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OPTIMAL CONTROL STRATEGY WITH DELAY IN STATE AND CONTROL VARIABLES OF TRANSMISSION OF COVID-19 PANDEMIC VIRUS

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Abstract. In this work, we are interested in the study of the dynamics of transmission of COVID-19 disease, described by SIR epidemic model, with time delay in state and control variables. Our goal is to characterize the optimal control pair, which minimizes the number of susceptible and infected individuals and maximizes the number of people recovered. Pontryagin's maximum principle with delay is used to characterize these optimal controls. The numerical resolution of the optimal system, has shown the effectiveness of our adopted strategy.

Keywords: optimal control; Pontryagin maximum with delay; SIR epidemic model; COVID-19; delay differential equations.

2010 AMS Subject Classification: 93A30.

1. INTRODUCTION

Coronaviruses are a large family of viruses that can be pathogenic in animals or humans. We know that several coronaviruses in humans can cause respiratory infections ranging from the

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common cold to more serious illnesses such as Middle East Respiratory Syndrome (MERS) and Severe Acute Respiratory Syndrome (SARS). The last coronavirus that was discovered is responsible for coronavirus disease 2019 (COVID-19). This new virus and disease was unknown before the outbreak in Wuhan (China) in December 2019. COVID-19 is mainly spread by respiratory droplets expelled by people who cough or have other symptoms, such as fever or fatigue. Many sufferers have only mild symptoms. Several mathematical models have been developed to simulate, analyze and understand the Coronavirus [1,2]. We can cite, Tahir and al [3] who proposed prevention strategies for the MERS-Coronavirus mathematical model with stability analysis and control optimal, Zhi-Qiang Xia et al. [4] modeled the transmission of the Middle East respiratory syndrome Coronavirus in the Republic of Korea. To describe the dynamics of a population infected with the COVID-19 virus, we consider a general SIR epidemic model with incubation time and a modified saturated incidence rate [5,6,7], the model used consists of three compartments SIR: susceptible, infected and recovered. We note their densities at time t by. S, I, R , respectively:

$$(1) \quad \left\{ \begin{array}{l} \frac{dS}{dt} = D - \beta \frac{S(t)I(t)}{1 + \alpha_1 S(t) + \alpha_2 I(t)} - dS \\ \frac{dI}{dt} = \beta \frac{S(t-\tau_1)I(t-\tau_1)}{1 + \alpha_1 S(t-\tau_1) + \alpha_2 I(t-\tau_1)} - (\gamma + d + \varepsilon)I(t) \\ \frac{dR}{dt} = \gamma I(t) - dR(t) \end{array} \right.$$

The initial condition for the above system is $S(\eta) = \psi_1(\eta)$, $I(\eta) = \psi_2(\eta)$, and $R(\eta) = \psi_3(\eta)$, $\eta \in [-\tau_1, 0]$. With $\psi = (\psi_1, \psi_2, \psi_3) \in (C^+)$, such that $\psi_i(\eta) \geq 0$ ($-\tau_1 \leq \eta \leq 0$, $i = 1, 2, 3$). Here C denotes the Banach space $C([-\tau_1, 0], IR)$ of continuous functions mapping the interval $[-\tau_1, 0]$ into IR , equipped with the supremum norm. The nonnegative cone of C is defined as $C^+ = C([-\tau_1, 0], IR^+)$. The parameters are defined as follows:

| Parameter | Definition |
|---------------------------|--|
| β | is the transmission rate |
| d | is the natural mortality rate |
| γ | is the recovery rate of the infectious individuals |
| α_1 and α_2 | the parameters that measure the inhibitory effect |
| ε | is the disease induced death rate |
| D | is the recruitment rate of susceptibles |
| τ_1 | is the incubation period |

Table 1:Parameter definition

To minimize the number of susceptible and infected individuals and to maximize the individuals recovered, We have adopted a control strategy consists of optimally combine, the raising people's awareness of the danger of covid 19 in order to take hygienic precautions necessary, and treatment of infected individuals, so that the cost of implementing the two interventions is minimized. The organization of this work is as follows. In Sect. 2, we formulate the optimal control problem and we use the principle of Pontryagin Maximum with delay given in Göllmann et al. [8] to characterize it. In Sect. 3, we give the numerical method and the simulation results. Finally, we draw conclusions in Sect. 4.

