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STABILITY OF A NONLINEAR DISCRETE-TIME MODEL OF INFORMATION DISSEMINATION UNDER STOCHASTIC PERTURBATIONS

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Abstract. A nonlinear discrete-time model of information dissemination is considered and conditions for existence of a positive equilibrium of this system are obtained. It is shown that asymptotically stable positive equilibrium saves his stability under the influence of stochastic perturbations of the different types: small multiplicative perturbations, quickly fading multiplicative perturbations and quickly fading additive perturbations. Stability conditions are obtained using Lyapunov functions, are formulated in the terms of linear matrix inequalities (LMIs) and are illustrated by numerical simulations of solutions of the considered system. As an unsolved problem it is proposed to investigate the situation when stochastic perturbations fade on the infinity, but not very quickly. The proposed research method can be applied to investigate many other nonlinear mathematical models in various applications.

Keywords: stochastic perturbations; Lyapunov functions; linear matrix inequalities (LMIs); asymptotic mean square stability; stability in probability; numerical simulations.

2010 AMS Subject Classification: 39A05, 39A45, 39A60, 93C35, 93C55.

1. Introduction

The model of information dissemination during the last years is enough popular in research (see, for instance, [1, 2, 3, 4, 5, 8] and references therein). This model consists of the following compartments: Ignorants (I), Sharers or Spreaders (S), and Removed (R) people. The term

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"Ignorant" means a person that does not know about the information. The word "Sharer" is used to denote that a person is attracted by the information and/or he finds it funny or interesting, then he decides to share it. The term "Removed" means a person who has seen the post and has decided for some personal reasons not to share it.

It is supposed that the quantity of members of each group depends on the time moment $i=0,1,...$, i.e., I_i, S_i, R_i, but the quantity of members of all groups is a constant, i.e., $I_i+S_i+R_i=N=const$. So, the model of information dissemination is described via the system of three nonlinear difference equations [3]

$$\begin{aligned}\Delta I_i =&\mu N-\mu I_i-\frac{\beta I_i S_i}{N},\\ \Delta S_i =&-\mu S_i+\frac{\beta_1 I_i S_i}{N}-\frac{\gamma R_i S_i}{N},\\ \Delta R_i =&-\mu R_i+\frac{\beta_2 I_i S_i}{N}+\frac{\gamma R_i S_i}{N},\end{aligned} \tag{1.1}$$

where $\Delta I_i = I_{i+1}-I_i$ and the same for all other variables.

Below conditions are obtained by which the system (1.1) has a positive equilibrium and stability of this equilibrium is investigated under different types of stochastic perturbations.

2. Existence of a Positive Equilibrium

Putting in (1.1) $I_i=I^*$, $S_i=S^*$, $R_i=R^*$, we obtain the system of algebraic equations for the system (1.1) equilibrium $E^*=\{I^*,S^*,R^*\}$

$$\begin{aligned}\mu N =& \frac{\beta I^* S^*}{N}+\mu I^*,\\ \frac{\beta_1 I^* S^*}{N} =& \frac{\gamma R^* S^*}{N}+\mu S^*,\\ \frac{\beta_2 I^* S^*}{N}+\frac{\gamma R^* S^*}{N} =& \mu R^*.\end{aligned} \tag{2.1}$$

From two first equations (2.1) we have

$$S^*=\frac{\mu N(N-I^*)}{\beta I^*}, \quad R^*=\frac{\beta_1 I^*-\mu N}{\gamma}. \tag{2.2}$$

Substituting (2.2) into the equality $N=I^*+S^*+R^*$, we obtain the equation for I^*

$$N=I^*+\frac{\mu N(N-I^*)}{\beta I^*}+\frac{\beta_1 I^*-\mu N}{\gamma}. \tag{2.3}$$

Using the notations

$$
\begin{aligned}
a_0 &= \beta(\beta_1+\gamma), \quad a_1 = (\beta\mu+\beta\gamma+\gamma\mu)N, \quad a_2 = \gamma\mu N^2, \\
b &= \frac{\mu N}{\beta_1}, \quad F(b) = a_0 b^2 - a_1 b + a_2,
\end{aligned}
\tag{2.4}
$$

the Eq. (2.3) can be transformed to the quadratic equation

$$
F(I^*) = a_0(I^*)^2 - a_1 I^* + a_2 = 0. \tag{2.5}
$$

Lemma 2.1. *Let us suppose that $a_1^2 > 4a_0a_2$, i.e., the Eq.* (2.5) *has two roots*

$$
I_1^* = \frac{a_1+\sqrt{a_1^2-4a_0a_2}}{2a_0}, \quad I_2^* = \frac{a_1-\sqrt{a_1^2-4a_0a_2}}{2a_0}, \tag{2.6}
$$

and the system (1.1) *has two equilibria $E_1^* = (I_1^*, S_1^*, R_1^*)$, $E_2^* = (I_2^*, S_2^*, R_2^*)$, where S_i^* and R_i^* are defined via* (2.2) *for appropriate I_i^*, $i = 1,2$. Then*

(1) if $a_1 > 2a_0b$ and $F(b) > 0$ then both equilibria E_1^* and E_2^* are positive;

(2) if $a_1 > 2a_0b$ and $F(b) \le 0$ or $a_1 \le 2a_0b$ and $F(b) < 0$ then there exist only one positive equilibrium E_1^*;

(3) if $a_1 \le 2a_0b$ and $F(b) \ge 0$ then a positive equilibrium does not exist.

