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A MATHEMATICAL MODEL ANALYSIS OF MARRIAGE DIVORCE

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Abstract. In this paper, a deterministic model for the marriage divorce in a population is proposed and analysed qualitatively using the stability theory of differential equations. The basic reproduction number with respect to the divorce free equilibrium was obtained using next generation matrix approach. The conditions for local and global asymptotic stability of divorce free and endemic equilibria were established. The model exhibits backward bifurcation and the sensitivity indices of the parameters with respect to eradicating or spreading divorce in marriage was determined. Numerical simulation was performed and displayed graphically to justify the analytical results.

Keywords: bifurcation analysis; marriage divorce; sensitivity analysis; numerical simulation.

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1. Introduction

Marriage is the sole means to enter family life in any community, and it is the foundational element that plays the primary goal of providing the needs and requirements of its members

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and society [1]. Marriage is a socially recognized and approved union of couples who dedicate to one another with the aspirations of a permanent and long intimate relationship[2]. There may be a disagreement inside a family that leads to separation and divorce for a variety of reasons. Divorces have become a common part of American life, affecting children of all ethnic backgrounds, religions, and social position [4]. Divorce rates have risen year after year in England and Wales [3] in Spain, a disparity in marital satisfaction, as well as the economy, has a significant impact on divorce [5].

Divorce is a common phenomenon in Africa, with immediate and long-term consequences [5, 6]. Divorce rates among young people have risen in South Africa, which has one of the highest rates in the world [7]. Divorce has wider societal, cultural, economic, psychological, and political consequences [10, 11] and divorce affects all the children [12, 14]. Stable families produce a wealthy nation and a stable world, whereas unstable families produce a country and a world in disorder. This must first be decided out of the family for a country and region or the world at large to be at peace [6].

Mathematical modeling is an important tool used in analyzing the dynamics of infectious diseases [15]. The mathematical modeling of marriage divorce as a social epidemiology has received relatively little attention. For example, the study by [16] proposed and analyzed a non-linear MSD (Married, Separated and Divorced) mathematical model to study the dynamics of divorce epidemic in Ghana. The existence and stability of the divorce free and endemic equilibria was proved using the computed basic reproduction number. They concluded that, reducing the contact rate between the marriaged and divorce, increasing the number of marriage that go into separation and educating separators to refrain from divorce can be useful in combating the divorce epidemic. Duato & L. J.odar [5] proposed and analyzed a mathematical modeling of divorce propagation allowing the estimation of the future divorced population. A sensitivity analysis of the growth of the divorced population with respect to the contagion rate is included. A discrete model of the marital status of the family dynamics also studied by [18].

In this paper, we proposed a mathematical model of marriage divorce considering divorce as a transmitted disease that transmit in between human propagated by divorced women/ man over married ones.

2. Model Description and Formulation

The model divides the entire population into four subpopulations: those who reach the age of getting married are single individuals, $S(t)$; those who got married individuals are denoted by, $M(t)$; those who separate bud not divorced are broken marriage individuals, $B(t)$ and those who are divorced marriage are denoted by $D(t)$.

π is the recruitment rate of individuals being single when he/she reaches the age of getting married. This individuals got married at rate of β. The married individuals got broken and move to the broken compartment due to the contact with the divorced individuals at a rate of α. The broken marriage recovered from their conflicts and renew their marriage at a rate of ε and live as the previous style. Some of these broken marriage got a permanent divorce at a rate of δ. this divorced people join the single subpopulation at a rate of ρ and some of the will die due to divorce at a rate of σ. The whole population has μ as an average death rate. In addition we assume that sex, race and social status do not affect the probability of being divorced and members mix homogeneously (have the same interaction to the same degree). The state variables of the model are represented and described in Table 1 and Table 2 shows the description of model parameters. The compartmental flow diagram for the model is shown in Figure 1.

With regards to the above assumptions, the model is governed by the following system of differential equation:

$$
\begin{aligned}
\frac{dS}{dt} &= \pi + \rho D - (\beta + \mu) S \\
\frac{dM}{dt} &= \beta S + \varepsilon B - (\alpha D + \mu) M \\
\frac{dB}{dt} &= \alpha M D - (\delta + \varepsilon + \mu) B \\
\frac{dD}{dt} &= \delta B - (\sigma + \rho + \mu) D
\end{aligned} \tag{1}
$$

with the initial condition

$$
S(0) = S_0 \geq 0\ , M(0) = M_0 \geq 0\ , B(0) = B_0 \geq 0\ , D(0) = D_0 \geq 0. \tag{2}
$$

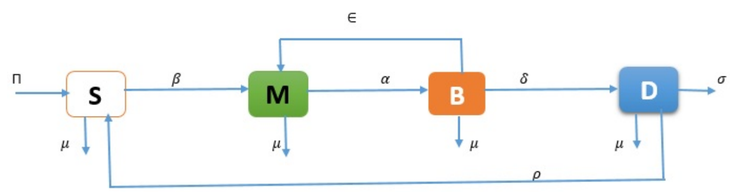


FIGURE 1. Compartmental Flow diagram of Divorce in marriage

TABLE 1. Description of variable states of the marriage divorce model (1).

