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COMPARATIVE ANALYSIS OF LOTKA-VOLTERRA TYPE MODELS WITH NUMERICAL METHODS USING RESIDUALS IN MATHEMATICA

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Abstract. This paper details the approximate solution to the Lotka-Volterra type models with no closed-form by employing the use of two numerical techniques; Runge-Kutta Fehlberg and the 4-stage Runge-Kutta method. From the numerical techniques used, a comparative analysis is carried out, and an approximate solution is obtained. Two cases were considered, case 1 details on indistinguishable graphs but different numerical values (not enough to conclude on the efficient technique) are obtained using the two techniques and while case 2 shows that both numerical techniques give same graphical representation and numerical values, hence this necessitate the reason for this investigation; by so doing the *log – plots* and residuals was introduced to obtain the most effective and efficient technique under such condition.

Keywords: Runge–Kutta–Fehlberg method; 4-stage Runge-Kutta method; Lotka-Volterra model; predator-prey-scavenger model; one-prey-one-predator; Mathematica[®].

2010 AMS Subject Classification: 65L05, 65L06.

1. INTRODUCTION

A simple predator-prey-model describes a system that details the interaction and communication among species for food, or the competition between two or three telecommunication companies and internet service providers. This is the reason why it is unrealistic for any species to

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live in isolation, instead it is observed that different types of interaction is exhibited among several species. Several researchers have investigated the simple predator-prey model [1, 2, 3, 4]. Over the years, it was observed that the simple Lotka-Volterra has some draw backs. To overcome this drawback, a logistic growth term was introduced, which details the quadratic models and depicts the saturation effect in feeding. In the process of time, a framework that extended the work of Lotka and Volterra on the simple one predator and one prey was extended to the study of three species (two preys and one predator or two predators and one preys) interaction by other researchers [5, 6, 7]. The investigation of this modified system was carried out numerically and analytically.

In this regard, other researchers have investigated and introduced scavenger species [8, 9], stability analysis has been considered [10, 11]. These species are threats to preys, they consume dead bodies of other animals. They are also well-known as carnivores. Although, their presence serves as great advantage to the ecosystem in the sense that they balance the ecosystem by eating up dead animals. This results into cleaning the environments and getting rid of odor or carcasses. These animals includes Vultures, Hyenas, Jackals e.t.c. Ben Nolting [12] developed a model where he incorporated scavenger species into the simple Lotka-Volterra model and he assumed that in the absence of other species, the scavenger would die exponentially by immensely profiting directly from the proportion of deaths of other species and predation. However, Pre-vite and Hoffman [13], investigated the presence of the scavenger which is also a predator to the prey x and consumes the death body of the predator y . They also studied a scenario where the presence of the scavenger is of non-effect to the prey and predator and a model and assumptions were presented to that effect. Predator-prey scavenger model in this paper, would be investigated using the Runge-Kutta-Fehlberg and 4 stage R-K method to obtain numerical approximations. The most effective numerical technique when the graphical and numerical experiments are the same when considering the formulated nonlinear differential equations. The Runge-Kutta-Fehlberg known as RKF45 [14] is part of the family of the Runge-Kutta family and its procedure to determine the accuracy of the step-size to obtain better results in comparison to other numerical methods. The RKF45 has its application to technological systems such as virus transmission [15], boundary value problems [16], [17]. The conventional known

methods with a constant step-size is the classical Runge-Kutta method of order four [18, 19] gives a good results and obtains its accuracy without the need of higher derivative calculations. Although, its error estimation terms is limited as the one-step method with an adaptive step size which gives better error estimation in comparison to one-step constant step-size of 4 stage R-K method. Conventionally, the RKF45 method is a method with an error estimator of order $O(h^5)$ and is expected to give better result than 4 stage R-K method, and this will be investigated in this paper and give proper conclusion by the aid of graphical and numerical representations. To achieve comparison, we introduce the log-plot by comparing the residual error because errors sometimes exhibits a range of orders of magnitude and the log-plot of Abs [residuals].

This paper is structured as follows: Section 2 details on the formulation of governing systems of equations, and assumptions which describes the interaction between the species. In section 3, the methodology is discussed. Under section 4, the numerical simulations of the respective generalized governing system of equations were visually illustrated and numerical approximation shown using computer algebra software. In section 5, the discussion of the numerical solutions was done and the conclusions of the study are stated. The aim of this paper is to show the comparison of the obtained results and numerical approximation solution between the Runge-Kutta Fehlberg method (RKF45) and 4-stage R-K method with respect to the Predator-Prey-Scavenger Model.

