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GLOBAL STABILITY OF A FRACTIONAL ORDER SIR EPIDEMIC MODEL WITH DOUBLE EPIDEMIC HYPOTHESIS AND NONLINEAR INCIDENCE RATE

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Abstract. In this paper, we consider a fractional order SIR epidemic model with double epidemic hypothesis and specific functional response, where the fractional derivative is defined in the Caputo sense. The nonnegativity and boundedness of solutions in this system are proved. The basic reproduction number is obtained. Qualitative results show that the model has four equilibria: one disease-free equilibrium and three endemic equilibrium points. Local and global stability analysis of the equilibria are established.

Keywords: fractional order SIR epidemic model; local stability; global stability.

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1. Introduction

Fractional calculus is the field of mathematical analysis aiming at the investigation of integrals and derivatives of arbitrary (non integer) orders. In recent years, with the continuous development of fractional calculus theory, fractional differential equations are increasingly used

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to modeling many phenomena in different fields see, e.g., [1, 2, 3, 4]. The importance of modeling real phenomena using the fractional differential equations is due to these fractional systems naturally include both memory and nonlocality effects [5]. These effects are quite relevant to epidemic spread. Therefore, large numbers of researchers have started to study the epidemic models using the fractional differential equations, see, e.g., [6, 7, 8, 9, 10, 11]. Mouaouine et al. [10] proposed the following fractional order SIR epidemic model with nonlinear incidence rate

$$\begin{cases} D^{\alpha}S(t) = & \Lambda - \mu S(t) - \frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)+\alpha_3 S(t)I(t)}, \quad t \geq 0, \\ D^{\alpha}I(t) = & \frac{\beta S(t)I(t)}{1+\alpha_1 S(t)+\alpha_2 I(t)+\alpha_3 S(t)I(t)} - (\mu+d+r)I(t), \\ D^{\alpha}R(t) = & rI(t) - \mu R(t), \end{cases} \tag{1}$$

where D^{α} is the Caputo fractional derivative of order $\alpha \in (0,1]$ defined for a function $f \in C^1(\mathbb{R}_+,\mathbb{R})$ as follows [12]

$$D^{\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}f'(s)ds,$$

where Γ is the Gamma function defined by the integral

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}dt.$$

In system (1), $S(t)$, $I(t)$, and $R(t)$ represent the numbers of susceptible, infective, and recovered individuals at time t, respectively. Λ is the recruitment rate of the population, μ is the natural death rate, d is the death rate due to disease and r is the recovery rate of the infective individuals. The incidence rate of disease in model (1) is modeled by the specific functional response $\beta SI/(1+\alpha_1 S+\alpha_2 I+\alpha_3 SI)$, where β is the infection rate and $\alpha_1,\alpha_2,\alpha_3$ are saturation factors measuring the psychological or inhibitory effect. This specific functional response was introduced by Hattaf et al. [13], and here it becomes to be, a bilinear incidence rate if $\alpha_1 = \alpha_2 = \alpha_3 = 0$, a saturated incidence rate if $\alpha_1 = \alpha_3 = 0$ or $\alpha_2 = \alpha_3 = 0$, a Beddington-DeAngelis functional response [14, 15] if $\alpha_3 = 0$, and a Crowley-Martin functional response [16] if $\alpha_1\alpha_2 = \alpha_3$. According to the theory in [10], the basic reproduction number of model (1) is $R_0 = \beta\Lambda/[(\mu+\alpha_1\Lambda)(\mu+d+r)]$. Moreover, if $R_0 \leq 1$, model (1) has only the disease-free equilibrium $E_0 = (\frac{\Lambda}{\mu}, 0, 0)$ which is globally asymptotical stable, and if $R_0 > 1$, E_0 becomes unstable and system (1) has an endemic equilibrium which is globally asymptotically stable.

In classical epidemic models, there exists only one epidemic disease caused by one virus. In fact, there might be two epidemic diseases caused by two different viruses. Recently, many authors studied the epidemic models with double epidemic hypothesis, see, e.g., [17, 18, 19, 20, 21, 22, 23, 24, 25, 26]. In this paper, we propose the fractional epidemic model (1) with double epidemic hypothesis written as follows

$$(2)\quad \begin{cases} D^{\alpha}S(t) = & \Lambda - \mu S(t) - \frac{\beta_1 S(t) I_1(t)}{1+\alpha_1 S(t)+\gamma_1 I_1(t)+\rho_1 S(t) I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{1+\alpha_2 S(t)+\gamma_2 I_2(t)+\rho_2 S(t) I_2(t)}, \\ D^{\alpha}I_1(t) = & \frac{\beta_1 S(t) I_1(t)}{1+\alpha_1 S(t)+\gamma_1 I_1(t)+\rho_1 S(t) I_1(t)} - (\mu + r_1 + \lambda_1) I_1(t), \\ D^{\alpha}I_2(t) = & \frac{\beta_2 S(t) I_2(t)}{1+\alpha_2 S(t)+\gamma_2 I_2(t)+\rho_2 S(t) I_2(t)} - (\mu + r_2 + \lambda_2) I_2(t), \\ D^{\alpha}R(t) = & \lambda_1 I_1(t) + \lambda_2 I_2(t) - \mu R(t), \end{cases}$$

where $S(t)$ is the number of susceptible individuals at time t, $I_1(t)$ and $I_2(t)$ are the numbers of infected individuals with virus V_1 and V_2 at time t, respectively, and $R(t)$ is the number of individuals who have recovered, β_i is the transmission coefficients between S and I_i, $i = 1,2$. α_i, γ_i and ρ_i are saturation factors, r_i is the disease related death rate caused by virus V_i, λ_i is the recovery rate of the disease caused by virus V_i. As in model (1), $\alpha \in (0,1]$ is the order of the fractional derivative, Λ is the recruitment rate of susceptible individuals, μ is the natural death rate of the population. All parameters in model (2) are positive constants.