Variable State	Description
$S(t)$	Number of Single individuals at time t
$M(t)$	Number of married individuals at time t
$B(t)$	Number of broken individuals at time t
$D(t)$	Number of divorced individuals at time t
$N(t)$	Total number of population at time t

TABLE 2. Description of parameters of the Marriage divorce model (1).

Parameter	Description
π	Recruitment rate of individuals to the age of single
β	Average rate of single individuals who got married
ρ	Rate of divorced individuals who become single
σ	Death rate of individuals due to divorce
ε	Rate of Broken individuals who renew their previous marriage
δ	Rate of broken individuals who got divorced
μ	Natural death rate of individuals
α	The contact rate of divorced individuals with married individuals

3. Model Analysis

3.1. Invariant Region. Let us determine a region in which the solution of model (1) is bounded. For this model the total population is $N(S,M,B,D) = S(t)+M(t)+B(t)+D(t)$. Then, differentiating N with respect to time we obtain:

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dM}{dt} + \frac{dB}{dt} + \frac{dD}{dt} = \pi - \sigma D - \mu N.$$

If there is no death due to the divorce, we get

$$\frac{dN}{dt} \leq \pi - \mu N. \tag{3}$$

After solving equation (3) and evaluating it as $t \longrightarrow \infty$, we got

$$\Omega = \{(S,M,B,D)\varepsilon\Re_+^4 : N(t) \leq \frac{\pi}{\mu}\},$$

which is the feasible solution set for the model (1) and all the solutions of the model are bounded.

3.2. Positivity of Solutions. In this section, we show all the solution of the model (1) remain positive for future time if their respective initial values are positive. This will be established by the following theorem:

Theorem 3.1. *If the initial conditions $S(0), M(0), B(0), D(0)$ are positive in the feasible set Ω. Then the solution set $(S(t), M(t), B(t), D(t))$ of system (1) is positive for all $t \geq 0$.*

Proof. : We let $\tau = sup\{t > 0 : S_0(\nu) \geq 0\ , M_0(\nu) \geq 0\ , B_0(\nu) \geq 0, D_0(\nu) \geq 0 \text{ for all } \nu \in [0,t]\}$. Since $S_0(t) \geq 0\ , M_0(t) \geq 0\ , B_0(t) \geq 0$ and $D_0(t) \geq 0$ then $\tau > 0$. If $\tau < \infty$, then automatically $S_0(t)$ or $M_0(t)$ or $B_0(t)$ or $D_0(t)$ is equal to zero at τ. Taking the first equation of the model (1)

$$\frac{dS}{dt} = \pi + \rho D - (\beta + \mu) S. \tag{4}$$

Then, using the variation of constants formula the solution of equation (4) at τ is given by:

$$S(\tau) = S(0)\exp\left[-\int_0^\tau (\beta+\mu)(s)ds\right] + \int_0^\tau (\pi+\rho D).\exp\left[-\int_s^\tau (\beta+\mu)(\nu)d\nu\right] ds > 0$$

Moreover, since all the variables are positive in $[0,\tau]$, hence, $S(\tau) > 0$. It can be shown in a similar way that $M(\tau) > 0, B(\tau) > 0, D(\tau) > 0$, which is a contradiction. Hence $\tau = \infty$. Therefore, all the solution sets are positive for $t \geq 0$. □

3.3. Divorce free equilibrium point(DFEP). When there is no divorce in marriage, I.e $D = B = 0$, the divorce free equilibrium occur and is obtained by taking the right side of Eq. (1) equal to zero. Therefore the divorce free equilibrium point is given by:

$$E_0 = \left(\frac{\pi}{\beta+\mu}, \frac{\pi\beta}{\mu(\beta+\mu)}, 0, 0\right). \tag{5}$$

3.4. Basic reproduction number. We calculate the basic reproduction number $\mathscr{R}_0$ of the system by applying the next generation matrix approach as laid out by [19] and so it is the spectral radius of the next-generation matrix. Hence,

$$\begin{aligned} \frac{dB}{dt} &= \alpha MD - (\delta+\varepsilon+\mu)B \\ \frac{dD}{dt} &= \delta B - (\sigma+\rho+\mu)D \end{aligned} \tag{6}$$

Then by the principle, we obtained:

$$F = \begin{bmatrix} \alpha MD \\ 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} (\delta+\varepsilon+\mu)B \\ -\delta B + (\sigma+\rho+\mu)D \end{bmatrix} \tag{7}$$

The Jacobian matrices at DFEP is given by

$$\mathscr{F} = \begin{bmatrix} 0 & \frac{\alpha\pi\beta}{\mu(\beta+\mu)} \\ 0 & 0 \end{bmatrix} \text{ and } \mathscr{V} = \begin{bmatrix} (\delta+\varepsilon+\mu) & 0 \\ -\delta & (\sigma+\rho+\mu) \end{bmatrix}$$

