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LIE SYMMETRY ANALYSIS OF STOCHASTIC SIRS MODEL

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Abstract. In this paper, we discuss the connection between Lie symmetry of a SIRS stochastic differential equations and its integrability. A derived determining equation is obtained and used to find the admitted Lie point symmetries of the stochastic model. A third dimensional system is reduced into a one dimensional to achieve a linear model by mean of Symmetry analysis techniques and obtained the corresponding Lie bracket.

Keywords: Lie group theory; SIRS model; Stochastic differential equations; Integrability.

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1. Introduction

In this paper, we introduce stochastic effects into a deterministic *SIRS* model. In nature, infection of disease are impacted by diverse complex biological process. Therefore one could believe on the existence of randomness in the transmission dynamics of the disease. Taking into consideration the uncertainty aspects, randomness can be presented to the model by replacing the parameters μ (birth rate), γ (infection rate) and ν (death rate) by $\mu \to \mu + \sigma \dot{W}(t)$, $\nu \to$

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$\nu+\sigma\dot{W}(t)$, $\gamma\to\gamma+\sigma\dot{W}(t)$. The system that governed a stochastic *SIRS* model is given by

$$
\begin{aligned}
dS(t) &= \Big(\mu N-\frac{\beta S(t)I(t)}{N}+\theta R(t)-\nu S(t)\Big)dt-\sigma_1 S(t)dW_1(t)\\
dI(t) &= \Big(\frac{\beta S(t)I(t)}{N}-\gamma I(t)-\nu I(t)\Big)dt-\sigma_2 I(t)dW_2(t) \qquad (1)\\
dR(t) &= \Big(\gamma I(t)-\theta R(t)-\nu R(t)\Big)dt-\sigma_3 R(t)dW_3(t)
\end{aligned}
$$

with suitable initial conditions, where $W(t)$ is a Wiener processes (or Brownian motions), β the transmission rate. In view of the fact that death rate is so unpredictable, it is therefore important to introduce environmental noise to the parameter as stated in [2, 3]. It should be acknowledged that the equivalent noise intensity σ and the Wiener process $W(t)$ are achieved for susceptible and infected classes. However, the genetic factors that have an impact on infectious individuals are divergent even in the absence of complete biological information. Given that infectious and noninfectious individuals required to have different biological system, it is realistic for $W(t)$ and σ to differ from each and other. In this paper, Kozlov's approach [8] is used to demonstrate that if a SIRS stochastic model admits a point symmetry, then it is integrable. In this regards, the point symmetry establishes a change of variables which linearizes the nonlinear model and provides explicit solutions of the stochastic differential equations.

This paper is organised as follows. In Section 2 we provide the fundamental characteristics and definitions for the Lie point symmetries of stochastic differential equations. In Section 3, we use the preliminary results to find Lie operators of the three dimensional nonlinear system. The model is reduced to one dimensional equation and performed the Lie symmetry analysis in Section 4. The Wiener processes term is considered into a reduced equation and study the resulting equation from the view point of Lie symmetry in Section 5. The integrability of the reduced stochastic SIRS model is established in Section 6.

2. PRELIMINARIES ON SYMMETRY OF STOCHASTIC DIFFERENTIAL EQUATIONS

This Section focuses on Lie point symmetry transformation of Itô stochastic differential equation as described in [1]. Furthermore, the idea was extended in [6, 7] by incorporating Brownian motion on the theory of Lie symmetry by mean of Itô lemma for Poisson Process.