2. OPTIMAL CONTROL PROBLEM

Our goal is to minimize the number of susceptible and infected individuals, and to maximize the recovered individuals. We include two controls (u, v) , which represents the percentage of susceptible and infected individuals sensitized by the danger of the covid 19 and treated respectively per unit of time. In order to have a realistic model, we take in account that the transfer of infected individuals treated from the class of infected to the recovered class is subject to delay. Thus, the delay is introduced into the system as follows: at the time t only a percentage of infected individuals who have been treated. There is τ_2 unit of time, i.e. at times $t - \tau_2$, are deleted from the infected class and added to the recovered class. So the model (1) becomes:

$$(2) \quad \left\{ \begin{array}{l} \frac{dS}{dt} = D - \beta \frac{S(t)I(t)}{1 + \alpha_1 S(t) + \alpha_2 I(t)} - dS - u(t)S(t) \\ \frac{dI}{dt} = \beta \frac{S(t-\tau_1)I(t-\tau_1)}{1 + \alpha_1 S(t-\tau_1) + \alpha_2 I(t-\tau_1)} - (\gamma + d + \varepsilon)I(t) - v(t - \tau_2)I(t - \tau_2) \\ \frac{dR}{dt} = \gamma I(t) - dR(t) + u(t)S(t) + v(t - \tau_2)I(t - \tau_2) \end{array} \right.$$

with initial data $(S_0, I_0, R_0) \in (C^+)^3$. It is easy to show that there exists a unique solution $(S(t), I(t), R(t))$ of system (2).

$$(3) \quad S_0 = S(0); I_0 = I(0); R_0 = R(0);$$

For biological reasons, we assume that the initial data for system (2) satisfy:

$$S(t) \geq 0; I(t) \geq 0; R(t) \geq 0.$$

The problem is to minimize the objective functional given by:

$$(4) \quad J(u, v) = \int_0^{t_f} [AS(t) + BI(t) - ER(t) + \frac{1}{2}c_1 u^2(t) + \frac{1}{2}c_2 v^2(t)] dt$$

Where A, B, F, c_1 and c_2 are positive constants to keep a balance size of the terms. We seek the optimal control (u^*, v^*) such that:

$$(5) \quad \{J(u^*, v^*) = \min\{J(u, v) / (u, v) \in U_{ad}\}$$

where $U_{ad} = \{(u, v) / u(t) \text{ and } v(t) \text{ are measurable } 0 \leq u(t) \leq u^{max}, 0 \leq v(t) \leq v^{max}, t \in [0, t_f]\}$.

2.1 The existence of a Optimal Control. The existence of the optimal control pair can be obtained using a result by Fleming and Rishel in [9] and by Lukes in [10].

Theorem 1. *There exists an optimal control pair $(u^*, v^*) \in U_{ad}$ such that*

$$J(u^*, v^*) = \min_{(u, v) \in U_{ad}} (J(u, v)).$$

Proof. To prove the existence of an optimal control pair it is easy to verify that:

1. The set of controls and corresponding state variables is nonempty.
2. The admissible set U_{ad} is convex and closed.
3. The right hand side of the state system (2) is bounded by a linear function in the state and control variables.
4. The integrand of the objective functional is convex on U_{ad} .
5. There exists constants $b_1 > 0$; $b_2 > 0$ and $\theta > 0$ such that the integrand $L(S, I, R, u, v)$ of the objective functional satisfies:

$$L(S, I, R, u, v) \geq b_1 + b_2(|u|^2 + |v|^2)^{\theta/2} \quad \square$$

2.1. Optimality System. Pontryagin's minimum Principle with delay given in [8] provides necessary conditions for an optimal control problem. This principle converts (2), (4), and (5) into a problem of minimizing an Hamiltonian, H , with

$$(6) \quad H = AS(t) + BI(t) - ER(t) + \frac{1}{2}c_1u^2(t) + \frac{1}{2}c_2v^2(t) + \sum_{i=1}^3 \lambda_i f_i$$

where f_i is the right side of the differential equation of the state variable, and λ_i , $i = 1, 2, 3$ are the adjoint functions.

