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DYNAMIC BEHAVIORS OF A HOLLING TYPE COMMENSAL SYMBIOSIS MODEL WITH THE FIRST SPECIES SUBJECT TO ALLEE EFFECT

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Abstract. A two species commensal symbiosis model with Holling type functional response and the first species subject to Allee effect takes the form

$$\begin{aligned}\frac{dx}{dt} &= x\Big(a_1 - b_1 x\Big)\frac{x}{\beta + x} + \frac{c_1 x y^p}{1 + y^p},\\ \frac{dy}{dt} &= y(a_2 - b_2 y)\end{aligned}$$

is investigated, where $a_i, b_i, i = 1,2$ p, β and c_1 are all positive constants, $p \geq 1$ is a positive integer. Local and global stability property of the equilibria are investigated. Our study indicates that the unique positive equilibrium is globally stable. Also, the final density of the first species is increasing with the Allee effect, this result differs from that obtained for the Holling type commensal symbiosis model with the second species subject to Allee effect.

Keywords: commensal symbiosis model; Holling type functional response; Allee effect; stability.

2000 Mathematics Subject Classification: 34C25, 92D25, 34D20, 34D40.

1. Introduction

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The aim of this paper is to investigate the dynamic behaviors of the following two species commensal symbiosis model with Holling type functional response and the first species subject to Allee effect:

$$\begin{aligned}\frac{dx}{dt} &= x\Big(a_1 - b_1x\Big)\frac{x}{\beta+x} + \frac{c_1xy^p}{1+y^p},\\ \frac{dy}{dt} &= y(a_2 - b_2y),\end{aligned} \tag{1.1}$$

where $a_i, b_i, i = 1, 2$ p, β and c_1 are all positive constants, $p \geq 1$ is a positive integer. $\frac{x}{\beta+x}$ represents the Allee effect of the first species, and β reflects the strength of the Allee effect.

During the lase decades, many scholars investigated the dynamic behaviors of the mutualism model or commensalism model ([1]-[27]). In [1]-[12], the authors paid their attention to the dynamic behaviors of the mutualism model, and many interesting results were obtained, for example, Yang et al [1] showed that single feedback control variable could lead some species in the system driven to extinction. Chen et al[2] showed that the stage structure plays import roles on the persistence and extinction of the cooperation system. However, only recently did scholars paid their attention to commensalism model, such topic as the stability of the positive equilibrium ([13, 15, 19, 21, 22, 24, 26, 27]), the persistent and extinction of the system ([18, 20, 25]), the existence of the positive periodic solution ([16, 17]) etc are extensively investigated.

Han and Chen[15] proposed the following commensalism model:

$$\begin{aligned}\frac{dx}{dt} &= x\Big(b_1 - a_{11}x\Big) + a_{12}xy,\\ \frac{dy}{dt} &= y\Big(b_2 - a_{22}y\Big).\end{aligned} \tag{1.2}$$

They showed that the system admits a unique positive equilibrium, which is globally asymptotically stable.

Wu, Li and Zhou[13] argued that it may be more suitable to assume the relationship between two species is nonlinear type, and they established the following two species commensal symbiosis model

$$\begin{aligned}\frac{dx}{dt} &= x\Big(a_1 - b_1x + \frac{c_1y^p}{1+y^p}\Big),\\ \frac{dy}{dt} &= y(a_2 - b_2y),\end{aligned} \tag{1.3}$$

where $a_i, b_i, i = 1,2$, p and c_1 are all positive constants, $p \geq 1$. They showed that the system also admits a unique positive equilibrium.