2. GOVERNING SYSTEM: PREDATOR-PREY-SCAVENGER MODEL

2.1. Predator-Prey-Scavenger Model. The system considered is patterned to the definition:

Definition 1. *Let a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be defined as nonlinear then by we introduce the definition; A system of nonlinear equation contains a set of equations such that*

$$\begin{aligned}
 f_1(y_1, y_2, \dots, x_n) &= 0 \\
 f_2(y_1, y_2, \dots, x_n) &= 0 \\
 &\vdots \\
 f_n(y_1, y_2, \dots, y_n) &= 0,
 \end{aligned}
 \tag{1}$$

where $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and f_i 's is nonlinear real function such that $i = 1, 2, 3, \dots, n$. In the problem considered, the nonlinear Predator-Prey-Scavenger Model is a system of first order differential equations. The model does not have a closed form solution and to obtain an approximate solution, we employed the use of numerical methods RKF45 and 4-stage R-K method to compute the systematic analysis using computer algebra system (CAS) to show the most effective and efficient method. The Predator-Prey-Scavenger Model is presented:

$$(2) \quad \dot{x} = x [1 - x - y - z - \alpha - \beta x^2]$$

$$(3) \quad \dot{y} = y [\gamma x - \delta - \eta - \mu y]$$

$$(4) \quad \dot{z} = z [v x + \omega y - \zeta - \varepsilon - \tau z]$$

$$(5) \quad x_0 = 10, y_0 = z_0 = 5$$

The scavenger model is modified on the classical Lotka-Volterra with the following assumptions:

- The parameters are non-negative for the predator, prey and scavenger respectively, according to [10], the parameters are defined in Table 1 as:

TABLE 1. Biological definition of parameters

<i>Parameters</i>	<i>Definition</i>
α	ratio of harvesting of prey to intrinsic growth rate of prey species
β	ratio of toxicity of prey to intrinsic growth rate of prey
γ	ratio of prey's attack from predator to intrinsic growth rate of prey
δ	ratio of predator's death rate to the intrinsic growth rate of the prey,
η	ratio of harvesting of predator to intrinsic growth rate of prey species
μ	ratio of toxicity coefficient of predator to prey's attack rate from predator
v	ratio of prey's attack rate from scavenger to intrinsic growth rate of prey,
ω	ratio of predator's attack rate from scavenger to the prey's attack rate from predator
ζ	ratio of scavenger death rate to the intrinsic growth rate of the prey
ε	ratio of harvesting of scavenger to intrinsic growth rate of prey species
τ	ratio of toxicity coefficient of scavenger to prey's attack rate from scavenger

- In the absence of the prey population with non-accidental occurrence, the predator and scavenger population die.
- The interaction functions of the model is assumed to be continuous and continuous partial derivative, therefore the model exists and is unique.
- The scavenger population benefits from the prey and predator that die naturally without the presence of external factors.
- It's assumed that no entering nor leaving the vicinity be of benefits to either predator, prey or scavenger.

In addition to the description of the model, we further assume that the model only accommodates three species in a balanced ecosystem. The prey population is described as the feeder to other population while the second population is the predator which feeds on the prey and the third population for survival as the environment is competitive. The last population is known as scavengers, they feed on the prey and eat the dead bodies of the predator, however, the scavenger population affects the predator population indirectly by reducing the prey for their own survival.

2.2. One prey and one predator. Adeniji et.al [20] and Noufe et.al [21], investigated on prey-predator model type using some numerical techniques, but for the purpose of this paper, the authors presented a modified prey-predator model type. This model is represented as follows: Consider the prey-predator system,

$$(6) \quad \dot{x} = a_{11}x - a_{12}x^2 - a_{13}xy$$

$$(7) \quad \dot{y} = -a_{12}y - a_{22}y^2 + a_{13}xy$$

with initial conditions

$$(8) \quad x_0 = X_0, y_0 = Y_0$$

From Equation 6 - Equation 8, the constant parameters are defined biologically for the purpose of the model [20] .