Theorem 2. *Given an optimal control $u^*, v^* \in U$ and solutions S^*, I^* and R^* of the corresponding state sysem (2 -4), there exist adjoint functions λ_1, λ_2 , and λ_3 satisfying*

$$(7) \quad \begin{cases} \frac{d\lambda_1(t)}{dt} = -A + \lambda_1(t)(\Phi_1 + d + u^*(t)) - \lambda_3(t)u^*(t) - \chi_{[0, t_f - \tau_1]}(t + \tau_1)\lambda_2(t + \tau_1)\Phi_1 \\ \frac{d\lambda_2(t)}{dt} = -B + \lambda_1(t)\Phi_2 + \lambda_2(t)(\gamma + d + \varepsilon) - \lambda_3(t)\gamma - \chi_{[0, t_f - \tau_1]}(t + \tau_1)\lambda_2(t + \tau_1)\Phi_2 \\ \quad + \chi_{[0, t_f - \tau_2]}(t + \tau_2)(\lambda_2(t + \tau_2) - \lambda_3(t + \tau_2))v^*(t) \\ \frac{d\lambda_3(t)}{dt} = E + \lambda_3(t)d \end{cases}$$

where $\Phi_1 = \frac{\beta I^*(1 + \alpha_2 I^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2}$ and $\Phi_2 = \frac{\beta S^*(1 + \alpha_1 S^*)}{(1 + \alpha_1 S^* + \alpha_2 I^*)^2}$ with transversality conditions $\lambda_i(t) = 0$, $i = 1, 2, 3$.

Furthermore, the optimal control pair (u^*, v^*) is given by

$$(8) \quad u^* = \min(u^{max}, \min(\frac{(\lambda_1 - \lambda_3)S^*}{c_1}, 0))$$

$$(9) \quad v^* = \min(v^{max}, \min(\chi_{[0,t_f-\tau_2]} \frac{(\lambda_2^+ - \lambda_3^+)I^*}{c_2}, 0))$$

Proof. Using the Pontryagin's Maximum principle with delay in state and control variables, we obtain the adjoint equations and transversality conditions such that:

$$(10) \quad \frac{d\lambda_1(t)}{dt} = -\frac{\partial H}{\partial S} - \chi_{[0,t_f-\tau_1]} \frac{\partial H(t+\tau_1)}{\partial S(t-\tau_1)} - \chi_{[0,t_f-\tau_2]} \frac{\partial H(t+\tau_2)}{\partial S(t-\tau_2)} \lambda_1(t_f) = 0$$

$$(11) \quad \frac{d\lambda_2(t)}{dt} = -\frac{\partial H}{\partial I} - \chi_{[0,t_f-\tau_1]} \frac{\partial H(t+\tau_1)}{\partial I(t-\tau_1)} - \chi_{[0,t_f-\tau_2]} \frac{\partial H(t+\tau_2)}{\partial I(t-\tau_2)} \lambda_2(t_f) = 0$$

$$(12) \quad \frac{d\lambda_3(t)}{dt} = -\frac{\partial H}{\partial R} - \chi_{[0,t_f-\tau_1]} \frac{\partial H(t+\tau_1)}{\partial R(t-\tau_1)} - \chi_{[0,t_f-\tau_2]} \frac{\partial H(t+\tau_2)}{\partial R(t-\tau_2)} \lambda_3(t_f) = 0$$

The optimal control (u^*, v^*) pair can be solved from the optimality condition:

$$(13) \quad \frac{\partial H}{\partial u} = c_1 u - \lambda_1 S + \lambda_3 S = 0$$

$$(14) \quad \frac{\partial H}{\partial v} + \chi_{[0,t_f-\tau_2]} \frac{\partial H(t+\tau_2)}{\partial v(t-\tau_2)} = c_2 v + \chi_{[0,t_f-\tau_2]} (\lambda_3^+ - \lambda_2^+) I = 0$$

with $\lambda_i^+ = \lambda_i^+(t+\tau_2)$ $i = 2, 3$. □

3. NUMERICAL RESULTS AND DISCUSSIONS

In this section, we solve numerically the optimality [11] system (7) which is a tow-point boundary value problem, with separated boundary conditions at times $t_0 = 0$, and t_f and we present the results found.

