are positive constants.

The paper is arranged as follows. In section 2 we investigate the persistent property of the system, based on this, we are able to investigate the locally stability property of the equilibrium solutions of the system (1.3). In section 3, by applying the Dulac criterion, we are able to show that under some assumption, the positive equilibrium is globally asymptotically stable. Section 4 presents some numerical simulations concerning the stability of our model.

2. Persistence and local stability of the equilibria

We need several Lemmas to prove the persistent property of the system.

Lemma 2.1. [24] *Consider the following equation*

$$\dot{N} = NF(N). \tag{2.1}$$

Assumed that function $F(N)$ satisfies the following conditions.

(1)There is a N^*, such that $F(N^*) = 0$.

(2)For all $N^* > N > 0, F(N) > 0$.

(3)For all $N > N^* > 0, F(N) < 0$.

Then N^* is global stability.

Lemma 2.2. *Consider the following equation*

$$\frac{dy}{dt} = a\alpha(y)y(b-y). \tag{2.2}$$

the unique positive equilibrium $y^* = b$ is global stability.

Proof. Set

$$F(y) = a\alpha(y)(b-y).$$

Then

(1)There is a $y^* = b$, such that $F(y^*) = 0$;

(2)For all $y^* > y > 0, F(y) > 0$;

(3)For all $y > y^* > 0, F(y) < 0$,

where $\alpha(y) = \frac{y}{\beta+y}$, β is positive constant. Therefore, all the conditions of Lemma 2.1 is satisfied, and it follows from Lemma 2.1 that y^* is global stability. This ends the proof of Lemma 2.2.

As a direct corollary of Lemma 2.2 of Chen[25], we have

Lemma 2.3. *If $a > 0, b > 0$ and $\dot{x} \geq x(b-ax)$, when $t \geq 0$ and $x(0) > 0$, we have*

$$\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}.$$

If $a>0, b>0$ and $\dot{x} \le x(b-ax)$, when $t \ge 0$ and $x(0)>0$, we have

$$\limsup_{t\to+\infty} x(t) \le \frac{b}{a}.$$

Theorem 2.1. *Assume that*

$$r > a \tag{2.3}$$

holds, system (1.3) is permanent.

Proof. It follows from (2.3) that there exists a $\varepsilon>0$ enough small such that

$$r > a(1+\varepsilon). \tag{2.4}$$

Let $(x(t), y(t))$ be any positive solution of system (1.3). From system (1.3) it follows that

$$\frac{dx}{dt} \le rx(1-x). \tag{2.5}$$

Thus, as a direct corollary of Lemma 2.3, one has

$$\limsup_{t\to+\infty} x(t) \le 1, \tag{2.6}$$

Hence, for enough small $\varepsilon>0$, which satisfies (2.4), it follows from (2.6) that there exists a $T_1>0$ such that

$$x(t) < 1+\tfrac{\varepsilon}{2}, \tag{2.7}$$

For $t>T_1$, it follows from the second equation of system (1.3) that

$$\frac{dy}{dt} \le a\alpha(y)y\big(1+\varepsilon-y\big). \tag{2.8}$$

Consider the equation

$$\frac{du}{dt} = a\alpha(u)u\big(1+\frac{\varepsilon}{2}-u\big). \tag{2.9}$$

It follows from Lemma 2.2 that (2.9) admits a unique globally stable positive equilibrium

$$u^* = 1+\varepsilon. \tag{2.10}$$

By differential inequality theory, any solution of (2.8) satisfies

$$\limsup_{t\to+\infty} y(t) \le 1+\frac{\varepsilon}{2}. \tag{2.11}$$

Hence, there exists a $T_2 > T_1$ such that

$$y(t) < 1+\varepsilon, \tag{2.12}$$

For $t > T_2$, it follows from the first equation of system (1.3) that

$$\begin{aligned} \frac{dx}{dt} &\geq rx(1-x) - a(1+\varepsilon)x \\ &= x\Big(r - a(1+\varepsilon) - rx\Big). \end{aligned} \tag{2.13}$$