Recently, Wu, Li and Lin[27] also incorporated Allee effect to the second species in system (1.3), and this leads to the following system

$$\begin{aligned} \frac{dx}{dt} &= x\Big(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\Big), \\ \frac{dy}{dt} &= y(a_2 - b_2 y)\frac{y}{u+y}. \end{aligned} \tag{1.4}$$

They showed that the unique positive equilibrium of system (1.4) is globally stable, and the Allee effect has no influence on the final density of the species. However, numeric simulations showed that the stronger the Allee effect, the longer the for the system to reach its stable steady-state solution. During the lase decade, many scholars ([28]-[32]) studied the influence of Allee effect to the ecosystem, and many interesting results were obtained. For example, C Çelik and Duman[29] showed that Allee effects increase the local stability of equilibrium points of the discrete-time predator-prey model, Merdan[31] showed the the system subject to an Allee effect takes a much longer time to reach its stable steady-state solution. Noting that Wu et al[27] was the first time to study the Allee effect to the commensalism model, and their result is differ to the known results as that of C Çelik and Duman[29] and Merdan[31]. It's necessary to propose some new commensalism model incorporating the Allee effect, and to study the influence of Allee effect. This motivated us to propose the system (1.1).

One may conjecture that system (1.1) has the similar dynamic behaviors as that of system (1.4), however, this is impossible, the reason is that as shown in [27], the second equation of system (1.4) admits a unique positive equilibrium $y = \frac{a_2}{b_2}$, which is globally attractive. However, in system (1.1), since the first equation is depend on the second species y, it is impossible for the point $x^* = \frac{a_1}{b_1}$ to be globally stable. Also, from above analysis, we could also see that the analysis technique of Wu et al.[27] could not be applied to system (1.1). Hence, it is very necessary to study the dynamic behaviors of the system (1.1).

The aim of this paper is to investigate the local and global stability property of the possible equilibria of system (1.1). We arrange the paper as follows: In the next section, we will investigate the existence and local stability property of the equilibria of system (1.1). In Section 3, by constructing some suitable Dulac function, we will investigate the global stability property of the positive equilibrium of the system; In Section 4, an example together with its numeric simulations is presented to show the feasibility of our main results. We end this paper by a briefly discussion.

2. The existence and local stability of the equilibria

The equilibria of system (1.1) is determined by the system

$$\begin{aligned} x\Big(a_1-b_1x\Big)\frac{x}{\beta+x}+\frac{c_1xy^p}{1+y^p} &= 0,\\ y(a_2-b_2y) &= 0. \end{aligned} \tag{2.1}$$

Hence, system (1.1) admits three boundary equilibria, $A_0(0,0)$, $A_1\big(\frac{a_1}{b_1},0\big)$, $A_2\big(0,\frac{a_2}{b_2}\big)$. The positive equilibrium is determined by the system

$$\begin{aligned} \Big(a_1-b_1x\Big)\frac{x}{\beta+x}+\frac{c_1y^p}{1+y^p} &= 0,\\ a_2-b_2y &= 0. \end{aligned} \tag{2.2}$$

From the second equation we have $y=\frac{a_2}{b_2}$, substituting this to the first equation of system (2.2), and simplify, we finally obtain

$$A_1x^2+A_2x+A_3=0, \tag{2.3}$$

where

$$\begin{aligned} A_1 &= b_1\left(\frac{a_2}{b_2}\right)^p+b_1>0,\\ A_2 &= -a_1\left(\frac{a_2}{b_2}\right)^p-c_1\left(\frac{a_2}{b_2}\right)^p-a_1<0,\\ A_3 &= -c_1\left(\frac{a_2}{b_2}\right)^p\beta<0. \end{aligned} \tag{2.4}$$

Hence, system (1.1) admits unique positive equilibrium $A_3\big(x^*,y^*\big)$, where

$$x^*=\frac{-A_2+\sqrt{A_2^2-4A_1A_3}}{2A_1}, \qquad y^*=\frac{a_2}{b_2}. \tag{2.5}$$

Remark 2.1. In [27], Wu, Lin and Li showed that the unique positive equilibrium of system (1.4) is the same as that of the system (1.3), hence in system (1.4), Allee effect has no influence on the final density of the species, however, noting that in (2.4), A_3 is relevant to β, hence, x^* is the function of β, which means that Allee effect could influent the final density of the first species in system (1.1).