3. METHODOLOGY

A general expression follows a system of nonlinear equation with initial value problem such that:

$$(9) \quad x'_i = f_1(t, x_1, x_2, \dots, x_n); x_i(t_0) = x_0,$$

$$(10) \quad y'_i = f_1(t, x_1, x_2, \dots, y_n); y_i(t_0) = y_0,$$

$$i = 1, 2, 3, \dots, n$$

where the x'_i 's and y'_i 's represent the prey and predator population. Solving the initial value problem in the Equation 9 and Equation 10, using the Runge-Kutta Fehlberg which is the combination of two different orders (order four and order five). The algorithm follows such that at different steps, order four and order five approximations for the solution of the nonlinear predator-prey-scavenger model are obtained by using order four method with five stages and order five method with six stages representation. This is obtained as follows by defining the general formulation of Runge-Kutta method of order five:

$$k_1 = hf_i(t, x_1, x_2, \dots, x_n),$$

$$k_2 = hf_i\left(t + \frac{1}{4}h, x_1 + \frac{1}{4}k_1, x_2 + \frac{1}{4}k_1, \dots, x_n + \frac{1}{4}k_1\right),$$

$$k_3 = hf_i\left(t + \frac{3}{8}h, x_1 + \frac{3}{32}k_1 + \frac{9}{32}k_2, x_2 + \frac{3}{32}k_1 + \frac{9}{32}k_2, \dots, x_n + \frac{3}{32}k_1 + \frac{9}{32}k_2\right),$$

$$k_4 = hf_i\left(t + \frac{12}{13}h, x_1 + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3, x_2 + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3 \dots\right) \\ \left(x_n + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right),$$

$$k_5 = hf_i\left(t + h, x_1 + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4, x_2 + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4, \dots\right) \\ \left(x_n + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right),$$

$$k_6 = hf_i\left(t + h, x_1 - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5, x_2 - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5 \dots\right) \\ (11) \quad \left(x_n - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right),$$

where $i = 1, 2, \dots, n$. By formulation of the fifth order of Runge-Kutta method in Equation 11, the approximation to the initial value problem is obtained using the Runge-Kutta method of order four as represented in Equation 12

$$(12) \quad x_{ik+1} = x_k + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5,$$

where $i = 1, 2, \dots, n$. In the same vein, the approximate solution to the initial value problem is likewise obtained by employing the fifth order of Runge-Kutta method and represented as

$$(13) \quad x_{ik+1} = x_k + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6.$$

Subtracting Equation 13 from Equation 12, a formula was generated to estimate the error of the Runge-Kutta Fehlberg method and represented as

$$(14) \quad E^* = \frac{1}{360}k_1 + \frac{128}{4275}k_3 + \frac{2197}{7524}k_4 - \frac{1}{50}k_5 + \frac{2}{55}k_6,$$

where the values of k_1, \dots, k_6 are known from each steps, such that $\lambda = 0.9 \left(\frac{\varepsilon}{E^*} \right)^{\frac{1}{4}}$. It's imperative to know that the optimal step size can be obtained from Equation 14, if $E^* \leq \varepsilon$, we keep x as the step solution and move to the next step size λh and if $E^* < \varepsilon$, the current step size is recalculated with the step size λh . It is quite important to note that RK5 method needs evaluation of six per each step and 4 stage R-K requires evaluation of four per each step which results to ten for both methods. As it were, for effective computation, Fehlberg requires only six evaluations using k values for RK5 and 4 stage R-K methods. For implementation procedure in relation to objective of this paper, 4 stage R-K and RKF45 approximate solution of the predator-prey-scavenger model will be compared and investigated.

4. NUMERICAL SIMULATION

This section details on the results, computational analysis, and implementation of the algorithm in Mathematica[®] with its built-in functions to obtain a numerical approximation for the Prey-Predator-Scavenger model. The model case closed form solution is unobtainable which necessitates the reason for the numerically approximate solution, comparison between the Runge-Kutta Fehlberg method and 4-stage R-K method. The investigation of the global dynamics of

the system was carried out using numerical simulations. To understand the purpose of this research when both numerical techniques gives graphs (which are visibly indistinguishable) and numerical approximations, we investigate by assuming certain parameters [10]. Two cases of prey-predator models are investigated for the numerical simulation. Case 1 considers indistinguishable graphs but different numerical values obtained using the two numerical techniques are not sufficient enough to identify the best technique, and case2 details on the solution obtained by the numerical techniques which result in identical graphical representation and numerical values.