Let there exists a step size $h > 0$, and integers $(n, m) \in \mathbb{N}^2$ with $\tau = mh$ and $t_f - t_0 = nh$.

For reasons of programming, we consider knots to left of t_0 and right of t_f , and we obtain the following partition:

$$\Delta = (t_{-m} = -\tau < \dots < t_{-1} < t_0 = 0 < t_1 < \dots < t_n = t_f < \dots < t_{n+m}).$$

Then, we have $t_i = ih$ $(-m \leq i \leq n+m)$. Next we define the state and adjoint variables $S(t), I(t), R(t), \lambda_1(t), \lambda_2(t), \lambda_3(t)$ and $u(t), v(t)$ in terms of nodal points $S_i, I_i, R_i, \lambda_1^i, \lambda_2^i, \lambda_3^i, u^i$ and v^i .

Now using combination of forward and backward difference approximations, we obtain the Algorithm 1.

Algorithm 1

Step1

$$m = \max(m_1, m_2)$$

for $i = -m, \dots, 0$, do

$$S_i = S_0, I_i = I_0, R_i = R_i, u^i = 0 \text{ and } v^i = 0, ,$$

end for

for $i = n, \dots, n + m$, do

$$\lambda_1^i = 0, \lambda_2^i = 0, \lambda_3^i = 0,$$

end for

Step2

$$S_{i+1} = S_i + h[D - \frac{\beta S_i I_i}{1 + \alpha_1 S_i + \alpha_2 I_i} - dS_i - u_i S_i]$$

$$I_{i+1} = I_i + h[\frac{\beta S_{i+m_1} I_{i+m_1}}{1 + \alpha_1 S_{i+m_1} + \alpha_2 I_{i+m_1}} - (\gamma + d + \varepsilon)I_i - v_{i+m_2} I_{i+m_2}]$$

$$R_{i+1} = R_i + h[\gamma I_i - dR_i + u_i S_i + v_{i+m_2} I_{i+m_2}]$$

$$\lambda_1^{n-i-1} = \lambda_1^{n-i} + h[-A + \lambda_1^{n-i}(\Phi_1^{i+1} + d + u^i) - \lambda_3^{n-i} u^i - \chi_{[0, t_f - \tau_1]}(t_{n-i}) \lambda_2^{n-i} \Phi_1^{i+1}]$$

$$\lambda_2^{n-i-1} = \lambda_2^{n-i} + h[-B + \lambda_1^{n-i}(\Phi_2^{i+1}) + \lambda_2^{n-i}(\gamma + d + \varepsilon) - \lambda_3^{n-i} \gamma - \chi_{[0, t_f - \tau_1]}(t_{n-i}) \lambda_2^{n-i} \Phi_2^{i+1} +$$

$$\chi_{[0, t_f - \tau_2]}(t_{n-i})(\lambda_2^{n-i} - \lambda_3^{n-i})v_{i+m_2}]$$

$$\lambda_3^{n-i-1} = \lambda_3^{n-i} + h[E + \lambda_3^{n-i} d]$$

$$u^i = \min(u^{\max}, \min(\frac{(\lambda_1^{n-i-1} - \lambda_3^{n-i-1})S_{i+1}}{c_1}, 0))$$

$$v^i = \min(v^{\max}, \min(\chi_{[0, t_f - \tau_2]} \frac{(\lambda_2^{n-i-1+m_2} - \lambda_3^{n-i-1+m_2})I_{i+1}}{c_2}, 0))$$