Remark 2.2. Since x^* is the function of β, it is nature to further investigate the relationship of the x^* and β. Noting that

$$\frac{dx^*}{d\beta} = \frac{c_1\left(\frac{a_2}{b_2}\right)^p}{\sqrt{A_2^2-4A_1A_3}} > 0. \tag{2.6}$$

(2.6) shows that with the increasing of the Allee effect, the final density of the first species is also increasing. Such an phenomenon is quite different to that of the the results of [27], [29] and [31].

Concerned with the local stability property of the above four equilibria, we have

Theorem 2.1. *$A_0(0,0)$, $A_1(\frac{a_1}{b_1},0)$ and $A_2\left(0,\frac{a_2}{b_2}\right)$ are all unstable; $A_3\left(x^*,y^*\right)$ is locally asymptotically stable.*

Proof. The Jacobian matrix of the system (1.1) is calculated as

$$J(x,y) = \begin{pmatrix} \Gamma & \dfrac{c_1pxy^{p-1}}{(1+y^p)^2} \\ 0 & a_2-2b_2y \end{pmatrix}, \tag{2.7}$$

where

$$\Gamma = \frac{(-2b_1x+a_1)x}{\beta+x} + \frac{-b_1x^2+a_1x}{\beta+x} - \frac{\left(-b_1x^2+a_1x\right)x}{(\beta+x)^2} + \frac{c_1y^p}{1+y^p}. \tag{2.8}$$

Then the Jacobian matrix of the system (1.1) about the equilibrium $A_0(0,0)$ is

$$\begin{pmatrix} 0 & 0 \\ 0 & a_2 \end{pmatrix}. \tag{2.9}$$

Obviously, the two eigenvalues of $J(A_0)$ are $\lambda_1 = 0$ and $\lambda_2 = a_2 > 0$. Hence, the equilibrium A_0 is non-hyperbolic. To determine the stability property of this equilibrium, now let's consider the transformation $X = x, Y = y, \tau = a_2t$, then the system (1.1) becomes

$$\begin{aligned} \frac{dX}{d\tau} &= X\left(a_1-b_1X\right)\frac{X}{a_2(\beta+X)} + \frac{c_1XY^p}{a_2(1+Y^p)}, \\ \frac{dY}{d\tau} &= Y - \frac{b_2}{a_2}Y^2. \end{aligned} \tag{2.10}$$

Expand the system (2.10) in power series up to the third order around the origin, we get:

$$\begin{aligned} \frac{dX}{d\tau} &= P_2(X,Y), \\ \frac{dY}{d\tau} &= Y + Q_2(X,Y), \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} P_2(X,Y) &= \frac{X^2 a_1}{\beta a_2} + \frac{c_1 X Y^p}{a_2} + \frac{X^3}{\beta a_2}\left(-b_1 - \frac{a_1}{\beta}\right) - \frac{c_1 X Y^{2p}}{a_2} + P_4(x,y), \\ Q_2(x,y) &= -\frac{b_{22} Y^2}{a_2}, \end{aligned} \tag{2.12}$$

here $P_4(X,Y)$ is power series with terms X^iY^j satisfying $i+j \geq 4$. Noting that in $P_2(X,Y)$, the coefficient of the term $\frac{X^2 a_1}{\beta a_2}$ is $\frac{a_1}{\beta a_2} > 0$, by the Theorem 7.1 in Chaper 2 of [33], the boundary equilibrium $(0,0)$ of system (2.11) is saddle-node, consequently, the equilibrium $A_0(0,0)$ of system (1.1) is saddle-node, hence, it is unstable.

Now let's consider the equilibrium $A_1(\frac{a_1}{b_1},0)$, the Jacobian matrix of the system (1.1) about the equilibrium $A_1(\frac{a_1}{b_1},0)$ is given by

$$\begin{pmatrix} -\frac{{a_1}^2}{b_1}\left(\beta + \frac{a_1}{b_1}\right)^{-1} & 0 \\ 0 & a_2 \end{pmatrix}. \tag{2.13}$$

(2.13) shows that the two eigenvalues of $J(A_1)$ are $\lambda_1 = -\frac{{a_1}^2}{b_1}\left(\beta + \frac{a_1}{b_1}\right)^{-1} < 0$ and $\lambda_2 = a_2 > 0$. Hence, the equilibrium A_1 is unstable.