Since the three first equations in system (2) are independent of the four equation, system (2) can be reduced to the following equivalent system

$$(3)\quad \begin{cases} D^{\alpha}S(t) = & \Lambda - \mu S(t) - \frac{\beta_1 S(t) I_1(t)}{1+\alpha_1 S(t)+\gamma_1 I_1(t)+\rho_1 S(t) I_1(t)} - \frac{\beta_2 S(t) I_2(t)}{1+\alpha_2 S(t)+\gamma_2 I_2(t)+\rho_2 S(t) I_2(t)}, \\ D^{\alpha}I_1(t) = & \frac{\beta_1 S(t) I_1(t)}{1+\alpha_1 S(t)+\gamma_1 I_1(t)+\rho_1 S(t) I_1(t)} - (\mu + r_1 + \lambda_1) I_1(t), \\ D^{\alpha}I_2(t) = & \frac{\beta_2 S(t) I_2(t)}{1+\alpha_2 S(t)+\gamma_2 I_2(t)+\rho_2 S(t) I_2(t)} - (\mu + r_2 + \lambda_2) I_2(t). \end{cases}$$

The rest of this paper is organized as follows. In the next section, the existence of equilibria and the well-posedness of the model including existence, nonnegativity and boundedness of the solutions are established. In Section 3, we discuss the local stability of the equilibria of model (3). By constructing suitable Lyapunov functionals, the global stability of the equilibria is investigated in Section 4. A conclusion is given in Section 5.

2. WELL-POSEDNESS AND EQUILIBRIA

Theorem 2.1. *For any nonnegative initial condition, system (3) has a unique solution. Moreover, this solution remains nonnegative and bounded.*

Proof. *By using Theorem 3.1 and Remark 3.2 in* [27]*, it is easy to prove that system (3) has a unique solution* (S, I_1, I_2) *with any nonnegative initial condition. Now, we show the nonnegativity of this solution. From (3), one has*

$$\begin{cases} D^{\alpha} S|_{S=0} = \Lambda > 0 \ \textit{for all}\ I_1, I_2 \geq 0, \\ D^{\alpha} I_1|_{I_1=0} = 0 \ \textit{for all}\ S, I_2 \geq 0, \\ D^{\alpha} I_2|_{I_2=0} = 0 \ \textit{for all}\ S, I_1 \geq 0. \end{cases}$$

By Lemma 2.1 and Corollary 2.1 in [28]*, one can deduce that the solution of the fractional order system (3) is nonnegative. Next, we prove the boundedness of solution. Summing all the equations of system (3) we find that the total population size* $N(t) = S(t) + I_1(t) + I_2(t)$ *satisfies the inequality*

$$D^{\alpha} N(t) = \Lambda - \mu N(t) - (r_1 + \lambda_1) I_1(t) - (r_2 + \lambda_2) I_1(t) \leq \Lambda - \mu N(t).$$

By Lemma 3 in [29]*, we have*

$$N(t) \leq \left(N(0) - \frac{\Lambda}{\mu} \right) E_{\alpha}(-\mu t^{\alpha}) + \frac{\Lambda}{\mu}.$$

where $E_{\alpha}(z) = \sum\limits_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ *is the Mittag-Leffler function of parameter* α [30]*. Therefore,*

$$\limsup_{t \to \infty} N(t) \leq \frac{\Lambda}{\mu},$$

which implies that $S(t)$*,* $I_1(t)$ *and* $I_2(t)$ *are bounded.*

It is obvious that model (3) always has a disease-free equilibrium $E_0 = (S_0, 0, 0)$, where $S_0 = \frac{\Lambda}{\mu}$, that is, there is no infection present in the population and all individuals are susceptible.

By applying the next generation matrix approach provided by van den Driessche and Watmough [31], the basic reproduction number of model (3) is defined as follows

$$R_0 = \max\{R_{01}, R_{02}\},$$

where

$$R_{01} = \frac{\beta_1 \Lambda}{(\mu + \alpha_1 \Lambda)(\mu + r_1 + \lambda_1)} \text{ and } R_{02} = \frac{\beta_2 \Lambda}{(\mu + \alpha_2 \Lambda)(\mu + r_2 + \lambda_2)}.$$

Now, if $R_{01} > 1$, then system (3) has the endemic equilibrium $E_1^* = (S_1^*, I_1^*, 0)$, where

$$\begin{aligned} S_1^* &= \frac{\Lambda - \varpi_1 I_1^*}{\mu}, \\ I_1^* &= \frac{2(\mu + \alpha_1 \Lambda)(R_{01} - 1)}{\beta_1 - \alpha_1 \varpi_1 + \gamma_1 \mu + \rho_1 \Lambda + \sqrt{\Delta_1}}, \end{aligned}$$

with $\varpi_1 = \mu + r_1 + \lambda_1$ and

$$\begin{aligned} \Delta_1 &= [\beta_1 - \alpha_1 \varpi_1 + \gamma_1 \mu + \rho_1 \Lambda]^2 - 4\rho_1 [\beta_1 \Lambda - (\mu + \alpha_1 \Lambda)\varpi_1] \\ &= [\beta_1 - \alpha_1 \varpi_1 + \gamma_1 \mu - \rho_1 \Lambda]^2 + 4\rho_1 \mu (\varpi_1 + \gamma_1 \Lambda). \end{aligned}$$

Further, if $R_{02} > 1$, then system (3) has the endemic equilibrium $E_2^* = (S_2^*, 0, I_2^*)$, where

$$\begin{aligned} S_2^* &= \frac{\Lambda - \varpi_2 I_2^*}{\mu}, \\ I_2^* &= \frac{2(\mu + \alpha_2 \Lambda)(R_{02} - 1)}{\beta_2 - \alpha_2 \varpi_2 + \gamma_2 \mu + \rho_2 \Lambda + \sqrt{\Delta_2}}, \end{aligned}$$

with $\varpi_2 = \mu + r_2 + \lambda_2$ and

$$\begin{aligned} \Delta_2 &= [\beta_2 - \alpha_2 \varpi_2 + \gamma_2 \mu + \rho_2 \Lambda]^2 - 4\rho_2 [\beta_2 \Lambda - (\mu + \alpha_2 \Lambda)\varpi_2] \\ &= [\beta_2 - \alpha_2 \varpi_2 + \gamma_2 \mu - \rho_2 \Lambda]^2 + 4\rho_2 \mu (\varpi_2 + \gamma_2 \Lambda). \end{aligned}$$