4.1. Case 1: Prey-Predator-Scavenger model. Parameters assumed are as follows:

$$\begin{aligned} \alpha &= 0.2, \beta = 0.3, \gamma = 0.7 \\ \delta &= 0.1, \eta = 0.1, \mu = 0.1, \nu = 0.5 \\ \omega &= 0.2, \zeta = 0.1, \varepsilon = 0.1, \tau = 0.1 \end{aligned} \tag{15}$$

The Prey-Predator-Scavenger model considered

$$\dot{x} = x [1 - x - y - z - \alpha - \beta x^2] \tag{16}$$

$$\dot{y} = y [\gamma x - \delta - \eta - \mu y] \tag{17}$$

$$\dot{z} = z [\nu x + \omega y - \zeta - \varepsilon - \tau z] \tag{18}$$

Taking the initial conditions

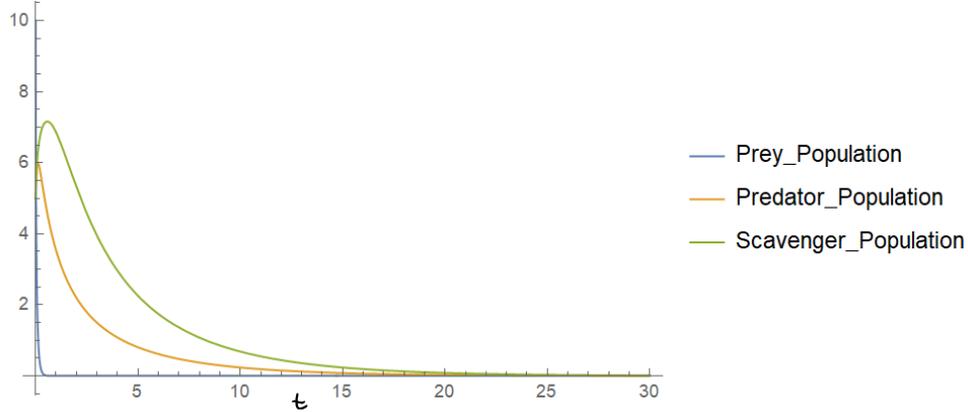
$$x_0 = 10, y_0 = z_0 = 5 \tag{19}$$

The domain of the system above is represented as

$$\mathbb{R}_+^3 = [(x, y, z \in \mathbb{R}^3 | x \geq 0, y \geq 0, z \geq 0)] \tag{20}$$

Case 1 is a special case investigated when the graphical representation of both numerical methods is indistinguishable but the numerical values differs as inference can't be drawn. We introduce the RKF45 method and 4 stage R-K method, by so doing, we integrate the system in Equation 16 - Equation 17 with the aid of built-in algorithm in Mathematica[®]. Investigating the

system with Runge-Kutta Fehlberg method and 4 stage R-K method, we obtain same graphical representation as seen in Figure 1.



(a)

FIGURE 1. Graphical solution of RKF45 and 4-stage R-K method

To further understand the case 1, the numerical values obtained using the numerical methods differs from each other as seen in Table 2

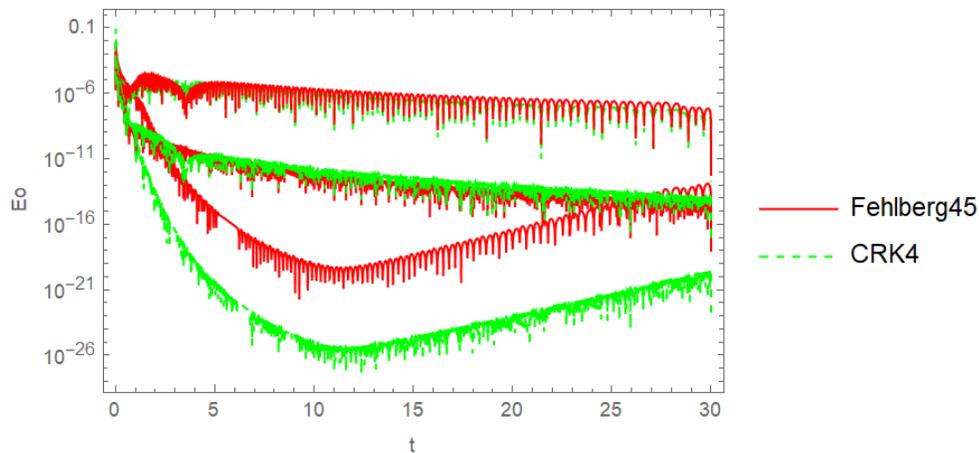
TABLE 2. Values of $x(t), y(t), z(t), t \in [0, 10]$ between RKF45 and 4-stage R-K method