Proof. (1) Following (2.2), (2.4), (2.6) and $I_1^* > I_2^*$, it is enough to show that from $a_1 > 2a_0b$ and $F(b) > 0$ it follows $I_2^* > b$, i.e., $\beta_1 I_2^* > \mu N$. Really, the inequality $F(b) = a_0b^2 - a_1b + a_2 > 0$ is equivalent to $(a_1-2a_0b)^2 > a_1^2 - 4a_0a_2$, from where it follows $a_1 - 2a_0b > \sqrt{a_1^2-4a_0a_2}$ or $I_2^* = \dfrac{a_1-\sqrt{a_1^2-4a_0a_2}}{2a_0} > b$.

(2) If $a_1 > 2a_0b$ then via (2.6) $I_1^* > \dfrac{a_1}{2a_0} > b$, i.e., $\beta_1 I_1^* > \mu N$, therefore, the equilibrium E_1^* is a positive one. Besides, the inequality $F(b) = a_0b^2 - a_1b + a_2 \le 0$ is equivalent to $(a_1-2a_0b)^2 \le a_1^2 - 4a_0a_2$, from where it follows $a_1 - 2a_0b \le \sqrt{a_1^2-4a_0a_2}$ or $I_2^* = \dfrac{a_1-\sqrt{a_1^2-4a_0a_2}}{2a_0} \le b$, i.e., the equilibrium E_2^* is not a positive one.

If $a_1 \le 2a_0b$ then $I_2^* < \dfrac{a_1}{2a_0} \le b$, i.e., $\beta_1 I_2^* < \mu N$, therefore, the equilibrium E_2^* is not a positive one. Besides, the inequality $F(b) = a_0b^2 - a_1b + a_2 < 0$ is equivalent to $(a_1-2a_0b)^2 < a_1^2 - 4a_0a_2$, from where it follows $2a_0b - a_1 < \sqrt{a_1^2-4a_0a_2}$ or $I_1^* = \dfrac{a_1+\sqrt{a_1^2-4a_0a_2}}{2a_0} > b$, i.e., $\beta_1 I_1^* > \mu N$, therefore, the equilibrium E_1^* is a positive one.

(3) The inequality $F(b) = a_0b^2 - a_1b + a_2 \geq 0$ is equivalent to $(a_1 - 2a_0b)^2 \geq a_1^2 - 4a_0a_2$, from where it follows $2a_0b - a_1 \geq \sqrt{a_1^2 - 4a_0a_2}$ or $I_1^* = \dfrac{a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_0} \leq b$, i.e., $\beta_1 I_1^* \leq \mu N$, therefore, the equilibrium E_1^* is not a positive one. Since $I_2^* < I_1^* \leq b$, then the equilibrium E_2^* is not a positive one too. □

Remark 2.1. *If $a_1^2 = 4a_0a_2$ then the Eq.* (2.5) *has one root $I^* = \dfrac{a_1}{2a_0}$ and by the condition $\beta_1 a_1 > 2a_0\mu N$ the system* (1.1) *has a positive equilibrium $E^* = (I^*, S^*, R^*)$, where S^* and R^* are defined in* (2.2).

Example 2.1. *Put $I_0 = 830$, $S_0 = 130$, $R_0 = 60$, $\beta_1 = 0.081$, $\beta_2 = 0.031$, $\gamma = 0.002$, $\mu = 2.6 \times 10^{-4}$. Via* (2.4) *in this case $b = 3.2741$, $a_0 = 0.0093$, $a_1 = 0.2587 > 2a_0b = 0.0609$, $a_2 = 0.5410$, $a_1^2 = 0.0669 > 4a_0a_2 = 0.0201$, $F(b) = -0.2064$. So, the conditions (2) of Lemma 2.1 hold and there are one positive equilibrium $E_1^* = (25.5530, 92.1499, 902.2971)$ and one not a positive equilibrium $E_2^* = (2.28, 1058.08, -40.36)$.*

Putting $\beta_1 = 0.000081$ with the same values of all other parameters, we obtain $a_1 - 2a_0b = -0.3514$ and $F(b) = 457.56$, i.e., the conditions (3) of Lemma 2.1 hold and both equilibria are not positive ones: $E_1^* = (1108.4, -0.6804, -87.7102)$, $E_2^* = (7.5465, 1144.7, -132.2944)$.