Applying Lemma 2.3 to (2.13) leads to

$$\liminf_{t\to+\infty} x(t) \geq \frac{r-a(1+\varepsilon)}{r}. \tag{2.14}$$

Setting $\varepsilon \to 0$ leads to

$$\liminf_{t\to+\infty} x(t) \geq \frac{r-a}{r}. \tag{2.15}$$

It follows from (2.14) that there exists an enough large $T_3 > T_2$ such that

$$x(t) > \frac{r-a(1+\varepsilon)}{r} - \varepsilon \ \text{ for all } \ t \geq T_3.$$

and so, from the second equation of system (1.3), we have

$$\frac{dy}{dt} \geq a\alpha(y)y\Big(\frac{r-a(1+\varepsilon)}{r} - \varepsilon - y\Big). \tag{2.16}$$

Consider the equation

$$\frac{du}{dt} = a\alpha(u)u\Big(\frac{r-a(1+\varepsilon)}{r} - \varepsilon - u\Big). \tag{2.17}$$

It follows from Lemma 2.2 that (2.17) admits a unique globally stable positive equilibrium

$$u^* = \frac{r-a(1+\varepsilon)}{r} - \varepsilon. \tag{2.18}$$

By differential inequality theory, any solution of (2.16) satisfies

$$\liminf_{t\to+\infty} y(t) \geq \frac{r-a(1+\varepsilon)}{r} - \varepsilon. \tag{2.19}$$

Setting $\varepsilon \to 0$ leads to

$$\liminf_{t\to+\infty} y(t) \geq \frac{r-a}{r}. \tag{2.20}$$

(2.6), (2.11), (2.15) and (2.20) shows that the system (1.3) is permanent. This ends the proof of Theorem 2.1.

Now we are in the position of investigate the stability property of steady-state solutions of the model (1.3). Defining

$$f(x,y) := x(1-x) - axy, \;\; g(x,y) := a\alpha(y)y(x-y).$$

The steady-state solutions of (1.3) are obtained by solving the equations $f(x,y) = 0$ and $g(x,y) = 0$. The model has three steady-state solutions: $A(0,0), B(1,1)$ and $C(x^*,y^*)$.

Theorem 2.2. *$C(x^*,y^*)$ is locally asymptotically stable. If $r > a$ hold, then $A(0,0)$ and $B(1,0)$ is locally unstable.*

Proof. The variation matrix of the continuous-time system (1.3) at an equilibrium solution (x,y) is

$$\begin{aligned} J(x,y) &= \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix} \\ &= \begin{pmatrix} r(1-x) - rx - ay & -ax \\ \dfrac{ay^2}{\beta + y} & \dfrac{ay\left(2\beta x - 3\beta y + xy - 2y^2\right)}{(\beta + y)^2} \end{pmatrix} \end{aligned}$$

Specially, at $C(x^*,y^*)$

$$J(x^*,y^*) = \begin{pmatrix} r\left(1 - \dfrac{r}{a+r}\right) - \dfrac{r^2}{a+r} - \dfrac{ra}{a+r} & -\dfrac{ar}{a+r} \\ \dfrac{ar^2}{(a+r)^2}\left(\beta + \dfrac{r}{a+r}\right)^{-1} & -\dfrac{ar^2}{(a+r)^2}\left(\beta + \dfrac{r}{a+r}\right)^{-1} \end{pmatrix}$$

Noting that

$$\begin{aligned} tr(J(x^*,y^*)) &= r\left(1 - \frac{r}{a+r}\right) - \frac{r^2}{a+r} - \frac{ar}{a+r} - \frac{ar^2}{(a+r)^2}\left(\beta + \frac{r}{a+r}\right)^{-1} \\ &= -\frac{(\beta+1)r^2}{\beta a + \beta r + r} \\ &< 0, \end{aligned}$$

and

$$
\begin{aligned}
& det(J(x^*,y^*)) \\
= & -\frac{ar^2}{(a+r)^2}\left(r\left(1-\frac{r}{a+r}\right)-\frac{r^2}{a+r}-\frac{ra}{a+r}\right)\left(\beta+\frac{r}{a+r}\right)^{-1} \\
& +\frac{r^3a^2}{(a+r)^3}\left(\beta+\frac{r}{a+r}\right)^{-1} \\
= & \frac{ar^3}{(\beta a+\beta r+r)(a+r)} \\
> & \ 0.
\end{aligned}
$$

So that both eigenvalues of $J(x^*,y^*)$ have negative real parts, and hence this steady-state solution is locally asymptotically stable.

From Theorem 2.1 we know that under the assumption $r > a$, system (1.3) is permanent, hence no solution could approach to $A(0,0)$ and $B(1,0)$, which means that $A(0,0)$ and $B(1,0)$ are locally unstable.

This ends the proof of Theorem 2.2.