Now, we investigate the existence of the positive endemic equilibrium $E_* = (S_*, I_*^1, I_*^2)$. For this, we rearranged system (3) to get I_1^* and I_2^* as follows

$$\begin{aligned} I_*^1 &= \frac{\left(\frac{\mu}{\Lambda} R_{01} + \alpha_1 (R_{01} - 1)\right) S_* - 1}{\gamma_1 + \rho_1 S_*}, \\ I_*^2 &= \frac{\left(\frac{\mu}{\Lambda} R_{02} + \alpha_2 (R_{02} - 1)\right) S_* - 1}{\gamma_2 + \rho_2 S_*}. \end{aligned}$$

In addition, S_* is given by the following cubic equation

$$C_0 S_*^3 + C_1 S_*^2 + C_2 S_* - C_3 = 0, \tag{4}$$

where

$$\begin{aligned}
C_0 &= \mu \rho_1 \rho_2 > 0, \\
C_1 &= \mu(\gamma_1 \rho_2 + \gamma_2 \rho_1) - \Lambda \rho_1 \rho_2 + \rho_1(\beta_2 - \alpha_2 \varpi_2) + \rho_2(\beta_1 - \alpha_1 \varpi_1), \\
C_2 &= \mu \gamma_1 \gamma_2 - \Lambda(\gamma_1 \rho_2 + \gamma_2 \rho_1) + \gamma_1(\beta_2 - \alpha_2 \varpi_2) - \rho_1 \varpi_2 + \gamma_2(\beta_1 - \alpha_1 \varpi_1) - \rho_2 \varpi_1, \\
C_3 &= \Lambda \gamma_1 \gamma_2 + \gamma_1 \varpi_2 + \gamma_2 \varpi_1 > 0.
\end{aligned}$$

With the help of Descartes' rule of signs [32], Eq. (4) has a unique positive real root S_* if any one of the following holds

(i) $C_1 > 0, C_2 > 0$.

(ii) $C_1 > 0, C_2 < 0$.

(iii) $C_1 < 0, C_2 < 0$.

Hence the positive endemic equilibrium E_* exists when $R_{01} > 1$, $R_{02} > 1$, one of the conditions (i), (ii) and (iii) hold true and $S_* > \max\limits_{i=1,2} \left\{ \frac{\Lambda}{\mu R_{0i} + \alpha_i \Lambda (R_{0i} - 1)} \right\}$.

3. Local Stability

In this section, we discuss the local stability of the equilibria of system (3).

Lemma 3.1. [33]. *Consider the fractional order system*

$$D^\alpha x(t) = f(x(t)), \;\; x(0) = x_0,$$

where $\alpha \in (0,1]$, $x(t) \in \mathbb{R}^n$ *and* $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ *is a nonlinear function. An equilibrium point of the above system is locally asymptotically stable if all the eigenvalues* ξ_j $(j = 1, 2, \ldots, n)$ *of the Jacobian matrix* $J = \frac{\partial f}{\partial x}$ *evaluated at the equilibrium satisfy* $\left|\arg(\xi_j)\right| > \frac{\alpha\pi}{2}$, *and unstable if there exist an eigenvalue* ξ_j *such that* $\left|\arg(\xi_j)\right| < \frac{\alpha\pi}{2}$.

The Jacobian matrix of system (3) at the equilibrium E_0 is as follows

$$J_{E_0} = \begin{pmatrix} -\mu & \frac{-\beta_1\Lambda}{\mu+\alpha_1\Lambda} & \frac{-\beta_2\Lambda}{\mu+\alpha_2\Lambda} \\ 0 & \frac{\beta_1\Lambda}{\mu+\alpha_1\Lambda} - (\mu + r_1 + \lambda_1) & 0 \\ 0 & 0 & \frac{\beta_2\Lambda}{\mu+\alpha_2\Lambda} - (\mu + r_2 + \lambda_2) \end{pmatrix}.$$

The three eigenvalues of J_{E_0} are $\xi_1 = -\mu < 0$, $\xi_2 = (\mu + r_1 + \lambda_1)(R_{01} - 1)$ and $\xi_3 = (\mu + r_2 + \lambda_2)(R_{02} - 1)$. So that all eigenvalues ξ_j $(j = 1,2,3)$ of J_{E_0} satisfy $\left|\arg(\xi_j)\right| = \pi > \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1]$ if $R_0 < 1$. Further, $|\arg(\xi_2)| = 0 < \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1]$ if $R_{01} > 1$ and $|\arg(\xi_3)| = 0 < \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1]$ if $R_{02} > 1$. Consequently, by Lemma 3.1, we have the following result.

Theorem 3.1. *If $R_0 < 1$, then the disease-free equilibrium E_0 is locally asymptotically stable. E_0 unstable if $R_0 > 1$.*

The Jacobian matrix of system (3) at the equilibrium E_1^* is determined by

$$J_{E_1^*} = \begin{pmatrix} -m_1 & -m_2 & -m_3 \\ m_4 & m_2 - m_5 & 0 \\ 0 & 0 & m_3 - m_6 \end{pmatrix},$$

where

$$\begin{aligned} m_1 &= \mu + \frac{\beta_1 I_1^*(1+\gamma_1 I_1^*)}{(1+\alpha_1 S_1^* + \gamma_1 I_1^* + \rho_1 S_1^* I_1^*)^2}, \\ m_2 &= \frac{\beta_1 S_1^*(1+\alpha_1 S_1^*)}{(1+\alpha_1 S_1^* + \gamma_1 I_1^* + \rho_1 S_1^* I_1^*)^2}, \\ m_3 &= \frac{\beta_2 S_1^*}{1+\alpha_2 S_1^*}, \\ m_4 &= \frac{\beta_1 I_1^*(1+\gamma_1 I_1^*)}{(1+\alpha_1 S_1^* + \gamma_1 I_1^* + \rho_1 S_1^* I_1^*)^2}, \\ m_5 &= \mu + r_1 + \lambda_1, \\ m_6 &= \mu + r_2 + \lambda_2. \end{aligned}$$