2.1. One parameter Lie group of transformations.

Let [13]

$$dx_i = f_i(x,t)dt + \sigma_i(x,t)dW_i \tag{2}$$

be a Itô stochastic differential equations driven by Wiener process, with $f_i(x,t)$ and $\sigma_i(x,t)$ representing drift vector and Poisson diffusion coefficients respectively. The infinitesimal generator

$$G = \eta(t,x)\frac{\partial}{\partial t} + \xi_i(t,x)\frac{\partial}{\partial x_i} \tag{3}$$

admits a one-parameter Lie group of transformations

$$\tilde{t} = t + \varepsilon\xi_i(t,x),\ \tilde{x} = x + \varepsilon\eta(t,x) \tag{4}$$

with

$$\frac{d\tilde{t}}{d\varepsilon} = \xi_i(\tilde{t},\tilde{x}),\ \frac{d\tilde{x}}{d\varepsilon} = \eta(\tilde{t},\tilde{x}) \tag{5}$$

satisfying the initial conditions

$$\tilde{t}\mid_{\varepsilon=0} = t, \tilde{x}\mid_{\varepsilon=0} = x. \tag{6}$$

Furthermore, equation (3) admits the given determining equations

$$\left(f_i\Gamma_{(\eta)} + \frac{\lambda\sigma_i}{2}\Gamma_{(\eta)} + G(f_i) - \Gamma_{\xi_i}\right)(t,W(t)) = 0 \tag{7}$$

$$\left(\frac{\sigma_i}{2}\Gamma_{(\eta)} + G(\sigma_i) - \Gamma^*_{\xi_i}\right)(t,W(t)) = 0 \tag{8}$$

satisfying the conditions

$$\Gamma^*_{\eta}(t,W(t)) = 0,\ \Gamma_{\eta}(t,W(t)) = constant \tag{9}$$

where

$$\Gamma_{(X_j)} = \frac{\partial X_j}{\partial t} + f_i\frac{\partial X_j}{\partial x_i} \tag{10}$$

$$\Gamma^*_{(X_j)} = X_j(t,x_i(t) + \sigma_j(t,x(t))) - X_j(t,x(t)) \tag{11}$$

The Itô Poisson process of an arbitrary function $X(t,x)$ defined above is given by

$$dX_j(t,x(t)) = \Big(\frac{\partial X_j}{\partial t} + f_i \frac{\partial X_j}{\partial x_i}\Big)dt + \Big(X_j(t,x_i) + \sigma_j(t,x_i) - X_j(t,x_i)\Big)dW(t). \tag{12}$$

The substitution of (10) and (11) into (12) gives

$$dX_j(t,x(t)) = \Gamma_{(X_j)}dt + \Gamma^*_{(X_j)}dW(t). \tag{13}$$

2.2. Prolongation of an infinitesimal generator. A second order ordinary differential equation

$$u_t - F(t,u,u_{(1)}) = 0 \tag{14}$$

admits a one-parameter Lie group of transformations

$$\begin{aligned} \bar{t} &\approx t + a\xi^0(t,u) \\ \bar{u} &\approx u + a\eta(t,u) \end{aligned} \tag{15}$$

with infinitesimal generator

$$G = \xi^0(t,u)\frac{\partial}{\partial t} + \eta(t,u)\frac{\partial}{\partial u} \tag{16}$$

if

$$\bar{u}_{\bar{t}} - F(\bar{t},\bar{u},\bar{u}_{(1)}) = 0 \tag{17}$$

The group transformations $\bar{t}$ and $\bar{u}$ are obtained by solving the following Lie equations [5, 12]

$$\begin{aligned} \frac{d\bar{t}}{da} &= \xi^0(\bar{t},\bar{u}) \\ \frac{d\bar{u}}{da} &= \eta(\bar{t},\bar{u}) \end{aligned} \tag{18}$$

with initial conditions

$$\bar{t}\mid_{a=0} = t, \bar{u}\mid_{a=0} = u. \tag{19}$$

The infinitesimal form of $\bar{u}_{\bar{t}}, \bar{u}_{(1)}$ are found by the given formulas [5, 9]:

$$\begin{aligned} \bar{u}_{\bar{t}} &\approx u_t + a\zeta_0(t,u,u_t,u_{(1)}) \\ \bar{u}_{\bar{x}^i} &\approx u_{x^i} + a\zeta_i(t,u,u_t,u_{(1)}) \end{aligned} \tag{20}$$