2.1. Centralization and linearization. Consider the nonlinear differential equation

$$\Delta x_i = F(x_i), \tag{2.7}$$

where $x_i \in \mathbf{R}^n$ and the equation $F(x_i) = 0$ has a solution x^* that is an equilibrium of the difference equation (2.7). Using the new variable $y_i = x_i - x^*$, represent the Eq. (2.7) in the form

$$\Delta y_i = F(x^* + y_i). \tag{2.8}$$

It is clear that stability of the zero solution of the Eq. (2.8) is equivalent to stability of the equilibrium x^* of the Eq. (2.7).

Let $J_F = \left\| \dfrac{\partial F_i}{\partial x_j} \right\|$, $i, j = 1, ..., n$, be the Jacobian matrix of the function $F = \{F_1, ..., F_n\}$ and $\lim_{|y| \to 0} \dfrac{|o(y)|}{|y|} = 0$, where $|y|$ is the Euclidean norm in $\mathbf{R}^n$. Using Taylor's expansion in the form $F(x^*+y)=F(x^*)+J_F(x^*)y+o(y)$ and the equality $F(x^*)=0$, we obtain the linear approximation

$$\Delta z_i = J_F(x^*)z_i \tag{2.9}$$

of the Eq. (2.8). So, a condition for asymptotic stability of the zero solution of the linear Eq. (2.9) is also a condition for local stability of the equilibrium x^* of the initial nonlinear Eq. (2.7).

3. STOCHASTIC PERTURBATIONS

Let $\{\Omega,\mathfrak{F},\mathbf{P}\}$ be a basic probability space, $\mathfrak{F}_i\in\mathfrak{F}$, $i\in Z=\{0,1,...\}$, be a family of σ-algebras, $\mathbf{E}$ be the expectation, $(\xi_{ji})_{i\in Z}$, $j=1,2,3$, be three mutually independent sequences of $\mathfrak{F}_i$-adapted mutually independent identically distributed random variables such that

$$\begin{aligned}
&\mathbf{E}\xi_{ji}=0,\quad \mathbf{E}\xi_{ji}^2=1,\quad j=1,2,3,\quad i\in Z,\\
&\mathbf{E}\xi_{ji}\xi_{km}=0\quad \text{if}\quad j\neq k\quad \text{or}\quad i\neq m.
\end{aligned} \tag{3.1}$$

3.1. Multiplicative perturbations. Let us suppose that the system (1.1) influences by stochastic perturbations that are proportional to the deviation of the system state $E_i=(I_i,S_i,R_i)$ from the equilibrium $E^*=(I^*,S^*,R^*)$, i.e., the system (1.1) takes the form

$$\begin{aligned}
\Delta I_i &= \mu N-\mu I_i-\frac{\beta I_iS_i}{N}+\sigma_{1i}(I_i-I^*)\xi_{1,i+1},\\
\Delta S_i &= -\mu S_i+\frac{\beta_1 I_iS_i}{N}-\frac{\gamma R_iS_i}{N}+\sigma_{2i}(S_i-S^*)\xi_{2,i+1},\\
\Delta R_i &= -\mu R_i+\frac{\beta_2 I_iS_i}{N}+\frac{\gamma R_iS_i}{N}+\sigma_{3i}(R_i-R^*)\xi_{3,i+1},
\end{aligned} \tag{3.2}$$

where $(\sigma_{ji})_{i\in Z}$, $j=1,2,3$, are three number sequences. Note that the equilibrium $E^*=(I^*,S^*,R^*)$ of the system (1.1) is also the solution of the system (3.2).

Calculating the Jacobian matrix of the system (3.2) and using (2.1), we obtain for the system (3.2) the linear approximation of the type (2.9) in the form

$$\Delta z_i \;=\; Az_i+\sum_{j=1}^{3}C_{ji}z_i\xi_{j,i+1}, \tag{3.3}$$

where $z_i=(z_{1i},z_{2i},z_{3i})'$, $'$ is the transpose sign, the 3×3-dimensional matrix C_{ji} has all zero elements besides of the diagonal element $c_{jj,i}=\sigma_{ji}$, $j=1,2,3$, $i\in Z$, the matrix $A=J_F(I^*,S^*,R^*)$, i.e.,

$$A=\begin{bmatrix} -\left(\mu+\dfrac{\beta S^*}{N}\right) & -\dfrac{\beta I^*}{N} & 0 \\ \dfrac{\beta_1 S^*}{N} & \dfrac{\beta_1 I^*-\gamma R^*}{N}-\mu & -\dfrac{\gamma S^*}{N} \\ \dfrac{\beta_2 S^*}{N} & \dfrac{\beta_2 I^*+\gamma R^*}{N} & \dfrac{\gamma S^*}{N}-\mu \end{bmatrix}. \tag{3.4}$$