Therefore, the basic reproduction number is given us

$$\mathscr{R}_0 = \frac{\alpha\beta\pi\delta}{\mu(\mu+\beta)(\mu+\rho+\sigma)(\mu+\varepsilon+\delta)}. \tag{8}$$

3.5. Local Stability of DFEP.

Theorem 3.2. *The Divorce free equilibrium point is locally asymptotically stable if $\mathscr{R}_0 < 1$ and unstable if $\mathscr{R}_0 > 1$.*

Proof. The Jacobian matrix of system (1) evaluated at the divorce-free equilibrium, we get

$$J = \begin{pmatrix} -(\beta+\mu) & 0 & 0 & \rho \\ \beta & -\mu & \varepsilon & -\frac{\alpha\pi\beta}{\mu(\beta+\mu)}+\sigma \\ 0 & 0 & -(\delta+\varepsilon+\mu) & \frac{\alpha\pi\beta}{\mu(\beta+\mu)} \\ 0 & 0 & \delta & -(\sigma+\rho+\mu) \end{pmatrix}$$

The characteristic polynomial is given as

$$(-\lambda-(\beta+\mu))(-\lambda-\mu)(\lambda^2+\psi_1\lambda+\psi_2)=0. \tag{9}$$

Where

$$\psi_1 = \delta+\varepsilon+\rho+\sigma+2\mu$$
$$\psi_2 = (\mu+\rho+\sigma)(\mu+\varepsilon+\delta)-\frac{\alpha\beta\pi\delta}{\mu(\mu+\beta)}$$

From the Equation (9), we see that

$$-\lambda-(\beta+\mu)\Rightarrow\lambda_1=-(\beta+\mu)<0 \; and \; -\lambda-\mu\Rightarrow\lambda_2=-\mu<0$$

And

$$\lambda^2+\psi_1\lambda+\psi_2=0 \tag{10}$$

We applied Routh-Hurwitz criteria and by the principle equation (10) has strictly negative real root iff $\psi_1>0$, $\psi_2>0$ and $\psi_1\psi_2>0$. Clearly we see that $\psi_1>0$ because it is the sum of positive parameters and

$$\psi_2 = (\mu+\rho+\sigma)(\mu+\varepsilon+\delta)-\frac{\alpha\beta\pi\delta}{\mu(\mu+\beta)} = (\mu+\rho+\sigma)(\mu+\varepsilon+\delta)(1-\mathscr{R}_0)$$

Hence the DFE is locally asymptotically stable if $\mathscr{R}_0<1$. □

3.6. Global Stability of DFEP.

Theorem 3.3. *The divorce free equilibrium point E_0 of the model (1) is globally asymptotically stable if $\mathscr{R}_0<1$.*

Proof. Consider the following Lyapunov function

$$V = c_1 B + c_2 D. \tag{11}$$

Differentiating equation (11) with respect to t gives

$$\frac{dV}{dt} = c_1 \frac{dB}{dt} + c_2 \frac{dD}{dt}. \tag{12}$$

Substituting $\frac{dB}{dt}$ and $\frac{dD}{dt}$ from the model (1), we get:

$$\begin{aligned}\frac{dV}{dt} &= c_1[\alpha MD - (\delta+\varepsilon+\mu)B] + c_2[\delta B - (\sigma+\rho+\mu)D] \\ &= c_1\alpha MD - c_2(\sigma+\rho+\mu)D - c_1(\delta+\varepsilon+\mu) + c_2\delta B\end{aligned}$$

Here take $c_1 = \frac{\delta}{\delta+\varepsilon+\mu} c_2$, then we have

$$\frac{dV}{dt} = \frac{\delta}{\delta+\varepsilon+\mu} c_2 \alpha MD - c_2(\sigma+\rho+\mu)D \leq \left(\frac{\alpha\delta\pi\beta}{\mu(\beta+\mu)(\delta+\varepsilon+\mu)} - (\sigma+\rho+\mu)\right)c_2 D.$$

Taking $c_2 = 1$, and substituting $\mathscr{R}_0$ we get

$$\frac{dV}{dt} \leq (\sigma+\rho+\mu)(\mathscr{R}_0 - 1)D.$$

for $M \leq M^0 = \frac{\beta\pi}{\mu(\beta+\mu)}$ and $\frac{dV}{dt} \leq 0$ for $\mathscr{R}_0 < 1$ and $\frac{dV}{dt} = 0$ if and only if $D = 0$. This implies that the only trajectory of the system (1) on which $\frac{dV}{dt} \leq 0$ is E^0. Therefore by Lasalle's invariance principle, E^0 is globally asymptotically stable in Ω. Socially, this implies that divorce can be eliminated irrespective of the initial population of divorced humans provided $\mathscr{R}_0 < 1$. □