The functions ζ_0 and ζ_i are found by using the prolongation formulas below [10]

$$\begin{aligned} \zeta_0 &= D_t(\eta) - u_t D_t(\xi^0) \\ \zeta_i &= D_i(\eta) - u_t D_i(\xi^0) \end{aligned} \tag{21}$$

In [11], Matadi claimed that the equation (14) possesses the symmetry (group generator)

$$G = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^i(t,x,u)\frac{\partial}{\partial x^i} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{22}$$

iff

$$G^{[N]}N|_{N=0} = 0 \tag{23}$$

with $G^{[N]}$ the n-th extension of G. Hence, the determining equation (8) is used to obtain the following in case of stochastic differential equations

$$G^{[N]}(\dot{x}_i - f_i)|_{(\dot{x}_i - f_i)} - \frac{\sigma_i}{2}\Big(\frac{\partial X_j}{\partial t} + f_i\frac{\partial X_j}{\partial x_i}\Big) = 0. \tag{24}$$

3. Symmetry for three dimensional stochastic *SIRS* model

We will refer the reader to the proven Theorem 3 in [4] stated below

Theorem 1. *Suppose the system (1) admits an r-parameter solvable algebra g of simple deterministic symmetries, with generators*

$$G_k = \sum_{i=1}^{n} \phi_k^i(x,t)\frac{\partial}{\partial_i},\ k = 1,...,r \tag{25}$$

acting regularly with r-dimensional orbits.
Then it can be reduced to a system of m=(n-r) equations,

$$dy^i = g^i(y^1,...,y^m;t)dt + \sigma_k^i(y^1,...,y^m;t)dw^k,\ (i,k = 1,...,m) \tag{26}$$

and r "reconstruction equations", the solutions of which can be obtained by quadratures from the solution of the reduced (n-r) order system. In particular, if n=n, the general solution of the system can be found by quadratures.

In [8], it has been shown that nonlinear system of the form of equation (1) admits symmetries

$$G=\phi_1(t)\frac{\partial}{\partial S(t)}+\phi_2(t)\frac{\partial}{\partial I(t)}+\phi_3(t)\frac{\partial}{\partial R(t)} \tag{27}$$

with ϕ_1, $\phi_2$and ϕ_3 arbitrary smooth functions. The corresponding change of variables is given by

$$\begin{aligned}
y_1 &= \frac{x_1(B_2C_3-C_2B_3)+x_2(A_3C_2-C_3A_2)+x_3(A_2B_3-A_3B_2)}{\Delta},\\
y_2 &= \frac{x_1(B_1C_3-C_1B_3)+x_2(A_3C_1-C_3A_1)+x_3(A_1B_3-A_3B_1)}{\Delta},\\
y_3 &= \frac{x_1(B_1C_2-C_1B_2)+x_2(A_2C_1-C_2A_1)+x_3(A_1B_2-A_2B_1)}{\Delta}.
\end{aligned}$$

with $\Delta=A_1(B_2C_3-C_2B_3)-B_1(A_2C_3-C_2A_3)+C_1(A_2B_3-B_2A_3)\neq 0$ and the three dimensional vector fields given by

$$G_i=A_i(t)\frac{\partial}{\partial S(t)}+B_i(t)\frac{\partial}{\partial I(t)}+C_i(t)\frac{\partial}{\partial R(t)}. \tag{28}$$