3. Global stability

We had showed that the positive equilibrium is locally stable, in this section, we further give sufficient conditions to ensure the global stability of the positive equilibrium.

Theorem 3.1. *Assume that $r > a$ holds, then the unique positive equilibrium is globally asymptotically stable.*

Proof. Set

$$
P_1 = rx(1-x)-axy,\ Q_1 = a\frac{y^2}{\beta+y}(x-y). \tag{3.1}
$$

From Theorem 2.2 system (1.3) admits an unique local stable positive equilibrium $C(x^*,y^*)$. Also, from Theorem 2.2, $A(0,0)$ and $B(1,0)$ is unstable. To ensure $C(x^*,y^*)$ is globally stable, we consider the Dulac function $u_1(x,y) = x^{-1}y^{-2}$, then

$$
\begin{aligned}
&\frac{\partial(u_1P_1)}{\partial x}+\frac{\partial(u_1Q_1)}{\partial y}\\
=\ &\frac{r(1-x)-rx-ay}{xy^2}-\frac{rx(1-x)-axy}{x^2y^2}-\frac{a}{(\beta+y)x}-\frac{a(x-y)}{(\beta+y)^2x}\\
=\ &-\frac{a\beta y^2+axy^2+\beta^2rx+2\beta rxy+rxy^2}{xy^2(\beta+y)^2}<0.
\end{aligned}
$$

By Dulac Theorem, there is no closed orbit in area R_2^+. So $C(x_1^*,y_1^*)$ is globally asymptotically stable. This completes the proof of Theorem 3.1.

4. Numeric simulations

To find out the different influence of Allee effect on the predator and prey species, we will take $r=2.5, a=17$ as that of [2].

Figure 1 and 2 shows the impact of Allee effects on the local stability of the positive equilibrium solution of the population model defined by (1.3). One can see that the Allee effect has no influence on the population densities of both species at the stable steady-state solutions. However, both graphs show that the trajectories of the system subject to an Allee effect approach the equilibrium solution more slowly than in the model without the Allee effect. Figure 1 and 2 also shows that with the increasing of Allee effect (the increasing of β), the trajectories of the system take more time to approach the equilibrium solution.

The results obtained here is different from that obtained for the prey species with Allee effect by H*ü*seyin Merdan [2], who have illustrated that Allee effects reduces the population densities of both species at the stable-state solutions.

5. Discussion

During the last decades, many scholars([2]-[9]) investigated the influence of Allee effect on the stability analysis of equilibrium points. As was pointed out by H*ü*seyin Merdan [2], the Allee effect may have a stabilizing or a destabilizing effects on population dynamics. However, most of the scholars are focus their attention to the prey species.

In this paper, we incorporating the Allee effect to the predator species, and we find some

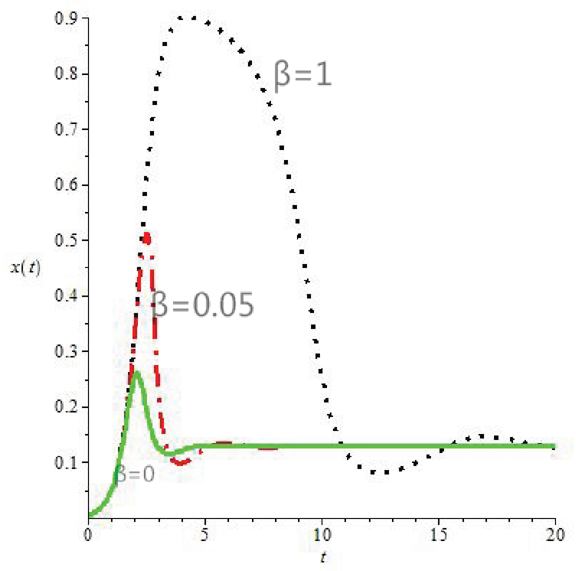

FIGURE 1. Dynamic behavior of the prey species in system (1.3) with $r = 2.5, a = 17$, $\beta = 0, 0.05$ and 1, respectively, here the initial condition $(x(0), y(0)) = (0.005, 0.01)$.