3.7. The Endemic Equilibrium Point. In the presence of divorce in the population,$(S(t) \geq 0; M(t) \geq 0; B(t) \geq 0, D(t) \geq 0)$, there exist an equilibrium point called endemic equilibrium point denoted by $E^* = (S^*, M^*, B^*, D^*) \neq 0$. It can be obtained by equating each equation of the model equal to zero, i.e

$$\frac{dS}{dt} = \frac{dM}{dt} = \frac{dB}{dt} = \frac{dD}{dt} = 0.$$

Then we obtain

$$S^* = \frac{(\pi\alpha - \mu\rho)(\delta+\varepsilon+\mu)(\rho+\sigma+\mu)}{\alpha\kappa},$$
$$M^* = \frac{(\rho+\sigma+\mu)(\delta+\varepsilon+\mu)}{\alpha\delta},$$
$$B^* = \frac{(\rho+\sigma+\mu)[\mathscr{R}_0 - 1]}{\alpha\delta\kappa},$$
$$D^* = \frac{\mathscr{R}_0 - 1}{\alpha\kappa},$$

where

$$\kappa = \beta\delta(\sigma+\mu) + (\rho+\sigma+\mu)[\beta(\varepsilon+\mu) + \mu(\delta+\varepsilon+\mu)].$$

3.8. Local stability of Endemic equilibrium.

Theorem 3.4. *The endemic equilibrium E^* of system (1) is locally asymptotically stable in Ω if $\mathscr{R}_0 > 1$.*

Proof. : Let's first obtain the Jacobian matrix of system (1):

$$J = \begin{pmatrix} -(\beta+\mu) & 0 & 0 & \rho \\ \beta & -(\alpha D+\mu) & \varepsilon & -\alpha M+\sigma \\ 0 & \alpha D & -(\delta+\varepsilon+\mu) & \alpha M \\ 0 & 0 & \delta & -(\sigma+\rho+\mu) \end{pmatrix} \tag{13}$$

Evaluating this at the Endemic equilibrium $E^* = (S^*, I^*, H^*, Y^*)$, we get following a characteristic polynomial as

$$\lambda^4 + \varphi_1\lambda^3 + \varphi_2\lambda^2 + \varphi_3\lambda + \varphi_4 = 0. \tag{14}$$

where

$$\varphi_1 = \frac{\mathscr{R}_0 - 1}{\kappa} + \beta + \delta + \varepsilon + 4\mu + \rho + \sigma,$$
$$\varphi_2 = (\delta+\varepsilon)\left[\frac{\mathscr{R}_0 - 1}{\kappa} + \beta\right] + (\rho+\sigma+3\mu)\left[\frac{\mathscr{R}_0 - 1}{\kappa} + \beta + \delta + \varepsilon + 3\mu\right]$$
$$- (\rho+\sigma+\mu)(\delta+\varepsilon+\mu) - 3\mu^2,$$

$$\varphi_3 = (\frac{\mathscr{R}_0 - 1}{\kappa})\,[\beta(\delta+\rho+\sigma+2\mu)+\delta(\rho+\sigma+2\mu)+\mu(2\rho+2\sigma)] - \beta(\delta+\varepsilon)(\rho+\sigma+2\mu)$$
$$+\mu(2\rho+2\sigma+3\mu)(\beta+\delta+\varepsilon)-(\beta+2\mu)(\rho+\sigma+\mu)(\delta+\varepsilon+\mu),$$
$$\varphi_4 = \left[\frac{\mu(\mathscr{R}_0-1)}{\kappa}\right][(\beta+\mu)(\sigma+\mu)+\delta(\rho+\sigma)+\mu(\delta+\rho)]+\mu(\rho+\sigma)(\beta+\mu)(\delta+\varepsilon)$$
$$+\beta\mu^2(\delta+\varepsilon\rho+\sigma)+\mu^3(\beta+\delta\rho+\sigma)-\mu(\beta\mu)(\rho+\sigma+\mu)(\delta+\varepsilon+\mu).$$

Using Routh-Hurwitz criterion all roots of characteristic polynomial have negative real parts if and only if $\varphi_1 > 0, \varphi_3 > 0, \varphi_4 > 0, \varphi_1\varphi_2\varphi_3 > \varphi_3^2 + \varphi_1^2\varphi_4$ for $\mathscr{R}_0 > 1$. Hence, the endemic equilibrium E^* is locally asymptotically stable.

□

3.9. Bifurcation analysis. A bifurcation is a qualitative change in the nature of the solution trajectories due to a parameter change. We investigate the nature of the bifurcation by using the method introduced in [17, 21], which is based on the use of the central manifold theory. In short, the method is summarized by Theorem 4.1 in [25]. In the theorem, there are two basic parameters a and b that decides the bifurcation type of the model.