Hence, the stochastic SIRS model (1) is mapped into

$$dy_1 = \frac{\mu N}{\Delta}dt+\frac{\sigma_1(B_2C_3-C_2B_3)}{\Delta}dW_1(t) \tag{29}$$

$$dy_2 = \frac{\sigma_2(A_2C_3-C_2A_3)}{\Delta}dW_2(t) \tag{30}$$

$$dy_3 = \frac{\sigma_3(A_2B_3-B_2A_3)}{\Delta}dW_3(t)$$

4. LIE SYMMETRY OF THE REDUCED STOCHASTIC *SIRS* MODEL

In this Section a three dimensional stochastic SIRS model is reduced into one dimensional equation and performed the Lie point symmetry. In this regard, the reduced form of system (1) is given by

$$dN(t)=N(t)[(\mu-\nu)dt-\sigma dW(t)] \tag{31}$$

with initial condition $N(t_0)=n_0>0$. Hence, the determining equations described in (7) and (8) are given by

$$\left[(\mu-\nu)n\Gamma_{(\eta)}+\frac{\lambda\sigma(\mu-\nu)n}{2}\Gamma_{(\eta)}+(\mu-\nu)\xi-\Gamma_{\xi_i}\right](t,N(t))=0 \tag{32}$$

substituting equation (10) into (32) we obtain

$$(\mu-\nu)n\frac{\partial(\eta)}{\partial t}+\frac{\lambda\sigma(\mu-\nu)n}{2}\frac{\partial(\eta)}{\partial t}+(\mu-\nu)\xi(t,n)-\frac{\xi(t,n)}{\partial t}-(\mu-\nu)n\frac{\xi(t,n)}{\partial n}=0 \tag{33}$$

and

$$\Big(\frac{\sigma(\mu-\nu)n}{2}\Gamma_{(\eta)}+(\mu-\nu)\xi(t,n)-\Gamma^{*}_{\xi_i}\Big)(t,N(t))=0 \tag{34}$$

The substitution of (11) into (34) gives

$$\frac{\sigma(\mu-\nu)n}{2}\frac{\partial(\eta)}{\partial t}+(\mu-\nu)\xi(t,n)-(\mu-\nu)\xi(t,n+n)+(\mu-\nu)\xi(t,n)=0 \tag{35}$$

From equation (9), we obtain the temporal infinitesimal below

$$\eta=At+B \tag{36}$$

The substitution of (36) into (33) and (35) gives

$$\frac{n(\mu-\nu)(2A+\lambda\sigma)}{2}+(\mu-\nu)\xi(t,n)-\frac{\partial\xi(t,n)}{\partial t}-(\mu-\nu)n\frac{\partial\xi(t,n)}{\partial n}=0 \tag{37}$$

and

$$\frac{\sigma A(\mu-\nu)n}{2}+2(\mu-\nu)\xi(t,n)-(\mu-\nu)\xi(t,2n)=0. \tag{38}$$

Differentiating equations (37) and (38) with respect to n, we obtain respectively

$$\frac{(\mu-\nu)(2A+\lambda\sigma)}{2}-\frac{\partial^2\xi(t,n)}{\partial n\partial t}-(\mu-\nu)n\frac{\partial^2\xi(t,n)}{\partial n^2}=0 \tag{39}$$

and

$$\frac{\sigma A(\mu-\nu)}{2}+2(\mu-\nu)\frac{\partial\xi(t,n)}{\partial n}-2(\mu-\nu)\frac{\partial\xi(t,2n)}{\partial n}=0. \tag{40}$$

Differentiating equation (40) with respect to t we obtain

$$\frac{\partial^2\xi(t,n)}{\partial n\partial t}=\frac{\partial^2\xi(t,2n)}{\partial n\partial t}. \tag{41}$$

Equation (41) can be written as

$$\frac{\partial^2\xi(t,n)}{\partial n\partial t}=\frac{dg(t)}{dt}. \tag{42}$$

with

$$g(t)=\frac{\partial\xi(t,2n)}{\partial n}. \tag{43}$$

Integrating twice equation (42) we obtain

$$\xi(t,n) = g(t)n + h(n). \tag{44}$$

The substitution of equation (44) into (39) gives

$$\frac{(\mu-\nu)(2A+\lambda\sigma)}{2} = \frac{dg(t)}{dt} + (\mu-\nu)n\frac{d^2h(n)}{dn^2}. \tag{45}$$