3.2. Additive perturbations. Let us suppose that the system (1.1) influences by additive stochastic perturbations of the form

$$\begin{aligned} \Delta I_i =&\mu N-\mu I_i-\frac{\beta I_i S_i}{N}+\sigma_{1i}\xi_{1,i+1}, \\ \Delta S_i =&-\mu S_i+\frac{\beta_1 I_i S_i}{N}-\frac{\gamma R_i S_i}{N}+\sigma_{2i}\xi_{2,i+1}, \\ \Delta R_i =&-\mu R_i+\frac{\beta_2 I_i S_i}{N}+\frac{\gamma R_i S_i}{N}+\sigma_{3i}\xi_{3,i+1}, \end{aligned} \tag{3.5}$$

where $(\sigma_{ji})_{i\in Z}$, $j=1,2,3$, are three number sequences. Note that in this case the equilibrium $E^*=(I^*,S^*,R^*)$ of the system (1.1) is not a solution of the system (3.5).

Similarly to (3.3) the linear approximation of the system (3.5) takes the form

$$\Delta z_i \;=\; Az_i+C_i\xi_{i+1}, \tag{3.6}$$

where z_i and A are the same as in (3.3), (3.4), $C_i=diag(\sigma_{1i},\sigma_{2i},\sigma_{3i})$, $\xi_i=(\xi_{1i},\xi_{2i},\xi_{3i})'$.

4. STABILITY

Definition 4.1. *The equilibrium* $E^*=(I^*,S^*,R^*)$ *of the system* (3.2) *is called stable in probability if for any* $\varepsilon>0$ *and* $\varepsilon_1>0$ *there exists a* $\delta>0$ *such that the solution* $E_i=(I_i,S_i,R_i)$ *of the system* (3.2) *satisfies the inequality* $\mathbf{P}\{\sup_{i\in Z}|E_i-E^*|>\varepsilon/\mathfrak{F}_0\}<\varepsilon_1$ *provided that* $\mathbf{P}\{|E_0-E^*|<\delta\}=1$.

Definition 4.2. *The solution of the Eq.* (3.3) *is called:*

- uniformly mean square bounded if $\sup_{i\in Z}\mathbf{E}|z_i|^2<\infty$*;*

- asymptotically mean square trivial if $\lim_{i\to\infty}\mathbf{E}|z_i|^2=0$*;*

- mean square summable if $\sum_{i=0}^{\infty}\mathbf{E}|z_i|^2<\infty$.

Remark 4.1. *Note that if the solution of the Eq.* (3.3) *is mean square summable then it is uniformly mean square bounded and asymptotically mean square trivial.*

Definition 4.3. *The zero solution of the Eq.* (3.3) *is called:*

- mean square stable if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathbf{E}|z_i|^2 < \varepsilon$, $i \in Z$, if $\mathbf{E}|z_0|^2 < \delta$;

- asymptotically mean square stable if it is mean square stable and for each initial value z_0 the solution z_i of the Eq. (3.3) *is asymptotically mean square trivial.*

Remark 4.2. *It is known that sufficient conditions for asymptotic mean square stability of the zero solution of the linear Eq.* (3.3) *at the same time are sufficient conditions for stability in probability of the equilibrium $E^* = (I^*, S^*, R^*)$ of the nonlinear system* (3.2) [6].

Remark 4.3. *Note that the Eq.* (3.6) *is the linear approximation of the nonlinear system* (3.5)*. So, if the solution z_i of the Eq.* (3.6) *is asymptotically mean square trivial then the solution $E_i = (I_i, S_i, R_i)$ of the system* (3.5) *satisfies the condition $\lim_{i\to\infty} \mathbf{E}|E_i - E^*|^2 = 0$ provided that $\mathbf{E}|E_0 - E^*|^2$ is small enough.*

Theorem 4.1. [6] *Let there exist a nonnegative function $V(x_i)$, which satisfies the conditions*

$$\mathbf{E}V(z_0) \le c_1 \mathbf{E}|z_0|^2, \quad \mathbf{E}\Delta V(z_i) \le -c_2 \mathbf{E}|z_i|^2, \quad i \in Z. \tag{4.1}$$

Then the zero solution of the Eq. (3.3) *is asymptotically mean square stable.*

Theorem 4.2. [7] *Let there exist a nonnegative function $V(x_i)$, which satisfies the conditions*

$$\mathbf{E}V(z_0) \le c_1 \mathbf{E}|z_0|^2, \quad \mathbf{E}\Delta V(z_i) \le -c_2 \mathbf{E}|z_i|^2 + \gamma_i, \quad i \in Z, \quad \sum_{i=0}^{\infty} \gamma_i < \infty. \tag{4.2}$$

Then the solution of the Eq. (3.6) *is mean square summable.*

4.1. Small multiplicative perturbations.

Everywhere below the inequality $R < 0$ means that the symmetric matrix R is negative definite.