Theorem 3.5. *(Castillo-Chavez & Song*[25]*) Let us consider a general system of ODE's with a parameter ϕ:*

$$\frac{dx}{dt} = f(x,\phi), f : R^n \times R \longrightarrow R^n, f \in C^2(R^n \times R) \tag{15}$$

Where $x = 0$ is an equilibrium point for the system in eq(15). That is $f(0,\phi) \equiv 0$ for all ϕ. Assume the following

M$_1$: $A = D_x f(0,0) = (\frac{\partial f}{\partial x_j}(0,0))$ *is the linearization matrix of the system given by (15) around the equilibrium 0 with ϕ evaluated at 0. Zero is a simple eigenvalue of A and other eigenvalues of A have negative real parts;*

M$_2$: *Matrix A has a nonnegative right eigenvector w and a left eigenvector v corresponding to the zero eigenvalue. Let f_k be the kth component of f and*

$$a = \sum_{k,i,j=1}^{n} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0)$$

$$b=\sum_{k,i=1}^{n} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0,0)$$

The local dynamics of (15) around 0 are totally determined by a and b.

i . $a>0, b>0$. *When* $\phi<0$ *with* $|\phi| \ll 1$, *0 is locally asymptotically stable and there exists a positive unstable equilibrium; when* $0<\phi \ll 1, 0$ *is unstable and there exists a negative, locally asymptotically stable equilibrium;*

ii . $a<0, b<0$. *When* $\phi<0$ *with* $|\phi| \ll 1$, *0 is unstable; when* $0<\phi \ll 1, 0$*is locally asymptotically stable equilibrium, and there exists a positive unstable equilibrium;*

iii . $a>0, b<0$. *When* $\phi<0$ *with*$|\phi| \ll 1, 0$ *is unstable, and there exists a locally asymptotically stable negative equilibrium; when* $0<\phi \ll 1, 0$ *is stable, and a positive unstable equilibrium appears;*

iv . $a<0, b>0$. *When* ϕ *changes from negative to positive,* 0 *changes its stability from stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable.*

In particular, if a <0 *and b* >0,*then the bifurcation is forward; if a* >0 *and b* >0,*then the bifurcation is backward. Using this approach, the following result may be obtained:*

Theorem 3.6. *The model in system (1) exhibits forward bifurcation at* $\mathscr{R}_0=1$.

Proof. : We proved the theorem using the concept center manifold theorem [25] the possibility of bifurcation at $\mathscr{R}_0=1$. Then the following change of variables was made $S=z_1$, $M=z_2$, $B=z_3$ and $D=z_4$. In addition, using vector notation $z=(z_1,z_2,z_3,z_4)^T$, and $\frac{dz}{dt}=G(x)$, with $G=(g_1,g_2,g_3,g_4)^T$, then model in system (1) re-written in the form:

$$\begin{aligned} \frac{dz_1}{dt} &= \pi+\rho z_4-(\beta+\mu)z_1 \\ \frac{dz_2}{dt} &= \beta z_1+\varepsilon z_3-(\alpha z_4+\mu)z_2 \\ \frac{dz_3}{dt} &= \alpha z_2 z_4-(\delta+\varepsilon+\mu)z_3 \\ \frac{dz_4}{dt} &= \delta z_3-(\sigma+\rho+\mu)z_4 \end{aligned} \tag{16}$$

We consider the contact and transmission rate α as a bifurcation parameters so that $\mathscr{R}_0=1$ iff

$$\alpha=\alpha^*=\frac{\mu(\beta+\mu)(\rho+\sigma+\mu)(\delta+\varepsilon+\mu)}{\beta\delta\pi}.$$

The divorce free equilibrium is given by $(z_1 = \frac{\pi}{\beta+\mu}, z_2 = \frac{\pi\beta}{\mu(\beta+\mu)}, z_3 = 0, z_4 = 0)$. Then the Jacobian matrix of the system (16) at a divorce free equilibrium is given by:

$$J = \begin{pmatrix} -(\beta+\mu) & 0 & 0 & \rho \\ \beta & -\mu & \varepsilon & -\frac{\alpha^*\pi\beta}{\mu(\beta+\mu)}+\sigma \\ 0 & 0 & -(\delta+\varepsilon+\mu) & \frac{\alpha^*\pi\beta}{\mu(\beta+\mu)} \\ 0 & 0 & \delta & -(\sigma+\rho+\mu) \end{pmatrix} \tag{17}$$

The right eigenvector, $w = (w_1, w_2, w_3, w_4)^T$, associated with this simple zero eigenvalue can be obtained from $Jw = 0$. The system becomes

$$\begin{aligned} -(\beta+\mu)w_1+\rho w_4 &= 0 \\ \beta w_1-\mu w_2+\varepsilon w_3-(\frac{\alpha^*\pi\beta}{\mu(\beta+\mu)}-\sigma)w_4 &= 0 \\ -(\delta+\varepsilon+\mu)w_3+\frac{\alpha^*\pi\beta}{\mu(\beta+\mu)}w_4 &= 0 \\ \delta w_3-(\rho+\sigma+\mu)w_4 &= 0 \end{aligned} \tag{18}$$