Now let's consider the equilibrium $A_2(0,\frac{a_2}{b_2})$, the Jacobian matrix of the system (1.1) about the equilibrium $A_2(0,\frac{a_2}{b_2})$ is given by

$$\begin{pmatrix} c_1\left(\frac{a_2}{b_2}\right)^p \left(1 + \left(\frac{a_2}{b_2}\right)^p\right)^{-1} & 0 \\ 0 & -a_2 \end{pmatrix}. \tag{2.14}$$

(2.14) shows that the two eigenvalues of $J(A_2)$ are $\lambda_1 = c_1\left(\frac{a_2}{b_2}\right)^p \left(1 + \left(\frac{a_2}{b_2}\right)^p\right)^{-1} > 0$ and $\lambda_2 = -a_2 < 0$. Hence, the equilibrium A_2 is unstable.

Now let's consider the stability property of the positive equilibrium $A_3(x^*,y^*)$. Noting that

(x^*, y^*) satisfies the equations

$$\begin{aligned} \left(a_1 - b_1 x^*\right)\frac{x^*}{\beta + x^*} + \frac{c_1 (y^*)^p}{1 + (y^*)^p} &= 0, \\ a_2 - b_2 y^* &= 0. \end{aligned} \tag{2.15}$$

Hence

$$\begin{aligned} &\Gamma(x^*, y^*) \\ = &\ \frac{(-2b_1 x^* + a_1)x^*}{\beta + x^*} + \frac{-b_1 x^{*2} + a_1 x^*}{\beta + x^*} - \frac{\left(-b_1 x^{*2} + a_1 x^*\right)x^*}{(\beta + x^*)^2} + \frac{c_1 (y^*)^p}{1 + (y^*)^p} \\ = &\ \frac{(-2b_1 x^* + a_1)x^*}{\beta + x^*} - \frac{\left(-b_1 x^{*2} + a_1 x^*\right)x^*}{(\beta + x^*)^2} \\ = &\ -\frac{b_1 (x^*)^2}{\beta + x^*} + \frac{(-b_1 x^* + a_1)x^*}{\beta + x^*} - \frac{(-b_1 x^* + a_1)(x^*)^2}{(\beta + x^*)^2} \\ = &\ -\frac{b_1 (x^*)^2}{\beta + x^*} - \frac{c_1 (y^*)^p}{1 + (y^*)^p} + \frac{c_1 (y^*)^p}{1 + (y^*)^p}\frac{x^*}{\beta + x^*} \\ = &\ -\frac{b_1 (x^*)^2}{\beta + x^*} - \frac{\beta}{\beta + x^*}\frac{c_1 (y^*)^p}{1 + (y^*)^p}, \end{aligned} \tag{2.16}$$

by using (2.7), (2.8), (2.15) and (2.16), the Jacobian matrix of the system (1.1) about the equilibrium $A_3(x^*, y^*)$ is given by

$$\begin{pmatrix} -\frac{b_1 (x^*)^2}{\beta + x^*} - \frac{\beta}{\beta + x^*}\frac{c_1 (y^*)^p}{1 + (y^*)^p} & \frac{c_1 p x^* (y^*)^{p-1}}{(1 + (y^*)^p)^2} \\ 0 & -b_2 y^* \end{pmatrix}. \tag{2.17}$$

(2.17) shows that the two eigenvalues of $J(A_3)$ are $\lambda_1 = -\frac{b_1 (x^*)^2}{\beta + x^*} - \frac{\beta}{\beta + x^*}\frac{c_1 (y^*)^p}{1 + (y^*)^p} < 0$ and $\lambda_2 = -b_2 y^* < 0$. Hence, the equilibrium A_3 is locally asymptotically stable.