Clearly, $\xi_1 = \frac{\beta_2 S_1^*}{1+\alpha_2 S_1^*} - (\mu + r_2 + \lambda_2)$ is an eigenvalue of $J_{E_1^*}$. Since $S_1^* < \frac{\Lambda}{\mu}$ because $\Lambda - \mu S_1^* = \varpi_1 I_1^* > 0$ and the function $g : x \in \mathbb{R}_+ \mapsto \frac{\beta_2 x}{1+\alpha_2 x}$ is increasing, then $\xi_1 \leq g(\frac{A}{\mu}) - (\mu + r_2 + \lambda_2) = \frac{\beta_2\Lambda}{\mu+\alpha_2\Lambda} - (\mu + r_2 + \lambda_2) = (\mu + r_2 + \lambda_2)(R_{02} - 1)$. Hence $\xi_1 < 0$ if $R_{02} < 1$, then $|\arg(\xi_1)| = \pi > \frac{\alpha\pi}{2}$ for all $\alpha \in (0,1]$ if $R_{02} < 1$. The other two eigenvalues of $J_{E_1^*}$ are determined by the following equation

$$\xi^2 + P_1\xi + P_0 = 0,$$

where

$$P_1 = m_1 + m_5 - m_2,$$

$$P_0 = m_1(m_5 - m_2) + m_2 m_4.$$

Since

$$m_5 - m_2 = \frac{\beta_1 S_1^* I_1^* (\gamma_1 + \rho_1 S_1^*)}{(1 + \alpha_1 S_1^* + \gamma_1 I_1^* + \rho_1 S_1^* I_1^*)^2} > 0,$$

then $P_1 > 0$ and $P_0 > 0$. Thus the eigenvalues ξ_j $(j = 2, 3)$ of $J_{E_1^*}$ have negative real part, so that $\left|\arg(\xi_j)\right| > \frac{\pi}{2} \geq \frac{\alpha\pi}{2}$ for all $\alpha \in (0, 1]$ if $R_{01} > 1$. Hence, we have the following theorem.

Theorem 3.2. *If $R_{02} < 1 < R_{01}$, then the equilibrium E_1^* is locally asymptotically stable.*

As in the stability analysis of previous case E_1^* we have following result.

Theorem 3.3. *If $R_{01} < 1 < R_{02}$, then the equilibrium E_2^* is locally asymptotically stable.*

The Jacobian matrix of system (3) at the positive equilibrium E_* is determined by

$$J_{E_*} = \begin{pmatrix} -p_1 & -p_2 & -p_3 \\ p_4 & p_2 - p_5 & 0 \\ p_6 & 0 & p_3 - p_7 \end{pmatrix},$$

where

$$\begin{aligned}
p_1 &= \mu + \frac{\beta_1 I_*^1 (1 + \gamma_1 I_*^1)}{(1 + \alpha_1 S_* + \gamma_1 I_*^1 + \rho_1 S_* I_*^1)^2} + \frac{\beta_2 I_*^2 (1 + \gamma_2 I_*^2)}{(1 + \alpha_2 S_* + \gamma_2 I_*^2 + \rho_2 S_* I_*^2)^2}, \\
p_2 &= \frac{\beta_1 S_* (1 + \alpha_1 S_*)}{(1 + \alpha_1 S_* + \gamma_1 I_*^1 + \rho_1 S_* I_*^1)^2}, \\
p_3 &= \frac{\beta_2 S_* (1 + \alpha_2 S_*)}{(1 + \alpha_2 S_* + \gamma_2 I_*^2 + \rho_2 S_* I_*^2)^2}, \\
p_4 &= \frac{\beta_1 I_*^1 (1 + \gamma_1 I_*^1)}{(1 + \alpha_1 S_* + \gamma_1 I_*^1 + \rho_1 S_* I_*^1)^2}, \\
p_5 &= \mu + r_1 + \lambda_1, \\
p_6 &= \frac{\beta_2 I_*^2 (1 + \gamma_2 I_*^2)}{(1 + \alpha_2 S_* + \gamma_2 I_*^2 + \rho_2 S_* I_*^2)^2}, \\
p_7 &= \mu + r_2 + \lambda_2.
\end{aligned}$$

Theorem 3.4. *The positive endemic equilibrium E_* is locally asymptotically stable if it is exists.*

Proof. *The characteristic equation of Jacobian matrix J_{E_*} can be given as*

$$\xi^3+Q_2\xi^2+Q_1\xi+Q_0=0, \tag{5}$$

where

$$\begin{aligned}
Q_2 &= p_1+(p_5-p_2)+(p_7-p_3),\\
Q_1 &= p_1(p_5-p_2+p_7-p_3)+(p_5-p_2)(p_7-p_3)+p_2p_4+p_3p_6,\\
Q_0 &= p_1(p_5-p_2)(p_7-p_3)+p_2p_4(p_7-p_3)+p_3p_6(p_5-p_2).
\end{aligned}$$

Note that

$$\begin{aligned}
p_5-p_2 &= \frac{\beta_1S_*I_*^1(\gamma_1+\mu_1S_*)}{(1+\alpha_1S_*+\gamma_1I_*^1+\rho_1S_*I_*^1)^2}>0,\\
p_7-p_3 &= \frac{\beta_2S_*I_*^2(\gamma_2+\mu_2S_*)}{(1+\alpha_2S_*+\gamma_2I_*^2+\rho_2S_*I_*^2)^2}>0,
\end{aligned}$$

then it is easy to show that $Q_2>0$, $Q_1>0$, $Q_0>0$ and $Q_2Q_1>Q_0$. Thus by the Routh-Hurwitz criterion, all roots ξ_i $(i=1,2,3)$, of (5) have negative real part. Therefore, the equilibrium E_* of the system (3) is asymptotically stable.