From Eq. (18) we obtain

$$\begin{aligned} w_1 &= \frac{\delta}{\beta+\mu}w_4, \\ w_2 &= \frac{\beta\delta\rho+\varepsilon(\beta+\mu)(\sigma+\rho+\mu)+\sigma\delta(\beta+\mu)-(\beta+\mu)(\sigma+\rho+\mu)(\varepsilon+\delta+\mu)}{\mu\delta(\beta+\mu)}w_4, \\ w_3 &= \frac{\sigma+\rho+\mu}{\delta}w_4, \\ w_4 &= w_4 > 0. \end{aligned}$$

Here we have taken into account the expression for α^* . Next we compute the left eigenvector, $v = (v_1, v_2, v_3, v_4)$, associated with this simple zero eigenvalue can be obtained from $vJ = 0$ and the system becomes

$$\begin{aligned} -(\beta+\mu)v_1+\beta v_2 &= 0 \\ -\mu v_2 &= 0 \\ \varepsilon v_2-(\delta+\varepsilon+\mu)v_3+\delta\mu v_4 &= 0 \\ \delta v_1+\sigma-\frac{\alpha^*\beta\Pi}{\mu(\beta+\mu}v_2+\frac{\alpha^*\beta\Pi}{\mu(\beta+\mu}v_2-(\sigma+\rho+\mu)v_4 &= 0 \end{aligned} \tag{19}$$

From equation of Eq.(19), we obtain

$$v_1 = v_2 = 0, v_3 = \frac{\delta}{\delta+\varepsilon+\mu}v_4, v_4 = v_4 > 0.$$

Since the first and second component of v are zero, we don't need the partial derivatives of f_1 and f_2. From the partial derivatives of f_3 and f_4, the only ones that are nonzero are :

$$\frac{\partial^2 f_3}{\partial z_2 \partial z_4} = \frac{\partial^2 f_3}{\partial z_4 \partial z_2} = \alpha^* \; and \; \frac{\partial^2 f_3}{\partial z_4 \partial \alpha} = z_2^*.$$

and all the other partial derivatives are zero. The signs of the bifurcation coefficients a and b, obtained from the above partial derivatives, given respectively by

$$\begin{aligned} a &= 2v_3 w_4 w_2 \frac{\partial^2 f_3}{\partial z_2 \partial z_4}(0,0) = 2v_3 w_4 w_2 \alpha^* \\ &= \frac{2(\beta+\mu)(\varepsilon+\delta+\mu)(\sigma+\rho+\mu)^2}{\beta\delta\Pi}\left[\frac{\beta\delta\rho+\varepsilon(\beta+\mu)(\sigma+\rho+\mu)+\sigma\delta(\beta+\mu)}{(\beta+\mu)(\varepsilon+\delta+\mu)(\sigma+\rho+\mu)} - 1\right] < 0, \end{aligned}$$

and

$$b = v_3 w_4 \frac{\partial^2 f_3}{\partial z_4 \partial \alpha} = v_3 w_4 z_2^* = \frac{\delta\beta\Pi}{\mu(\beta+\mu)(\delta+\varepsilon+\mu)} > 0.$$

Since the coefficient b is always positive and a is negative. Therefore, system (1) exhibits forward bifurcation at $\mathscr{R}_0 = 1$. □

3.10. Sensitivity Analysis. Sensitivity analysis notifies us how significant each parameter to divorce transmission. To go through sensitivity analysis, we used the normalized sensitivity index definition as defined in [20] as it has done in [13, 27].

Definition. The Normalized forward sensitivity index of a variable, $\mathscr{R}_0$, that depends differentiably on a parameter, p, is defined as:

$$\Lambda_p^{\mathscr{R}_0} = \frac{\partial \mathscr{R}_0}{\partial p} \times \frac{p}{\mathscr{R}_0}.$$

for p represents all the basic parameters and $\mathscr{R}_0 = \frac{\alpha\beta\pi\delta}{\mu(\mu+\beta)(\mu+\rho+\sigma)(\mu+\varepsilon+\delta)}$ For the sensitivity index of $\mathscr{R}_0$ to the parameters:

$$\Lambda_\alpha^{\mathscr{R}_0} = \frac{\partial \mathscr{R}_0}{\partial \alpha} \times \frac{\alpha}{\mathscr{R}_0} = 1 \geq 0.$$

$$\Lambda_\sigma^{\mathscr{R}_0} = \frac{\partial \mathscr{R}_0}{\partial \sigma} \times \frac{\sigma}{\mathscr{R}_0} = -\frac{\sigma}{\sigma+\rho+\mu} \leq 0.$$

And it is similar with respect to the remaining parameters.

The sensitivity indices of the basic reproductive number with respect to main parameters are found in Table 3. Those parameters that have positive indices $(\alpha, \beta, \; and \; \delta)$ show that they have great impact on expanding the disease in the community if their values are increasing.