Differentiating equation (45) with respect to t, we obtain

$$g(t) = Ct + D. \tag{46}$$

The spatial infinitesimal below is obtained by substituting equation (46) into (44)

$$\xi(t,n) = (Ct+D)n + h(n). \tag{47}$$

The substitution of equation (47) into (45) gives

$$\frac{(\mu-\nu)(2A+\lambda\sigma)}{2} = C + (\mu-\nu)n\frac{d^2h(n)}{dn^2}. \tag{48}$$

Therefore,

$$\frac{d^2h(n)}{dn^2} = \frac{(\mu-\nu)(2A+\lambda\sigma)-2C}{2n(\mu-\nu)}. \tag{49}$$

The solution of equation (49) is given by

$$h(n) = \frac{(\mu-\nu)(2A+\lambda\sigma)-2C}{2(\mu-\nu)}\Big[n\ln n - n\Big] + En + F. \tag{50}$$

The spatial infinitesimal describes in equation (47) becomes

$$\xi(t,n) = (Ct+D)n + \frac{(\mu-\nu)(2A+\lambda\sigma)-2C}{2(\mu-\nu)}\Big[n\ln\mid n\mid - n\Big] + En + F. \tag{51}$$

Substituting equation (51) into (38) and after simplification, we obtain

$$\frac{\sigma A(\mu-\nu)n}{2} + F = \frac{(\mu-\nu)(2A+\lambda\sigma)-2C}{2(\mu-\nu)}\Big[n\ln\mid 4\mid\Big] \tag{52}$$

By comparing the coefficients of powers of n in equation (52) we obtain

- $n : C = A\Big[(\mu-\nu) - \frac{\sigma(\mu-\nu)^2}{2\ln 4}\Big]$
- $n^0 : F = 0$

Finally, equation (51) can be written as

$$\xi(t,n) = A\Big[\Big((\mu-\nu) - \frac{\sigma(\mu-\nu)^2}{2\ln 4} + \frac{\lambda\sigma(\mu-\nu)}{2}\Big)nt + \frac{\lambda\sigma(\mu-\nu)}{2\ln 4}\Big] + (D+E)n \tag{53}$$

where A, D and E are arbitrary constants. Thus the Lie algebra of equation (31) is spanned by the following three infinitesimal generator:

$$\begin{aligned} G_1 &= t\frac{\partial}{\partial t} + \Big[\Big((\mu-\nu) - \frac{\sigma(\mu-\nu)^2}{2\ln 4} + \frac{\lambda\sigma(\mu-\nu)}{2}\Big)nt + \frac{\lambda\sigma(\mu-\nu)}{2\ln 4}\Big]\frac{\partial}{\partial n} \\ G_2 &= \frac{\partial}{\partial t} \\ G_3 &= 2n\frac{\partial}{\partial n} \end{aligned}$$

Computing the Lie bracket we obtain the given commutator table:

	G_1	G_2	G_3
G_1	0	$-\Big(G_2+\delta G_3\Big)$	$\frac{\lambda\sigma(\mu-\nu)}{\ln 4}G_3$
G_2	$G_2+\delta G_3$	0	0
G_3	$-\frac{\lambda\sigma(\mu-\nu)}{\ln 4}G_3$	0	0

TABLE 1. The commutator table of the infinitesimal generator

with $\delta = \Big((\mu-\nu) - \frac{\sigma(\mu-\nu)^2}{2\ln 4} + \frac{\lambda\sigma(\mu-\nu)}{2}\Big)$.