Step3

for $i = 1, \dots, n$, do

$$\text{write } S^*(t_i) = S_i, I^*(t_i) = I_i, R^*(t_i) = R_i, u^*(t_i) = u_i \text{ and } v^*(t_i) = v_i,$$

end for

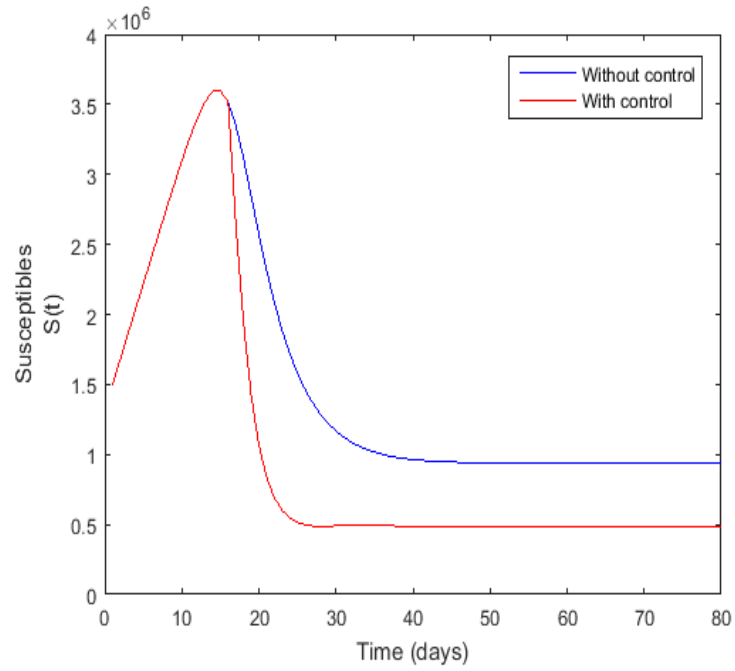


FIGURE 1. Susceptible behavior with and without control.