Also those parameters in which their sensitivity indices are negative $(\varepsilon, \rho, \sigma,\ and\ \mu)$ have an effect of minimizing the burden of the disease in the community as their values increase. Therefore, policy makers, stakeholders should work on decreasing the positive indices and increasing negative indices parameters.

TABLE 3. Sensitivity indices table.

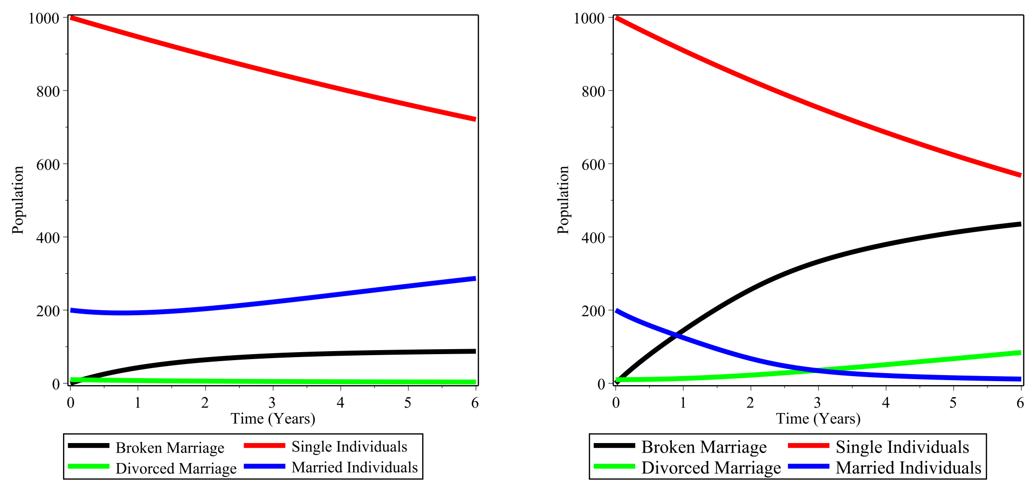
Parameter symbol	**Sensitivity indices**
α	+ve
δ	+ve
ε	-ve
ρ	-ve
μ	-ve
σ	-ve

4. NUMERICAL SIMULATION

Numerical simulations of the model (1) are carried out, in order to illustrate some of the analytical results of the study. A set of reasonable parameter values given in Table 3. These parameter values ware obtained from literature and some of them were assumed and estimated. We used S(0) = 10000, M(0) = 20, B(0) = 100, D(0) = 0, as initial values for simulation of marriage divorce model in addition to parameter values in Table 4.

TABLE 4. parameter values for the Marriage divorce model.

Parameter symbol	**Value for** $< \Re_0$	**Value for** $> \Re_0$	**Source**
π	0.4	0.4	Assumed
β	0.08	0.04	[18]
α	0.4	0.03	[14]
δ	0.05	0.01	[5]
μ	0.015	0.015	Assumed
ρ	0.001	0.0303	[5, 16]
σ	0.3	0.04	[5]
ε	0.3	0.09	Assumed



(A) (B)

FIGURE 2. (a) Simulation results of marriage divorce model when $\Re_0 = 0.0526 < 1$, (b) when $\Re_0 = 11.24 > 1$

Figure (2)(a), shows that when $\Re_0 < 1$, there is no divorced marriage even if their exists small break in a marriage. It also shows that a small number of divorce were got broken but do not move to the divorced compartment due to the reason that this individuals renewed their marriage and continue their life together.

Figure (2)(b), shows that the number of broken and divorced marriage individuals increases due to the reason that $\Re_0 > 1$. The figure also shows that the number of married individuals decreased in their number because of the contact of divorced individuals and married individuals.

4.1. Effect of α on the Married and Divorced individuals. As we see in **Figure** (3), the effect of was experimented by changing its value from 0.09 to 0.9. From the figure we see that it decreases the married individuals and increase the number of divorced marriage. So that, we should work on decreasing the contact rate of divorced individuals to married individuals to decease the number of divorce in a marriage.

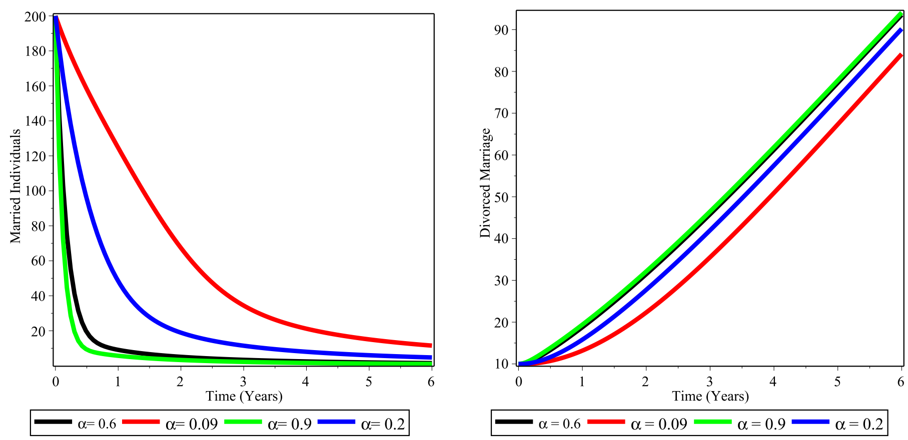


FIGURE 3. The effect of α on married individuals and divorced marriage.