5. W- SYMMETRY OF THE REDUCED STOCHASTIC SIRS MODEL

The Lie symmetry analysis performed in Section 4 involved only the total number of population and time, (t,n). However, in this Section, the Wiener process term $W(t)$ is also considered. The stochastic differential equation (2) driven by Wiener processes with infinitesimal operator

$$G = \eta(t,n,W)\frac{\partial}{\partial t} + \xi_i(t,n,W)\frac{\partial}{\partial n_i} + \phi_i(t,n,W)\frac{\partial}{\partial W} \tag{54}$$

admits a one-parameter Lie group of transformations

$$\tilde{t} = t + \varepsilon\eta(t,n,W),\ \tilde{n} = n + \varepsilon\xi_i(t,n,W),\ \tilde{W} = W + \varepsilon\phi_i(t,n,W) \tag{55}$$

with

$$\frac{d\tilde{t}}{d\varepsilon} = \eta(\tilde{t},\tilde{n},\tilde{W}),\ \frac{d\tilde{n}}{d\varepsilon} = \xi_i(\tilde{t},\tilde{n},\tilde{W}),\ \frac{d\tilde{W}}{d\varepsilon} = \phi_i(\tilde{t},\tilde{n},\tilde{W}) \tag{56}$$

satisfying the initial conditions

$$\tilde{t} \mid_{\varepsilon=0}= t, \tilde{n} \mid_{\varepsilon=0}= n, \tilde{W} \mid_{\varepsilon=0}= W. \tag{57}$$

The Lie group transformation (55) can be written in terms of Lie operator (54) as follows

$$\tilde{t} = \exp[\varepsilon G](t),\ \tilde{n} = \exp[\varepsilon n](n),\ \tilde{W} = \exp[\varepsilon W](W). \tag{58}$$

Furthermore, equation (54) admits the given determining equations

$$\Big(f_i \Gamma_{(\eta)} + \lambda \sigma_i [\Gamma_{(\eta)} - \Gamma^*(\phi_i)] + G(f_i) - \Gamma \xi_i\Big)(t, W(t)) = 0 \tag{59}$$

$$\Big(\sigma_i \Gamma^*(\phi_i) + G(\sigma_i) - \Gamma^*(\xi_i)\Big)(t, W(t)) = 0 \tag{60}$$

and

$$\Gamma(\phi_i) + \lambda \Gamma^*(\phi_i) = \lambda \Gamma(\eta) \tag{61}$$

with

$$\Gamma^*(\phi_i) = \frac{\Gamma(\eta)}{2} \tag{62}$$

$$\Gamma(\phi_i) = \lambda \frac{\Gamma(\eta)}{2} \tag{63}$$

satisfying the conditions

$$\Gamma^*_{\eta}(t, n, W(t)) = 0,\ \Gamma_{\eta}(t, n, W(t)) = constant \tag{64}$$

where

$$\Gamma_{(X_j)} = \frac{\partial X_j}{\partial t} + f_i \frac{\partial X_j}{\partial x_i} \tag{65}$$

$$\Gamma^*_{(X_j)} = X_j(t, x_i(t) + \sigma_j(t, x(t))) - X_j(t, x(t)) \tag{66}$$

The Itô Poisson process of an arbitrary function $X(t,x)$ defined above is given by

$$dX_j(t, x(t)) = \Big(\frac{\partial X_j}{\partial t} + f_i \frac{\partial X_j}{\partial x_i}\Big) dt + \Big(X_j(t, x_i) + \sigma_j(t, x_i) - X_j(t, x_i)\Big) dW(t). \tag{67}$$

The substitution of (65) and (66) into (67) gives

$$dX_j(t, x(t)) = \Gamma_{(X_j)} dt + \Gamma^*_{(X_j)} dW(t). \tag{68}$$

Now, considering equation (31), the determining equations (59) and (60) become

(69)
$$(\mu-\nu)n\Big(\frac{\partial\eta}{\partial t}+(\mu-\nu)n\frac{\partial\eta}{\partial n}\Big)+\frac{\lambda\sigma n}{2}\Big(\frac{\partial\eta}{\partial t}+(\mu-\nu)n\frac{\partial\eta}{\partial n}\Big)+(\mu-\nu)\eta=\frac{\partial\xi}{\partial t}+(\mu-\nu)n\frac{\partial\xi}{\partial n}.$$