4. GLOBAL STABILITY

In this section, we investigate the global stability of the four equilibria.

Theorem 4.1. *If $R_0\le 1$, then the disease-free equilibrium E_0 is globally asymptotically stable.*

Proof. *Let U_0 be the Lyapunov functional defined as*

$$U_0(t)=\frac{S_0}{(1+\alpha_1S_0)(1+\alpha_2S_0)}g\left(\frac{S}{S_0}\right)+\frac{1}{1+\alpha_2S_0}I_1+\frac{1}{1+\alpha_1S_0}I_2,$$

where $g(x)=x-1-\ln x$, $x>0$. According to Lemma 3.1 in [34], *one gets*

$$\begin{aligned}
D^\alpha U_0 \le\; & \frac{1}{(1+\alpha_1S_0)(1+\alpha_2S_0)}\left(1-\frac{S_0}{S}\right)D^\alpha S+\frac{1}{1+\alpha_2S_0}D^\alpha I_1+\frac{1}{1+\alpha_1S_0}D^\alpha I_2\\
=\; & -\frac{\mu}{(1+\alpha_1S_0)(1+\alpha_2S_0)}\frac{(S-S_0)^2}{S}-\frac{1}{(1+\alpha_1S_0)(1+\alpha_2S_0)}\left(1-\frac{S_0}{S}\right)\frac{\beta_1SI_1}{1+\alpha_1S+\gamma_1I_1+\rho_1SI_1}\\
& -\frac{1}{(1+\alpha_1S_0)(1+\alpha_2S_0)}\left(1-\frac{S_0}{S}\right)\frac{\beta_2SI_2}{1+\alpha_2S+\gamma_2I_2+\rho_2SI_2}+\frac{1}{1+\alpha_2S_0}\frac{\beta_1SI_1}{1+\alpha_1S+\gamma_1I_1+\rho_1SI_1}\\
& -\frac{\mu+r_1+\lambda_1}{1+\alpha_2S_0}I_1+\frac{1}{1+\alpha_1S_0}\frac{\beta_2SI_2}{1+\alpha_2S+\gamma_2I_2+\rho_2SI_2}-\frac{\mu+r_2+\lambda_2}{1+\alpha_1S_0}I_2
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\mu}{(1+\alpha_1 S_0)(1+\alpha_2 S_0)}\frac{(S-S_0)^2}{S}+\frac{\mu+r_1+\lambda_1}{1+\alpha_2 S_0}\left(R_{01}\frac{1+\alpha_1 S}{1+\alpha_1 S+\gamma_1 I_1+\rho_1 S I_1}-1\right)I_1\\
&\quad+\frac{\mu+r_2+\lambda_2}{1+\alpha_1 S_0}\left(R_{02}\frac{1+\alpha_2 S}{1+\alpha_2 S+\gamma_2 I_2+\rho_2 S I_2}-1\right)I_2\\
D^\alpha U_0 &\le -\frac{\mu}{(1+\alpha_1 S_0)(1+\alpha_2 S_0)}\frac{(S-S_0)^2}{S}+\frac{\mu+r_1+\lambda_1}{1+\alpha_2 S_0}(R_{01}-1)I_1+\frac{\mu+r_2+\lambda_2}{1+\alpha_1 S_0}(R_{02}-1)I_2.
\end{aligned}$$

Therefore, $R_0 \le 1$ ensures that $D^\alpha U_0 \le 0$. Furthermore, it is easy to verify that the singleton $\{E_0\}$ is the largest compact invariant set in $\left\{(S,I_1,I_2)\in\mathbb{R}^3_+ : D^\alpha U_0 = 0\right\}$. By Lemma 4.6 in [35]*, which generalized the integer order LaSalle's invariance principle to fractional order system, we conclude that E_0 is globally asymptotically stable if $R_0 \le 1$.*

Theorem 4.2. *If $R_{02} \le 1 < R_{01}$, then the equilibrium E_1^* is globally asymptotically stable.*

Proof. *Let U_1 be the Lyapunov functional defined as*

$$U_1(t)=\frac{1}{1+\alpha_2 S_0}\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}S_1^* g\left(\frac{S}{S_1^*}\right)+\frac{1}{1+\alpha_2 S_0}I_1^* g\left(\frac{I_1}{I_1^*}\right)+\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}I_2,$$

where $f_1(S,I_1)=1+\alpha_1 S+\gamma_1 I_1+\rho_1 S I_1$. We have

$$D^\alpha U_1 \le \frac{1}{1+\alpha_2 S_0}\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}\left(1-\frac{S_1^*}{S}\right)D^\alpha S+\frac{1}{1+\alpha_2 S_0}\left(1-\frac{I_1^*}{I_1}\right)D^\alpha I_1+\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}D^\alpha I_2.$$

Using

$$\Lambda=\mu S_1^*+\frac{\beta_1 S_1^* I_1^*}{f_1(S_1^*,I_1^*)},\quad \frac{\beta_1 S_1^* I_1^*}{f_1(S_1^*,I_1^*)}=(\mu+r_1+\lambda_1)I_1^*,$$

and

$$\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}\left(1-\frac{S_1^*}{S}\right)=\left(1-\frac{S_1^* f_1(S,I_1^*)}{S f_1(S_1^*,I_1^*)}\right),$$