4.2. Effect of δ on the Broken and Divorced individuals. Changing the value of δ from 0.05 to 0.6 the broken marriage and divorced marriage increase in number as Shown in the **Figure** (4). We can understand from the figure that as we increase α, the number of broken marriage decreases and divorced marriage increases. Hence, we should work on how to bring those broken individuals come to solution to renew their marriage and live together before they got divorce.

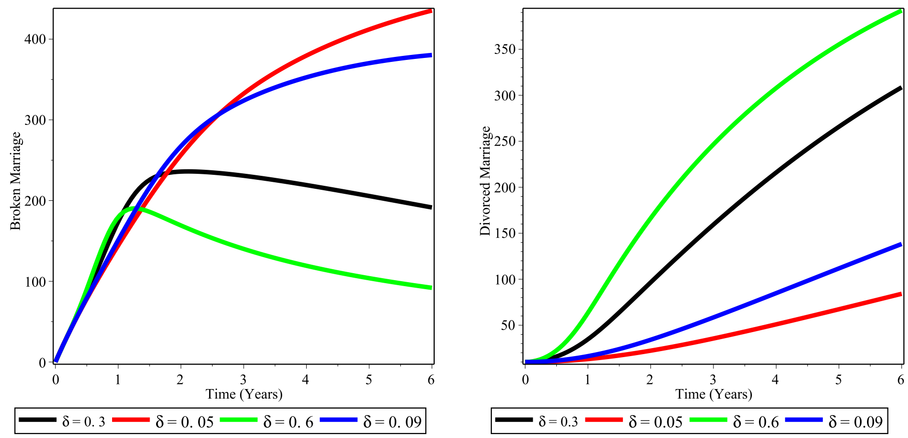


FIGURE 4. The effect of δ on broken marriage and divorced marriage.

4.3. Effect of ε on the married and Divorced individuals. In **Figure** (5), we can see that ε play an important role in bringing down the number of divorced individuals as its value increases. This indicates that if the broken individuals recovered from their conflict, then we got a decrease number of divorced marriage.

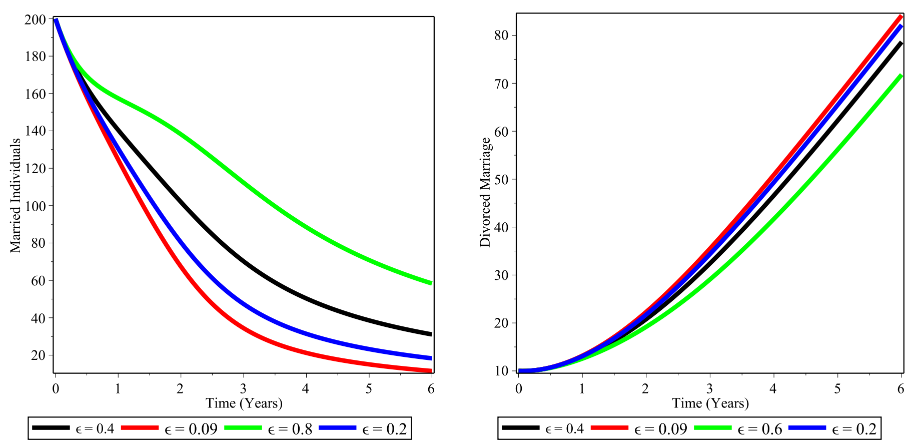


FIGURE 5. The effect of ε on married individuals and divorced marriage.

5. CONCLUSION

In this paper, we proposed and analysed a deterministic mathematical model for marriage divorce in a population. From the model analysis we obtained a region where the model is well-posed mathematically and epidemiologically meaningful. We determined the divorce free and endemic equilibra points and their local and global stability analysis in relation to $\mathscr{R}_0$. The model bifurcation analysis is done and exhibits forward bifurcation at $\mathscr{R}_0 = 1$. Sensitivity analysis of the model was performed and identified the positive and negative indices parameters. Numerical simulation was performed and displayed graphically to justify the analytical results.

Therefore, from the above results we recommend stakeholders and policy makers to give a positive feedback for parameters that has negative indices and put negative feedback on positive indices in order to control marriage divorce in a population. Doing on the parameters α and ε is an effective control to combat divorce. This mean that, reducing the contact rate between the marriage and divorce and educating separators to refrain from divorce and renew their marriage are important ways of control divorce in a population.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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