(70)
$$\sigma n\Big(\phi(t,n+\sigma n,W)-\phi(t,n,W)\Big)=\xi(t,n+\sigma+n)-\xi(t,n)$$

and

(71)
$$\phi(t,n+\sigma n,W)-\phi(t,n,W)=\frac{1}{2}\Big(\frac{\partial\eta}{\partial t}+(\mu-\nu)n\frac{\partial\eta}{\partial n}\Big)$$

(72)
$$\Big(\frac{\partial\phi}{\partial t}+(\mu-\nu)n\frac{\partial\phi}{\partial n}\Big)=\frac{\lambda}{2}\Big(\frac{\partial\eta}{\partial t}+(\mu-\nu)n\frac{\partial\eta}{\partial n}\Big)$$

From equation (64) we obtain

(73)
$$\frac{\partial\eta(t,n,W)}{\partial n}=0,\ \frac{\partial\eta(t,n,W)}{\partial W}=0.$$

Hence, the temporal infinitesimal is obtain by integrating equation (73) with respect to x

(74)
$$\eta(t,W)=At+B.$$

The substitution of (74) into (69), (71) and (72) gives

(75)
$$A\Big[\frac{\lambda\sigma n}{2}+(\mu-\nu)n+(\mu-\nu)t\Big]+(\mu-\nu)B=\frac{\partial\xi}{\partial t}+(\mu-\nu)n\frac{\partial\xi}{\partial n},$$

(76)
$$\phi(t,n+\sigma n,W)-\phi(t,n,W)=\frac{A}{2}$$

and

(77)
$$\frac{\partial\phi}{\partial t}+(\mu-\nu)n\frac{\partial\phi}{\partial n}=\frac{\lambda A}{2}$$

Differentiating equations (75) and (77) with respect to n with obtain

(78)
$$\frac{A\lambda\sigma}{2}+\mu-\nu=\frac{\partial^2\xi}{\partial t\partial n}+(\mu-\nu)\frac{\partial\xi}{\partial n}+(\mu-\nu)n\frac{\partial^2\xi}{\partial n^2}$$

(79)
$$\frac{\partial^2\phi}{\partial t\partial n}+(\mu-\nu)\frac{\partial\phi}{\partial n}+(\mu-\nu)n\frac{\partial^2\phi}{\partial n^2}=0$$

Differentiating equations (70) and (76) with respect to n we have

(80)
$$\frac{\partial\xi(t,2n+\sigma)}{\partial n}=\frac{\partial\xi(t,n)}{\partial n}=k(t)$$

$$\frac{\partial \phi(t,n+\sigma n,W)}{\partial n} = \frac{\partial \phi(t,n,W)}{\partial n} = h(t,W). \tag{81}$$

Integrating equation (80) with respect to n we obtain

$$\xi(t,n) = k(t)n + h(t) \tag{82}$$

The substitution of (82) into (78) gives

$$\frac{dk(t)}{dt} + (\mu - \nu)k(t) = \frac{A\lambda\sigma + \mu - \nu}{2} \tag{83}$$

Solving equation (83) we obtain

$$k(t) = \frac{A\lambda\sigma + \mu - \nu}{2(\mu - \nu)} + C\exp[-(\mu - \nu)t] \tag{84}$$

The substitution of equation (84) into (70) by using equation (76) gives the following relation

$$A = -\frac{2C}{\sigma}\exp[-(\mu - \nu)t] \tag{85}$$

Hence,

$$k(t) = \frac{(\mu - \nu - \lambda)C}{\mu - \nu}\exp[-(\mu - \nu)t] + \frac{1}{2} \tag{86}$$

Substituting equation (86) into (82) and using equation (75) we obtain

$$h(t) = A\frac{(\mu - \nu)}{2}t^2 + A\Big[\frac{\lambda\sigma + 2(\mu - \nu)}{2}\Big]nt + (\mu - \nu)t + D \tag{87}$$