Theorem 4.3. *Suppose that σ_{ji} does not depend on i, i.e., $\sigma_{ji} = \sigma_j$ and therefore $C_{ji} = C_j$, $j = 1, 2, 3$, $i \in Z$. Let there exist a positive definite 3×3-dimensional matrix P such that the LMI*

$$A'P + PA + A'PA + \sum_{j=1}^{3} C_j' P C_j < 0 \tag{4.3}$$

holds. Then the equilibrium (I^*, S^*, R^*) of the system (3.2) *is stable in probability.*

Proof. Following Remark 4.2, it is enough to prove that the zero solution of the linear Eq. (3.3) is asymptotically mean square stable. Really, using the properties of $\xi_{j,i+1}$ (3.1), for the function $V_i = z_i' P z_i$ and the Eq. (3.3) we have

$$
\begin{aligned}
\mathbf{E}\Delta V_i =& \mathbf{E}\left[z_{i+1}' P z_{i+1} - z_i' P z_i\right] \\
=& \mathbf{E}\left[\left(z_i + A z_i + \sum_{j=1}^{3} C_j z_i \xi_{j,i+1}\right)' P \left(z_i + A z_i + \sum_{j=1}^{3} C_j z_i \xi_{j,i+1}\right) - z_i' P z_i\right] \\
=& 2\mathbf{E} z_i' P \left(A z_i + \sum_{j=1}^{3} C_j z_i \xi_{j,i+1}\right) + \mathbf{E}\left(A z_i + \sum_{j=1}^{3} C_j z_i \xi_{j,i+1}\right)' P \left(A z_i + \sum_{j=1}^{3} C_j z_i \xi_{j,i+1}\right) \\
=& \mathbf{E}\left(2 z_i' P A z_i + z_i' A' P A z_i + \sum_{j=1}^{3} z_i' C_j' P C_j z_i\right) \\
=& \mathbf{E} z_i' \left(PA + A'P + A'PA + \sum_{j=1}^{3} C_j' P C_j\right) z_i.
\end{aligned}
$$

From here and the LMI (4.3) it follows that the function V_i satisfies the conditions (4.1) of Theorem 4.1. So, the zero solution of the Eq. (3.3) is asymptotically mean square stable and therefore the equilibrium (I^*, S^*, R^*) of the system (3.2) is stable in probability. The proof is completed. □

Example 4.1. *In Fig.1 the solution $E_i = (I_i, S_i, R_i)$ of the system* (3.2) *with $\sigma_{ji} = \sigma_j$, $j = 1,2,3$, is shown in the deterministic case ($\sigma_1 = \sigma_2 = \sigma_3 = 0$) with the initial condition $I_0 = 700$, $S_0 = 260$, $R_0 = 60$ and the values of all other parameters, given in Example 2.1. One can see that the solution converges to the positive equilibrium $E_1^* = (25.5530, 92.1499, 902.2971)$.*

Example 4.2. *Consider now the system* (3.2) *with constant σ_j, $j = 1,2,3$. Via MATLAB it was shown that by the values of all parameters, given in Example 2.1, the LMI* (4.3) *holds for maximal values of $\sigma_1 = 0.058$, $\sigma_2 = 0.05$, $\sigma_3 = 0.02$. In Fig.2 25 trajectories of the solution $E_i = (I_i, S_i, R_i)$ are shown with the initial conditions $I_0 = 230$, $S_0 = 60$, $R_0 = 730$. The equilibrium $E_1^* = (25.5530, 92.1499, 902.2971)$ of the system* (3.2) *is stable in probability, so, all trajectories converge to this equilibrium. Note that stability in probability is a local stability, so, the initial values are chosen closer to the equilibrium E_1^* than in Fig.1.*