we obtain

$$\begin{aligned}
D^\alpha U_1 &\le \frac{-\mu_0}{1+\alpha_2 S_0}\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}\frac{(S-S_1^*)^2}{S}+\frac{1}{1+\alpha_2 S_0}(\mu+r_1+\lambda_1)I_1^*\left(1-\frac{S_1^* f_1(S,I_1^*)}{S f_1(S_1^*,I_1^*)}\right)\left(1-\frac{S I_1 f_1(S_1^*,I_1^*)}{S_1^* I_1^* f_1(S,I_1)}\right)\\
&\quad-\frac{1}{1+\alpha_2 S_0}\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}\left(1-\frac{S_1^*}{S}\right)\frac{\beta_2 S I_2}{1+\alpha_2 S+\gamma_2 I_2+\rho_2 S I_2}\\
&\quad+\frac{1}{1+\alpha_2 S_0}(\mu+r_1+\lambda_1)I_1^*\left(1-\frac{I_1^*}{I_1}\right)\left(\frac{S I_1 f_1(S_1^*,I_1^*)}{S_1^* I_1^* f_1(S,I_1)}-\frac{I_1}{I_1^*}\right)\\
&\quad+\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}\frac{\beta_2 S I_2}{1+\alpha_2 S+\gamma_2 I_2+\rho_2 S I_2}-\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}(\mu+r_2+\lambda_2)I_2
\end{aligned}$$

$$\begin{aligned}
= \quad & \frac{-\mu}{1+\alpha_2 S_0}\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}\frac{(S-S_1^*)^2}{S}+\frac{1}{1+\alpha_2 S_0}(\mu+r_1+\lambda_1)I_1^*\left(2-\frac{S_1^* f_1(S,I_1^*)}{S f_1(S_1^*,I_1^*)}-\frac{S f_1(S_1^*,I_1^*)}{S_1^* f_1(S,I_1^*)}\right)\\
&+\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}\frac{\beta_2 I_2}{1+\alpha_2 S_0}\frac{S_1^*+\alpha_2 S S_0}{1+\alpha_2 S+\gamma_2 I_2+\rho_2 S I_2}-\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}(\mu+r_2+\lambda_2)I_2.
\end{aligned}$$

Since $S_1^* \leq S_0$ *because* $\Lambda-\mu S_1^*=\frac{\beta_1 S_1^* I_1^*}{f_1(S_1^*,I_1^*)}$, *then*

$$\begin{aligned}
D^\alpha U_1 \quad \leq \quad & \frac{-\mu}{1+\alpha_2 S_0}\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}\frac{(S-S_1^*)^2}{S}+\frac{1}{1+\alpha_2 S_0}(\mu+r_1+\lambda_1)I_1^*\left(2-\frac{S_1^* f_1(S,I_1^*)}{S f_1(S_1^*,I_1^*)}-\frac{S f_1(S_1^*,I_1^*)}{S_1^* f_1(S,I_1^*)}\right)\\
&+\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}(\mu+r_2+\lambda_2)\left(R_{02}\frac{1+\alpha_2 S}{1+\alpha_2 S+\gamma_2 I_2+\rho_2 S I_2}-1\right)I_2\\
\leq \quad & \frac{-\mu}{1+\alpha_2 S_0}\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}\frac{(S-S_1^*)^2}{S}+\frac{1}{1+\alpha_2 S_0}(\mu+r_1+\lambda_1)I_1^*\left(2-\frac{S_1^* f_1(S,I_1^*)}{S f_1(S_1^*,I_1^*)}-\frac{S f_1(S_1^*,I_1^*)}{S_1^* f_1(S,I_1^*)}\right)\\
&+\frac{f_1(0,I_1^*)}{f_1(S_1^*,I_1^*)}(\mu+r_2+\lambda_2)(R_{02}-1)I_2.
\end{aligned}$$

Using the arithmetic-geometric inequality, we have

$$2-\frac{S_1^* f_1(S,I_1^*)}{S f_1(S_1^*,I_1^*)}-\frac{S f_1(S_1^*,I_1^*)}{S_1^* f_1(S,I_1^*)}\leq 0.$$

Therefore, $R_{01}>1$ *and* $R_{02}\leq 1$ *ensures that* $D^\alpha U_1 \leq 0$. *Furthermore, it is easy to verify that the singleton* $\{E_1^*\}$ *is the largest compact invariant set in* $\left\{(S,I_1,I_2)\in\mathbb{R}_+^3 : D^\alpha U_1=0\right\}$. *By applying the LaSalle's invariance principle, we conclude that* E_1^* *is globally asymptotically stable if* $R_{02}\leq 1<R_{01}$.

Theorem 4.3. *If* $R_{01}\leq 1<R_{02}$, *then the equilibrium* E_2^* *is globally asymptotically stable.*

Proof. *It is analogue to the previous proof.*

Theorem 4.4. *The endemic equilibrium* E_* *is globally asymptotically stable if it is exists.*

Proof. *Consider the Lyapunov functional*

$$V(t)=\frac{f_1(0,I_*^1)f_2(0,I_*^2)}{f_1(S_*,I_*^1)f_2(S_*,I_*^2)}S_* g\left(\frac{S}{S_*}\right)+\frac{f_2(0,I_*^2)}{f_2(S_*,I_*^2)}I_*^1 g\left(\frac{I_1}{I_*^1}\right)+\frac{f_1(0,I_*^1)}{f_1(S_*,I_*^1)}I_*^2 g\left(\frac{I_2}{I_*^2}\right),$$

where $f_2(S,I_2)=1+\alpha_2 S+\gamma_2 I_2+\rho_2 S I_2$. *By Lemma 3.1 in* [34], *we have*

$$D^\alpha V \leq \frac{f_1(0,I_*^1)f_2(0,I_*^2)}{f_1(S_*,I_*^1)f_2(S_*,I_*^2)}\left(1-\frac{S_*}{S}\right)D^\alpha S+\frac{f_2(0,I_*^2)}{f_2(S_*,I_*^2)}\left(1-\frac{I_*^1}{I_1}\right)D^\alpha I_1+\frac{f_1(0,I_*^1)}{f_1(S_*,I_*^1)}\left(1-\frac{I_*^2}{I_2}\right)D^\alpha I_2.$$