This ends the proof of Theorem 2.1.

3. Global stability of the positive equilibrium

Theorem 2.1 shows that the system always admits a positive equilibrium, and this equilibrium is locally stable, and all the other three boundary equilibria are unstable. The aim of this section is to investigate the existence or non-existence of the limit cycle.

Theorem 3.1. *$A_3(x^*, y^*)$ is globally stable.*

Proof. Theorem 2.1 shows that there is a unique local stable positive equilibrium $A_3(x^*, y^*)$. To show that $A_3(x^*, y^*)$ is globally stable, it's enough to show that the system admits no limit cycle in the first quadrant. Let's consider the Dulac function $u(x,y) = x^{-2}y^{-1}$, then

$$\begin{aligned}
&\frac{\partial(uP)}{\partial x} + \frac{\partial(uQ)}{\partial y} \\
= & \frac{1}{x^2 y}\left(\frac{(-2b_1 x + a_1)x}{\beta + x} + \frac{-b_1 x^2 + a_1 x}{\beta + x} - \frac{\left(-b_1 x^2 + a_1 x\right)x}{(\beta + x)^2} + \frac{c_1 y^p}{1 + y^p}\right) \\
& -2\frac{1}{x^3 y}\left(\frac{\left(-b_1 x^2 + a_1 x\right)x}{\beta + x} + \frac{c_1 x y^p}{1 + y^p}\right) + \frac{-2b_2 y + a_2}{x^2 y} - \frac{-b_2 y^2 + a_2 y}{x^2 y^2} \\
= & -\frac{\Delta(x,y)}{(\beta + x)^2 (1 + y^p) x^2 y} < 0,
\end{aligned}$$

where

$$\begin{aligned}
P(x,y) &= x\Big(a_1 - b_1 x\Big)\frac{x}{\beta + x} + \frac{c_1 x y^p}{1 + y^p}, \\
Q(x,y) &= y(a_2 - b_2 y) \\
\Delta(x,y) &= y^p b_1 \beta x^2 + y^p b_2 \beta^2 y + 2y^p b_2 \beta xy + y^p b_2 x^2 y \\
&\quad + b_1 \beta x^2 + b_2 \beta^2 y + 2b_2 \beta xy + b_2 x^2 y + a_1 x^2 \\
&\quad + y^p \beta^2 c_1 + 2y^p \beta c_1 x + y^p c_1 x^2 + y^p a_1 x^2.
\end{aligned}$$

By Dulac Theorem[28], there is no closed orbit in the first quadrant. Consequently, $A_3(x^*, y^*)$ is globally asymptotically stable. This completes the proof of Theorem 3.1.

4. Numeric simulations

Now let us consider the following example.

Example 4.1. Consider the following system

$$\begin{aligned}
\frac{dx}{dt} &= x(1 - 2x)\frac{x}{1 + x} + \frac{xy}{1 + y}, \\
\frac{dy}{dt} &= y(1 - 2y).
\end{aligned} \tag{4.1}$$

In this system, corresponding to system (1.1), we take $a_1 = a_2 = c_1 = 1, b_1 = b_2 = 2$. From Theorem 3.1, the unique positive equilibrium $(\frac{2+\sqrt{7}}{6}, \frac{1}{2})$ is globally asymptotically stable. Numeric simulation (Fig.1) also support this assertion. Now let's take $\beta = 0, 1$ and 2, respectively, Fig. 2 shows that with the increasing of the β (i. e., the increasing of the Allee effect), the final density of the first species is also increasing.