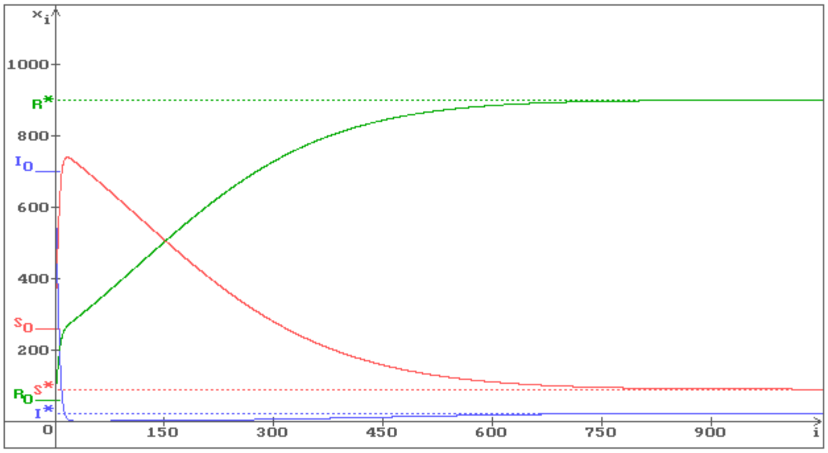

FIGURE 1. The solution of the system (3.2) in deterministic case (all σ_j are zeros) with the initial condition $I_0 = 700$, $S_0 = 260$, $R_0 = 60$. The solution converges to the equilibrium $E_1^* = (25.5530, 92.1499, 902.2971)$

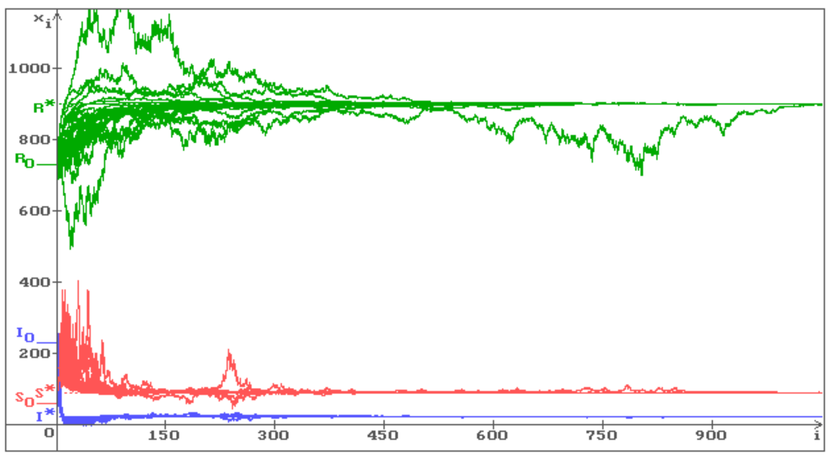

FIGURE 2. 25 trajectories of the solution of the system (3.2) with $\sigma_1 = 0.058$, $\sigma_2 = 0.05$, $\sigma_3 = 0.02$ and the initial conditions $I_0 = 230$, $S_0 = 60$, $R_0 = 730$. All trajectories converge to the equilibrium $E_1^* = (25.5530, 92.1499, 902.2971)$ which is stable in probability

4.2. Fading multiplicative perturbations.

Theorem 4.4. *Suppose that*

$$\sum_{i=0}^{\infty} \sigma_i^2 < \infty, \quad \sigma_i^2 = \max_{j=1,2,3} \sigma_{ji}^2, \tag{4.4}$$

and there exist a positive definite 3×3*-dimensional matrix* P *such that the LMI*

$$A'P + PA + A'PA < 0 \tag{4.5}$$

holds. Then the solution of the Eq. (3.3) *is asymptotically mean square trivial.*

Proof. Let p be the maximal diagonal element of the matrix P and p_m be the minimal eigenvalue of the matrix P, i.e., $z_i'Pz_i \geq p_m|z_i|^2$. Using the properties of $\xi_{j,i+1}$ (3.1), for the Lyapunov function

$$V_i = \exp\left(-\sum_{k=0}^{i-1}\lambda_k\right) z_i'Pz_i, \qquad \lambda_i = \frac{p}{p_m}\sigma_i^2, \tag{4.6}$$

we obtain

(4.7)

$$\begin{aligned}
\mathbf{E}\Delta V_i =& \mathbf{E}\left[\exp\left(-\sum_{k=0}^{i}\lambda_k\right) z_{i+1}'Pz_{i+1} - \exp\left(-\sum_{k=0}^{i-1}\lambda_k\right) z_i'Pz_i\right] \\
=& \exp\left(-\sum_{k=0}^{i}\lambda_k\right)\mathbf{E}\left[z_{i+1}'Pz_{i+1} - e^{\lambda_i}z_i'Pz_i\right] \\
=& \exp\left(-\sum_{k=0}^{i}\lambda_k\right)\mathbf{E}\left[\left(z_i + Az_i + \sum_{j=1}^{3}C_{ji}z_i\xi_{j,i+1}\right)' P\left(z_i + Az_i + \sum_{j=1}^{3}C_{ji}z_i\xi_{j,i+1}\right) - e^{\lambda_i}z_i'Pz_i\right] \\
=& \exp\left(-\sum_{k=0}^{i}\lambda_k\right)\mathbf{E}z_i'\left[(I + A')P(I + A) + \sum_{j=1}^{3}C_{ji}'PC_{ji} - e^{\lambda_i}P\right] z_i \\
=& \exp\left(-\sum_{k=0}^{i}\lambda_k\right)\mathbf{E}z_i'\left[A'P + PA + A'PA + \sum_{j=1}^{3}C_{ji}'PC_{ji} + (1 - e^{\lambda_i})P\right] z_i.
\end{aligned}$$