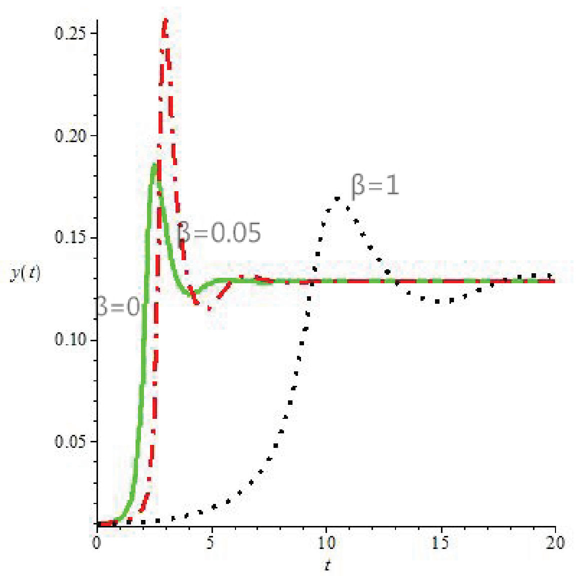

FIGURE 2. Dynamic behavior of the predator species in system (1.3) with $r = 2.5, a = 17$, $\beta = 0, 0.05$ and 1, respectively, here the initial condition $(x(0), y(0)) = (0.005, 0.01)$.

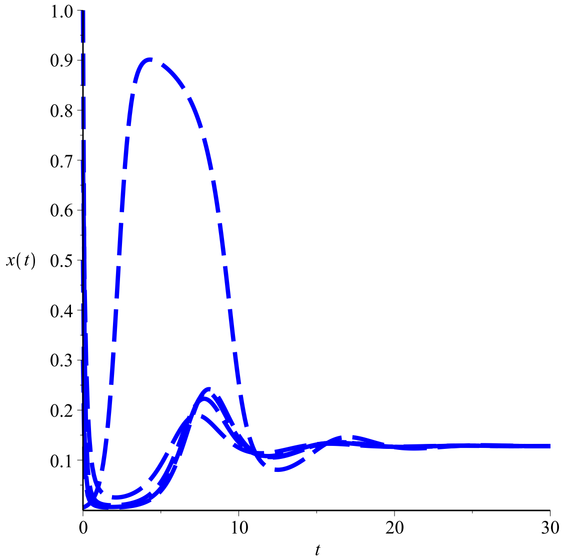

FIGURE 3. Dynamic behavior of the prey species in system (1.3) with $\beta = 1, r = 2.5, a = 17$ and the initial condition $(x(0), y(0)) = (0.005, 0.01)$, $(0.5, 0.1)$, $(0.7, 0.3)$, $(1, 0.7)$ and $(0.3, 1)$, respectively.

interesting phenomenon: if $r > a$ hold, the Allee effect on the predator species has no influence on the existence and stability of the positive equilibrium, also, Allee effect has no influence on the final density of the predator and prey species. Numeric simulations (Fig. 1 and 2) show that the system subject to an Allee effect takes much longer time to reach its stable steady-state solution, this is coincidence with the finding of H*ü*seyin Merdan [2].

At the end of the paper, we would like to point out that it is interesting to find out the influence of Allee effect under the assumption $r \leq a$. We leave this for future study.

Conflict of Interests

The authors declare that there is no conflict of interests.

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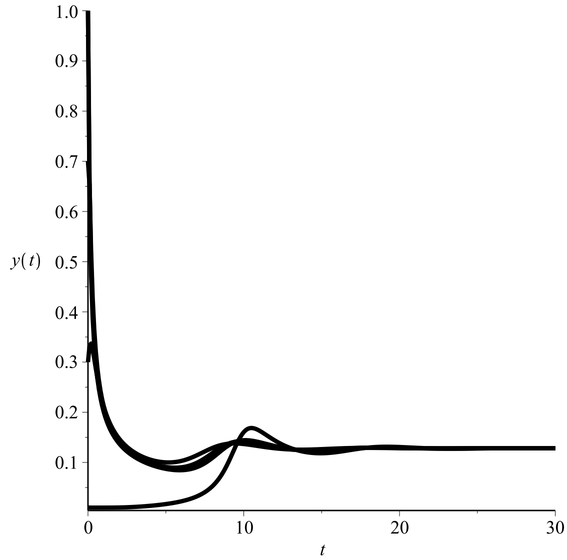

FIGURE 4. Dynamic behavior of the predator species in system (1.3) with $\beta = 1, r = 2.5, a = 17$ and the initial condition $(x(0), y(0)) = (0.005, 0.01)$, $(0.5, 0.1)$, $(0.7, 0.3)$, $(1, 0.7)$ and $(0.3, 1)$, respectively.

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