Using

$$\Lambda=\mu S_*+\frac{\beta_1 S_* I_*^1}{f_1(S_*,I_*^1)}+\frac{\beta_2 S_* I_*^2}{f_2(S_*,I_*^2)},\ \ \frac{\beta_1 S_* I_*^1}{f_1(S_*,I_*^1)}=(\mu+r_1+\lambda_1)I_*^1,\ \ \frac{\beta_2 S_* I_*^2}{f_2(S_*,I_*^2)}=(\mu+r_2+\lambda_2)I_*^2,$$

and

$$\frac{f_1(0,I_*^1)}{f_1(S_*,I_*^1)}\left(1-\frac{S_*}{S}\right)=\left(1-\frac{S_*f_1(S,I_*^1)}{Sf_1(S_*,I_*^1)}\right),\ \ \frac{f_2(0,I_*^2)}{f_2(S_*,I_*^2)}\left(1-\frac{S_*}{S}\right)=\left(1-\frac{S_*f_2(S,I_*^2)}{Sf_2(S_*,I_*^2)}\right),$$

we obtain

$$\begin{aligned}
D^\alpha V \ \leq\ & -\mu\frac{f_1(0,I_*^1)f_2(0,I_*^2)}{f_1(S_*,I_*^1)f_2(S_*,I_*^2)}\frac{(S-S_*)^2}{S}\\
&+\frac{f_2(0,I_*^2)}{f_2(S_*,I_*^2)}(\mu+r_1+\lambda_1)I_*^1\left(1-\frac{S_*f_1(S,I_*^1)}{Sf_1(S_*,I_*^1)}\right)\left(1-\frac{SI_1f_1(S_*,I_*^1)}{S_*I_*^1f_1(S,I_1)}\right)\\
&+\frac{f_1(0,I_*^1)}{f_1(S_*,I_*^1)}(\mu+r_2+\lambda_2)I_*^2\left(1-\frac{S_*f_2(S,I_*^2)}{Sf_2(S_*,I_*^2)}\right)\left(1-\frac{SI_2f_2(S_*,I_*^2)}{S_*I_*^2f_2(S,I_2)}\right)\\
&+\frac{f_2(0,I_*^2)}{f_2(S_*,I_*^2)}(\mu+r_1+\lambda_1)I_*^1\left(1-\frac{I_*^1}{I_1}\right)\left(\frac{SI_1f_1(S_*,I_*^1)}{S_*I_*^1f_1(S,I_1)}-\frac{I_1}{I_*^1}\right)\\
&+\frac{f_1(0,I_*^1)}{f_1(S_*,I_*^1)}(\mu+r_2+\lambda_2)I_*^2\left(1-\frac{I_*^2}{I_2}\right)\left(\frac{SI_2f_2(S_*,I_*^2)}{S_*I_*^2f_2(S,I_2)}-\frac{I_2}{I_*^2}\right)\\
=\ & -\mu\frac{f_1(0,I_*^1)f_2(0,I_*^2)}{f_1(S_*,I_*^1)f_2(S_*,I_*^2)}\frac{(S-S_*)^2}{S}+\frac{f_2(0,I_*^2)}{f_2(S_*,I_*^2)}(\mu+r_1+\lambda_1)I_*^1\left(2-\frac{S_*f_1(S,I_*^1)}{Sf_1(S_*,I_*^1)}-\frac{Sf_1(S_*,I_*^1)}{S_*f_1(S,I_*^1)}\right)\\
&+\frac{f_1(0,I_*^1)}{f_1(S_*,I_*^1)}(\mu+r_2+\lambda_2)I_*^2\left(2-\frac{S_*f_2(S,I_*^2)}{Sf_2(S_*,I_*^2)}-\frac{Sf_2(S_*,I_*^2)}{S_*f_2(S,I_*^2)}\right).
\end{aligned}$$

Using the arithmetic-geometric inequality, we have

$$\begin{aligned}
2-\frac{S_*f_1(S,I_*^1)}{Sf_1(S_*,I_*^1)}-\frac{Sf_1(S_*,I_*^1)}{S_*f_1(S,I_*^1)} &\leq 0,\\
2-\frac{S_*f_2(S,I_*^2)}{Sf_2(S_*,I_*^2)}-\frac{Sf_2(S_*,I_*^2)}{S_*f_2(S,I_*^2)} &\leq 0.
\end{aligned}$$

Hence, $D^\alpha V\leq 0$. *Further, the largest invariant set of* $\left\{(S,I_1,I_2)\in\mathbb{R}_+^3 : D^\alpha V=0\right\}$ *is the singleton* $\{E_*\}$. *By applying the LaSalle's invariance principle, we can obtain that the endemic equilibrium* E_* *of model (3) is globally asymptotically stable.*

5. CONCLUSION

This paper presents a mathematical study on the dynamical behavior of a fractional order SIR epidemic model with double epidemic hypothesis and specific functional response. First, we have proved the existence, nonnegativity and boundedness of the solution for any nonnegative initial value. It also identified four equilibria, viz, an disease-free equilibrium-free E_0, disease-free equilibrium for I_2, E_1^*, disease-free equilibrium for I_1, E_2^* and both-endemic equilibrium E_*. We have derived sufficient conditions for local asymptotic stability of the equilibria. Next, we have established the global asymptotic stability of the equilibria. Finally, from our theoretical analysis, it can be concluded that the fractional order parameter α has no effect on the stability of the equilibria.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] A.M.A. El-Sayed, S.Z. Rida, and A.A.M. Arafa, Exact solutions of fractional-order biological population model, Commun. Theor. Phys. 52 (2009), 992-996.

[2] F.C. Meral, T.J. Royston, and R. Magin, Fractional calculus in viscoelasticity: An experimental study, Commun. Nonlinear Sci. Numer. Simul. 15 (2010), 939-945.

[3] D.A. Benson, M.M. Meerschaert, and J. Revielle, Fractional calculus in hydrologic modeling, a numerical perspective, Adv. Water Resour. 51 (2013), 479-497.

[4] J. Korbel and Y. Luchko, Modelling of financial processes with a space-time fractional diffusion equation of varying order, Fract. Calc. Appl. Anal. 19 (6) (2016), 1414-1433.