Note that $e^{\lambda_i} \geq 1 + \lambda_i$ or $1 - e^{\lambda_i} \leq -\lambda_i$. So, via (4.6)

$$(1 - e^{\lambda_i})z_i'Pz_i \leq -\lambda_i p_m|z_i|^2 = -p\sigma_i^2|z_i|^2. \tag{4.8}$$

Besides, using the definition of the matrix C_{ji}, we have

$$\sum_{j=1}^{3} z_i'C_{ji}'PC_{ji}z_i = z_i'\begin{bmatrix} p_{11}\sigma_{1i}^2 & 0 & 0 \\ 0 & p_{22}\sigma_{2i}^2 & 0 \\ 0 & 0 & p_{33}\sigma_{3i}^2 \end{bmatrix} z_i \leq p\sigma_i^2|z_i|^2. \tag{4.9}$$

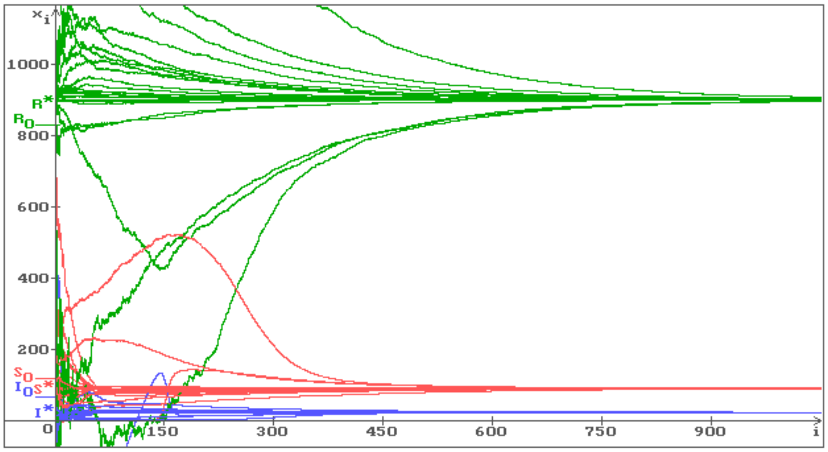

FIGURE 3. 25 trajectories of the solution of the system (3.5) with $\sigma_{1i} = \dfrac{2.8}{1+i}$, $\sigma_{2i} = \dfrac{2.2}{1+i}$, $\sigma_{3i} = \dfrac{4.6}{1+i}$, and the initial conditions $I_0 = 70$, $S_0 = 120$, $R_0 = 830$. All trajectories converge to the equilibrium $E_1^* = (25.5530, 92.1499, 902.2971)$.

From (4.7), (4.8), (4.9) and (4.4) it follows

$$\mathbf{E}\Delta V_i \le \exp\left(-\sum_{k=0}^{\infty}\lambda_k\right)\mathbf{E}z_i'\left[A'P + PA + A'PA\right]z_i. \tag{4.10}$$

From here and the LMI (4.5) it follows that the function V_i satisfies the conditions (4.1) of Theorem 4.1. So, the zero solution of the Eq. (3.3) is asymptotically mean square stable and therefore the equilibrium (I^*, S^*, R^*) of the system (3.2) is stable in probability. The proof is completed. □

Example 4.3. *In Fig.3 25 trajectories of the solution* $E_i = (I_i, S_i, R_i)$ *of the system* (3.2) *are shown with* $\sigma_{1i} = \dfrac{2.8}{1+i}$, $\sigma_{2i} = \dfrac{2.2}{1+i}$, $\sigma_{3i} = \dfrac{4.6}{1+i}$, *and the initial conditions* $I_0 = 70$, $S_0 = 120$, $R_0 = 830$, *all other parameters are the same as in Example 2.1. In correspondence with Theorem 4.4 and Remark 4.3 all trajectories converge to the equilibrium*
$E_1^* = (25.5530, 92.1499, 902.2971)$.