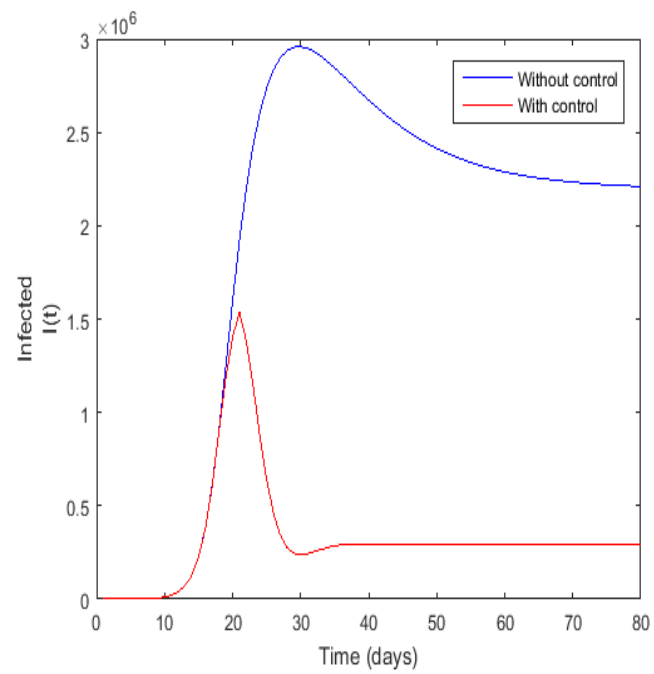


FIGURE 2. Infected behavior with and without control

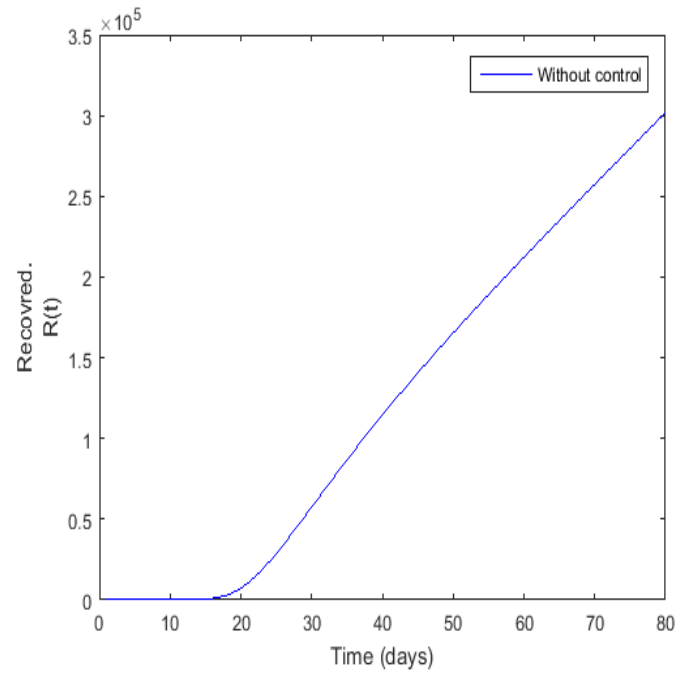


FIGURE 3. Recovered behavior without control

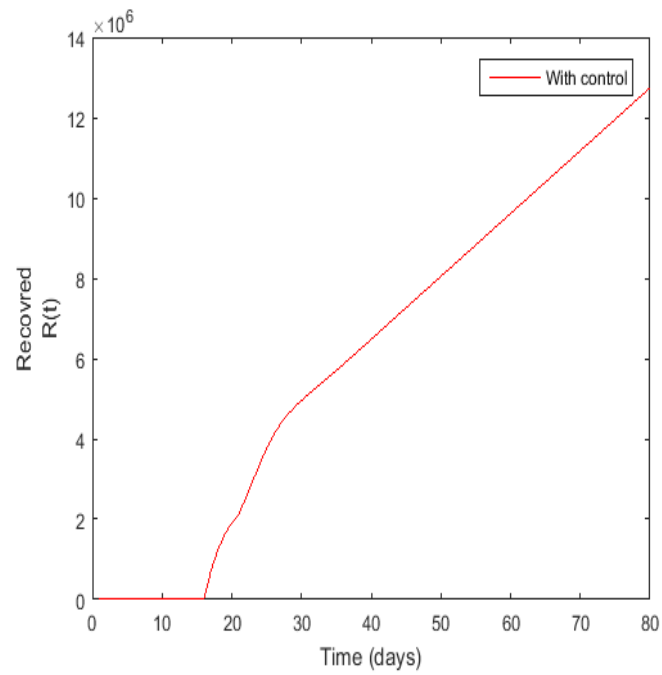
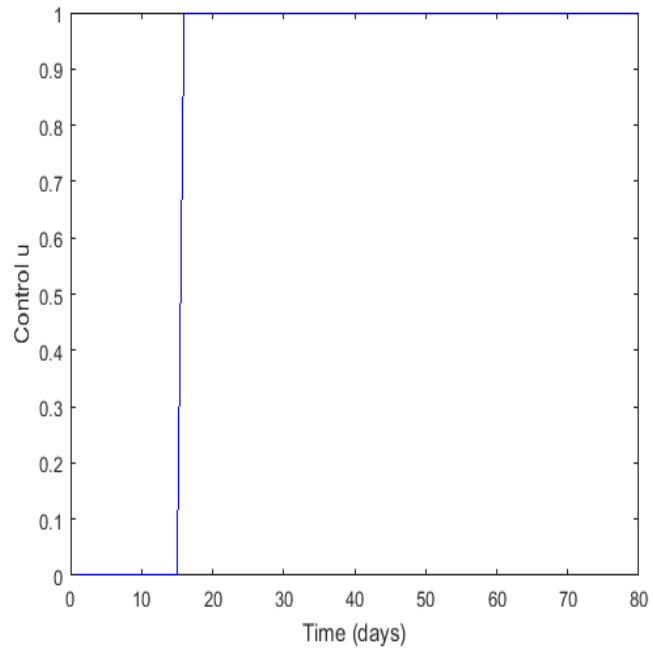
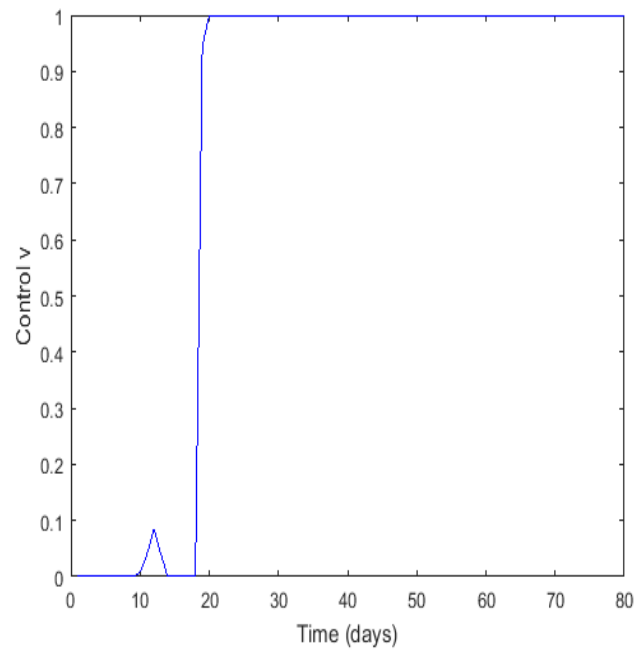