[5] E. Ahmed, A.M.A. El-Sayed, and H.A.A. El-Saka, Equilibrium points, stability and numerical solution of fractional-order predator-prey and rabies models, J. Math. Anal. Appl. 325 (2007), 542-553.

[6] N. ÖZalp and E. Demirci, A fractional order SEIR model with vertical transmission, Math. Comput. Model. 54(1-2) (2011), 1-6.

[7] A.M. Yousef and S.M. Salman, Backward bifurcation in a fractional-order SIRS epidemic model with a nonlinear incidence rate, Int. J. Nonlinear Sci. Numer. Simul. 17(7-8) (2016), 401-412.

[8] D. Rostamy and E. Mottaghi, Forward and backward Bifurcation in a fractional-order SIR epidemic model with vaccination, Iran. J. Sci. Technol. 42(2) (2018), 663-671.

[9] X. Wang, Z. Wang, X. Huang, and Y. Li, Dynamic analysis of a fractional-order delayed SIR model with saturated incidence and treatment functions, Int. J. Bifurc. Chaos, 28(14) (2018), 1850180.

[10] A. Mouaouine, A. Boukhouima, K. Hattaf, and Y. Noura, A fractional order SIR epidemic model with nonlinear incidence rate, Adv. Difference Equ. 2018 (2018), 160.

[11] M. Naim, F. Lahmidi, and A. Namir, Stability analysis of a delayed fractional order SIRS epidemic model with nonlinear incidence rate, Int. J. Appl. Math. 32(5) (2019), 733-745.

[12] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.

[13] K. Hattaf, N. Yousfi, and A. Tridane, Stability analysis of a virus dynamics model with general incidence rate and two delays, Appl. Math. Comput. 221 (2013), 514-521.

[14] J.R. Beddington, Mutual interference between parasites or predators and its effect on searching efficiency, J. Animal Ecol. 44(1) (1975), 331-341.

[15] D.L. DeAngelis, R.A. Goldsten, and R. Neill, A model for trophic interaction, Ecology 56(4) (1975), 881-892.

[16] P.H. Crowley and E. K. Martin, Functional responses and interference within and between year classes of a dragonfly population, J. North Amer. Benth. Soc. 8(3) (1989), 211-221.

[17] X. Meng, Z. Li, and X. Wang, Dynamics of a novel nonlinear SIR model with double epidemic hypothesis and impulsive effects, Nonlinear Dyn. 59(3) (2010), 503-513.

[18] X. Meng, S. Zhao, T. Feng, and T. Zhang, Dynamics of a novel nonlinear stochastic SIS epidemic model with double epidemic hypothesis, J. Math. Anal. Appl. 433 (1) (2016), 227-242.

[19] A. Miao, X. Wang, T. Zhang, W. Wang, and B. Sampath Aruna Pradeep, Dynamical analysis of a stochastic SIS epidemic model with nonlinear incidence rate and double epidemic hypothesis, Advances in Difference Equations 2017 (2017), 226.

[20] H. Qi, L. Liu, and X. Meng, Dynamics of a non-autonomous stochastic SIS epidemic model with double epidemic hypothesis, Complexity 2017 (2017), Article ID 4861391.

[21] X. Zhang, D. Jiang, T. Hayat, and B. Ahmad, Dynamics of a stochastic SIS model with double epidemic diseases driven by Lévy jumps, Physica A, Stat. Mech. Appl. 471 (2017), 767-777.

[22] Q. Liu and D. Jiang, Stationary distribution of a stochastic SIS epidemic model with double diseases and the Beddington-DeAngelis incidence, Chaos 27 (8) (2017), 083126.

[23] J. Zhao, L. Wang, and Z. Han, Stability analysis of two new SIRS models with two viruses, Int. J. Comput. Math. 95 (10) (2017), 2026-2035.

[24] S.P. Rajasekar and M. Pitchaimani, Qualitative analysis of stochastically perturbed SIRS epidemic model with two viruses, Chaos Solitons Fractals 118 (2019), 207-221.

[25] M. Naim, F. Lahmidi, and A. Namir, Mathematical analysis of a fractional order SIS epidemic model with double diseases, Beddington-DeAngelis functional response and time delay, Int. J. Nonlinear Sci. 29(1) (2020), 47-59.

[26] M. Naim and F. Lahmidi, Analysis of a deterministic and a stochastic SIS epidemic model with double epidemic hypothesis and specific functional response, Discrete Dyn. Nat. Soc. 2020 (2020), Article ID 5362716.

[27] W. Lin, Global existence theory and chaos control of fractional differential equations, J. Math. Anal. Appl. 332 (2007), 709-726.

[28] S. Mondal, M. Biswas, and N. Bairagi, Local and global dynamics of a fractional-order predator-prey system with habitat complexity and the corresponding discretized fractional-order system, J. Appl. Math. Comput. 63 (2020), 311-340.

[29] H.L. Li, L. Zhang, C. Hu, Y.L. Jiang, and Z. Teng, Dynamical analysis of a fractional-order prey-predator model incorporating a prey refuge, J. Appl. Math. Comput. 54 (2017), 435-449.

[30] K. Diethelm. The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type. Lecture Notes in Mathematics, Springer-Verlag, Berlin, Germany, 2010.

[31] P. van den Driessche and I. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180(1) (2002), 29-48.

[32] X. Wang, A simple proof of Descartes's rule of signs, Amer. Math. Mon. 111(6) (2004), 525-526.

[33] I. Petras, Fractional-order Nonlinear Systems: Modeling Analysis and Simulation, Higher Education Press, Beijing, 2011.

[34] C. Vargas-De-León, Volterra-type Lyapunov functions for fractional-order epidemic systems, Commun. Nonlinear Sci. Numer. Simul. 24 (2015), 75-85.

[35] J. Huo, H. Zhao, and L. Zhu, The effect of vaccines on backward bifurcation in a fractional order HIV model, Nonlinear Anal., Real World Appl. 26 (2015), 289-305.