4.3. **Fading additive perturbations.**

Theorem 4.5. *Suppose that*

$$\sum_{i=0}^{\infty}\gamma_i < \infty, \quad \gamma_i = \sum_{j=1}^{3}\sigma_{ji}^2, \tag{4.11}$$

and there exist a positive definite 3×3*-dimensional matrix* P *such that the LMI*

$$A'P+PA+A'PA<0 \tag{4.12}$$

holds. Then the solution of the Eq. (3.6) *is asymptotically mean square trivial.*

Proof. Using the properties of $\xi_{j,i+1}$ (3.1) and (4.11), for the function $V_i=z_i'Pz_i$ and the Eq. (3.6) we have

$$\begin{aligned}
\mathbf{E}\Delta V_i =&\mathbf{E}\left[z_{i+1}'Pz_{i+1}-z_i'Pz_i\right]\\
=&\mathbf{E}\left[(z_i+Az_i+C_i\xi_{i+1})'P(z_i+Az_i+C_i\xi_{i+1})-z_i'Pz_i\right]\\
=&2\mathbf{E}z_i'P(Az_i+C_i\xi_{i+1})+\mathbf{E}(Az_i+C_i\xi_{i+1})'P(Az_i+C_i\xi_{i+1})\\
\leq&\mathbf{E}\left(2z_i'PAz_i+z_i'A'PAz_i+\|P\|\sum_{j=1}^{3}\sigma_{ji}^2\xi_{j,i+1}^2\right)\\
=&\mathbf{E}z_i'\left(PA+A'P+A'PA\right)z_i+\|P\|\gamma_i.
\end{aligned}$$

From here and (4.12) it follows that the conditions (4.2) of Theorem 4.2 hold. Therefore, the solution of the Eq. (3.6) is asymptotically mean square summable and asymptotically mean square trivial. The proof is completed. □

Example 4.4. *In Fig.4 25 trajectories of the solution* $E_i=(I_i,S_i,R_i)$ *of the system* (3.5) *are shown with* $\sigma_{1i}=\dfrac{150}{1+i}$, $\sigma_{2i}=\dfrac{30}{1+i}$, $\sigma_{3i}=\dfrac{100}{1+i}$, *and the initial conditions* $I_0=230$, $S_0=60$, $R_0=730$, *all other parameters are the same as in Example 2.1. In correspondence with Theorem 4.5 and Remark 4.3 all trajectories converge to the equilibrium*
$E_1^*=(25.5530,92.1499,902.2971)$.

Remark 4.4. *There is an unsolved problem: is it possible to weaken the conditions* (4.4) *and* (4.11)*? For example, to consider a situation with fading stochastic perturbations in the case when the sequence* $(\sigma_{ji})_{i\geq 0}$ *converges to the zero but is not a square summable, i.e.* $\lim\limits_{i\to\infty}\sigma_{ji}^2=0$ *but* $\sum\limits_{i=0}^{\infty}\sigma_{ji}^2=\infty$, $j=1,2,3$.

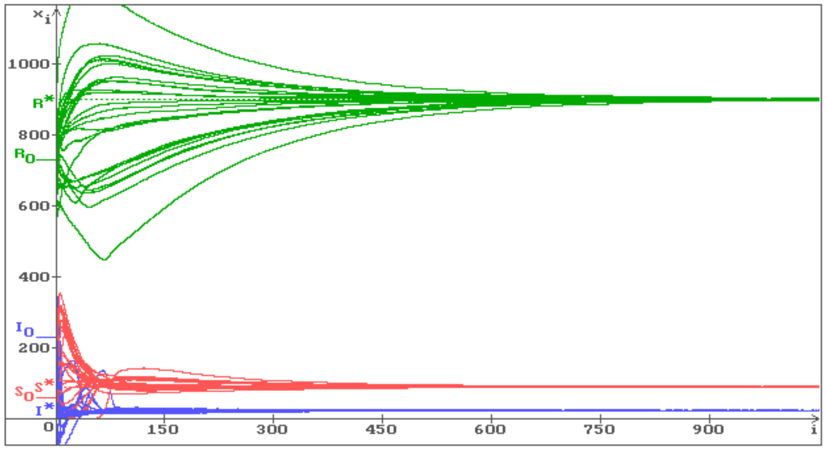

FIGURE 4. 25 trajectories of the solution of the system (3.5) with $\sigma_{1i} = \dfrac{150}{1+i}$, $\sigma_{2i} = \dfrac{30}{1+i}$, $\sigma_{3i} = \dfrac{100}{1+i}$, and the initial conditions $I_0 = 230$, $S_0 = 60$, $R_0 = 730$. All trajectories converge to the equilibrium $E_1^* = (25.5530, 92.1499, 902.2971)$.

5. Conclusions

Impact of different types of stochastic perturbations on a nonlinear discrete-time model of information dissemination is studied. It is shown that asymptotically stable positive equilibrium of the considered model saves his stability under influence of small multiplicative perturbations, quickly fading multiplicative perturbations or quickly fading additive perturbations. For getting appropriate stability conditions the method of Lyapunov functions and method of linear matrix inequalities (LMIs) are used. The levels of fading stochastic perturbations are defined by square summable number sequences. For future investigation the unsolved problem is proposed to get stability conditions for stochastic perturbations that fade on the infinity not so quickly. Similar research can be applied for many other nonlinear models in different applications.

Conflict of Interests

The author declares that there is no conflict of interests.

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