FIGURE 4. Recovered behavior with control

FIGURE 5. The optimal control u^* FIGURE 6. The optimal control v^*

For this simulation, we use the parameter values given in Table 2.

| Parameter | Definition | Values |
|---------------|--|---|
| α_1 | Parameter that measure the inhibitory effect | 0.01 people^{-1} |
| α_2 | Parameter that measure the inhibitory effect | 0.09 people^{-1} |
| β | is the transmission rate | $0.09 \text{ people}^{-1} \text{ day}^{-1}$ |
| d | is the natural mortality rate | 10^{-4} day^{-1} |
| γ | is the recovery rate of the infectious individuals | 0.01 day^{-1} |
| ε | is the disease induced death rate | 0.4 day^{-1} |
| D | is the recruitment rate of susceptibles | $9 \cdot 10^5 \text{ day}^{-1}$ |
| τ_1 | is the incubation period | 2 day |
| τ_2 | is the delay time | 3 day |

Table 2: Parameter defintion

We choose the following initial values: $S(0) \doteq 15 \cdot 10^5 \text{ peoples}$, $I(0) \doteq 30 \text{ peoples}$ and $R(0) \doteq 0 \text{ peoples}$.

The weight constant values in the objective functional are $A \doteq 90$, $B \doteq 1200$, $E \doteq 20$ and $c_1 \doteq 10^9$, $c_2 \doteq 10^9$.

For the time interval from $t = 1$ to $t = 80$ days, in figure (1-2) when we use control strategies, we can see that the number of susceptible and infected individuals controlled decreases in important way compared to uncontrolled cases.

This result is confirmed numerically, since the maximum number of infected individuals in the case with control is 1.5×10^6 and 3×10^6 in the case without control. And after the 30 days, the number of susceptible individuals decreases around 0.5×10^6 in the case with control and 10^6 in the case without control. In figure (4), in the presence of the control, the number of individuals recovered (R) increases sharply, compared to the case not controlled in figure (3). Finally, Figures 5 and 6 show the optimal controls u and v obtained to block the spread of the disease.

4. CONCLUSION

In this work, we propose a more realistic controlled model by including, on the one hand, the time delay which represents the time necessary for the transfer from the susceptible class to the infected class, and on the other hand, take into consideration the time delay in going from the infected class to recovered class after the proposed treatment. In this context, we applied the optimal control theory to prove the existence of the optimal control pair, and its explicit expression was obtained by using Pontryagin's maximum principle with delay. For the numerical simulation, we propose an algorithm based on the forward and backward difference approximation. The numerical results show that our optimal strategy is effective in reducing the number of susceptible and infected individuals and maximizing the recovered individuals.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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