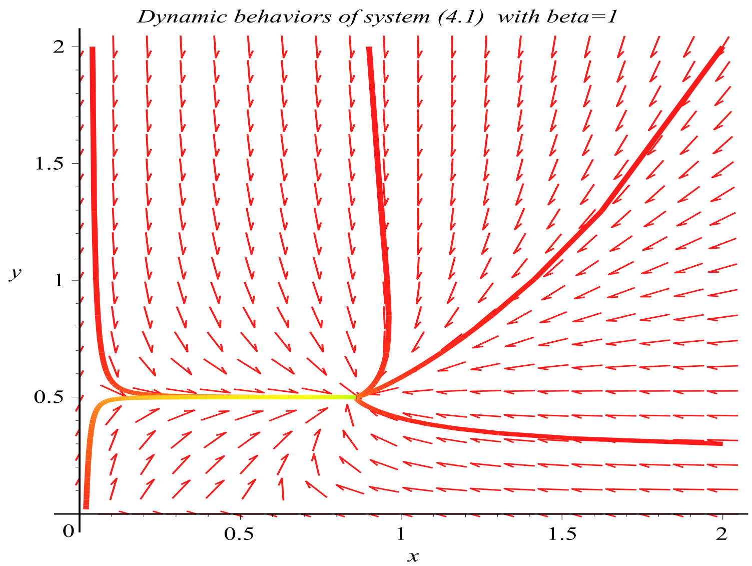

FIGURE 1. Numeric simulations of system (4.1) with $\beta = 1$, the initial conditions $(x(0), y(0)) = (0.04, 2), (2, 0.3), (0.02, 0.02), (2, 2)$ and $(0.9, 2)$, respectively.

5. Conclusion

Previously, Wu, Li and Lin[27] proposed a two species commensal symbiosis model with Holling type functional response and Allee effect to the second species, they showed that the dynamic behaviors of the system is similar to the system without Allee effect. The system always admits a unique globally stable positive equilibrium. Their numeric simulations showed that the stronger the Allee effect, the system takes a longer time to reach its steady-state solution.

In this paper, stimulated by the work of [27], we proposed a two species commensal symbiosis model with Holling type functional response and Allee effect to the first species. Our

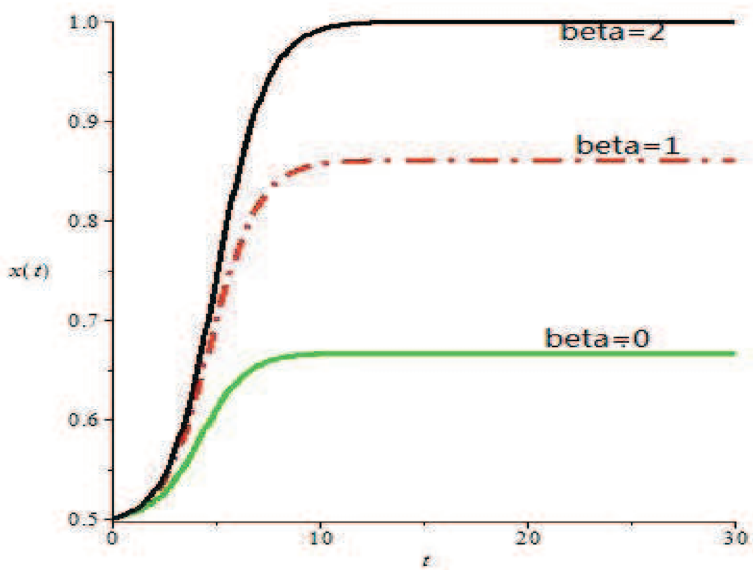

FIGURE 2. Numeric simulations of $x(t)$, with $\beta = 0,1,2$ and $(x(0),y(0)) = (0.5,0.01)$, where black curve is the solution of $\beta = 2$, green curve is the solution of $\beta = 0$, and red curve is the solution of $\beta = 1$.

study shows that the system also admits a unique positive equilibrium which is globally asymptotically stable. However, different to the results of [27], in system (1.1), the final density of the first species is relevant to the Allee effect, with the increasing of the Allee effect, the final density of the first species is also increasing.

At the end of the paper, we would like to mention that since Allee effect has different influence on the predator-prey model and commensalism model, we conjecture that competitive system subject to Allee effect may also have some new property, we leave this for the future study.

Conflict of Interests

The authors declare that there is no conflict of interests.

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