t	RKF45			CRK4		
	$x(t)$	$y(t)$	$z(t)$	$x(t)$	$y(t)$	$z(t)$
0	10	5	5	10	5	5
1	0.0000688132	3.53403	6.8561	0.0000688382	3.53401	6.85604
2	2.0863×10^{-8}	2.19149	5.31316	2.12017×10^{-8}	2.19148	5.31312
3	7.513×10^{-11}	1.49691	3.94668	7.77512×10^{-11}	1.49691	3.94665
4	1.51233×10^{-12}	1.07916	2.95899	1.57858×10^{-12}	1.07915	2.95897
5	9.91147×10^{-14}	0.80482	2.25412	1.03951×10^{-13}	0.804818	2.25411
6	1.49824×10^{-14}	0.614133	1.74207	1.57196×10^{-14}	0.614132	1.74207
7	4.14553×10^{-15}	0.476298	1.36221	4.35016×10^{-15}	0.476297	1.36221
8	1.79915×10^{-15}	0.373822	1.07511	1.8878×10^{-15}	0.373822	1.0751
9	1.09855×10^{-15}	0.29603	0.854738	1.1526×10^{-15}	0.296029	0.854735
10	8.72525×10^{-16}	0.236036	0.683475	9.15453×10^{-16}	0.236035	0.683473

For the purpose of the article, the numerical values of the approximation and the graphical representations do not produced the same results as in case 2. This necessitates the reason to

compare and identify which technique is better in approximation when encountered with such model with no closed form solution. So we introduce the residual formulation to investigate the errors in numerical technique through numerical integration and the time integration starts from the initial condition of the prey, predator and scavenger population in Equation 16 - Equation 18. We investigate this by the use of residuals in the software, Mathematica 12.2[®], through identification of the numerical technique to obtain better approximation, we introduce the use of *log – plot*, and *Absolute* in the coding and the use of residuals in the software and E_o is denoted as error order.

The *log – plot* through the spikes help to identify the efficient and accurate numerical technique by the use of residual, the graphical representation of the approximation between the two techniques and the results of the experiment is shown in Figure 2.

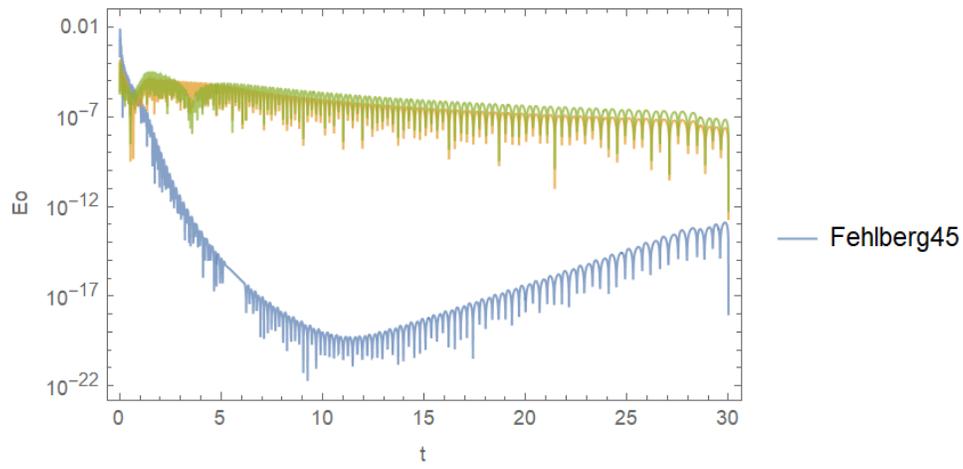


(a)