Therefore,

$$\xi(t,n) = A\Big[\frac{(\lambda\sigma + 2(\mu - \nu))nt}{2} + \frac{(\mu - \nu)t^2}{2} - \frac{\sigma(\mu - \nu - \lambda)n}{2(\mu - \nu)}\Big] + B(\mu - \nu)t + \frac{(\mu - \nu)n}{2} + D \tag{88}$$

From (81) we obtain

$$\phi(t,n,W) = h(W)n + h_1(t,W) \tag{89}$$

The substitution of (89) into (77) gives

$$h_1(t,W) = \frac{\lambda A}{2}t - (\mu - \nu)h(W)nt + h_2(W). \tag{90}$$

Therefore,

$$\phi(t,n,W) = h(W)n + \frac{\lambda A}{2}t - (\mu - \nu)h(W)nt + h_2(W). \tag{91}$$

The substitution of (91) into (76) produces

$$h(W) = \frac{A}{2\sigma n} \tag{92}$$

Finally, the infinitesimal of the Wiener processes is obtained by substituting equation (92) into (91)

$$\phi(t,n,W) = A\Big[\frac{1}{2\sigma} + \frac{(\lambda\sigma-\mu+\nu)t}{2\sigma}\Big] + h_2(W). \tag{93}$$

The infinitesimal Lie symmetry operators are obtained from equations (74), (88) and (93)

$$\begin{aligned} G_1 &= t\frac{\partial}{\partial t} + \Big[\frac{(\lambda\sigma+2(\mu-\nu))nt}{2} + \frac{(\mu-\nu)t^2}{2} - \frac{\sigma(\mu-\nu-\lambda)n}{2(\mu-\nu)}\Big]\frac{\partial}{\partial n} \\ &+ \Big[\frac{1}{2\sigma} + \frac{(\lambda\sigma-\mu+\nu)t}{2\sigma}\Big]\frac{\partial}{\partial W} \\ G_2 &= \frac{\partial}{\partial t} + (\mu-\nu)t\frac{\partial}{\partial n} \\ G_3 &= \frac{\partial}{\partial n} \\ G_4 &= h_2(W)\frac{\partial}{\partial W}. \end{aligned} \tag{94}$$

The corresponding Lie bracket is shown in the commutator table below where $G_5 = \Big(G_2 + \frac{\lambda\sigma}{2}(\sigma G_3 + G_4)\Big)$ and $G_6 = G_3 - \frac{G_4}{\lambda\sigma}$.

	G_1	G_2	G_3	G_4
G_1	0	$-G_5$	$-G_6$	0
G_2	G_5	0	0	0
G_3	G_6	0	0	0
G_4	0	0	0	0

TABLE 2. The commutator table of the infinitesimal generator

6. INTEGRABILITY OF THE REDUCED STOCHASTIC SIRS MODEL

According to Kozlov [8] the reduced equation (31) can be transformed into a deterministic map below

$$dy = n(t)dt + \sigma(t)dw. \tag{95}$$

Kozlov claimed that the transformation described in (95) is integrable if and only if it admits the following Lie operator

$$Y = \phi(n,t)\partial_y. \tag{96}$$

Furthermore, equation (31) is transform into (95) by mean of the given change of variables

$$x = F(n,t) \tag{97}$$

where equation (97) is an inverse to the map $y = \Phi(n,t)$ as indicated below

$$\Phi(n,t) = \int \frac{1}{\phi(n,t)} dn. \tag{98}$$

7. CONCLUSION

Lie symmetry analysis for the stochastic SIRS model driven by the Wiener processes was achieved, infinitesimals generator of the Wiener processes $dW(t)$ were obtained by the mean of the moments invariance properties of the process. Determining equations of the stochastic model were described to have the similar property as for the deterministic model. Ultimately, the Lie bracket relations obtained shown that every infinitesimal Lie operators found are closed under the Lie bracket and therefore the Lie generators have formed a Lie algebra.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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