FIGURE 2. Residual solution and comparison between RKF45 and 4-stage R-K method

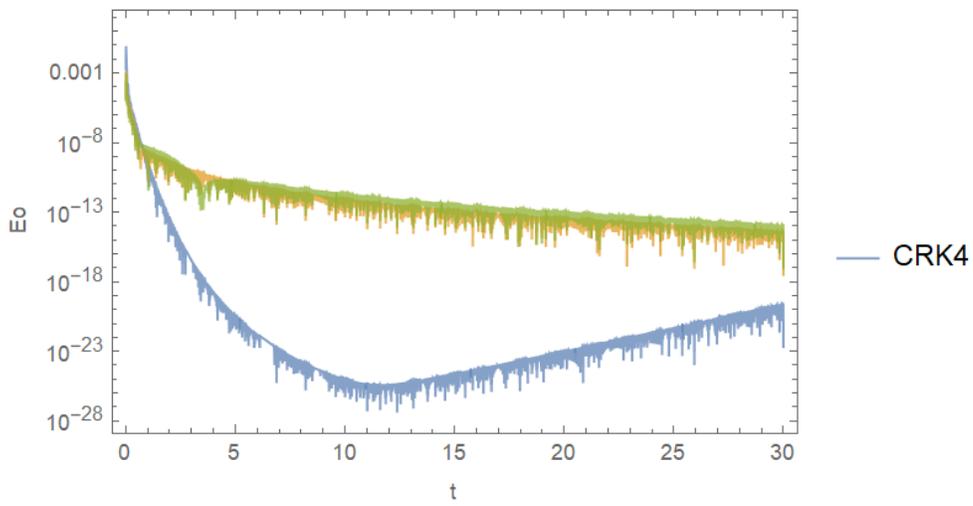
The Figure 3 shows an independent investigative plot of the *log – plot*, the deepest spike, graphical approximation of each population and the order of error for the Runge-Kutta Fehlberg method of solution.

The Figure 4 shows an approximation solution of the Predator,prey and scavenger population, *log – plot*, spike, and the order of error for the 4 stage R-K method of solution



(a)

FIGURE 3. Residual solution: RKF45 plot of $x(t), y(t), z(t)$



(a)

FIGURE 4. Residual solution: 4-stage R-K method plot of $x(t), y(t), z(t)$

4.2. Case 2 Predator-Prey type model. Consider the prey-predator system,

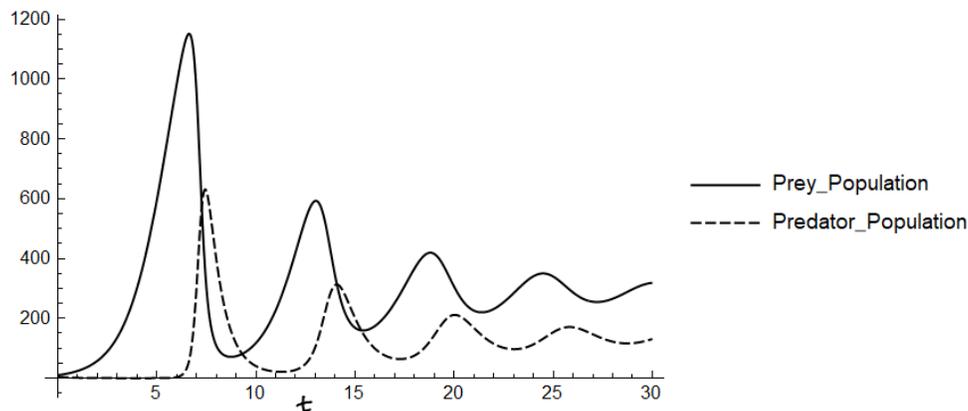
$$(21) \quad \dot{x} = a_{11}x - a_{12}x^2 - a_{13}xy$$

$$(22) \quad \dot{y} = -a_{12}y - a_{22}y^2 + a_{13}xy$$

with initial conditions

$$(23) \quad x_0 = 10, y_0 = 5$$

It is assumed that the parameters $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}$ are positive constants. The values assigned to the parameters are as follows: $a_{11} = 0.9, a_{12} = 0.0005, a_{13} = 0.0055, a_{21} = 1.5000, a_{22} = 0.00075$. The model investigated in this section is considered in the presence of an approximate solution whose numerical values are the same and graphical display can't be distinct from each other. In this section, we consider Prey-Predator model with a quadratic term, the model is investigated using known numerical methods (RKF45 and 4-stage R-K) and the results obtained are indistinguishable for both the graphs in Figure 5 and numerical values in Table 3 using the built-in algorithm in Mathematica and are presented.



(a)

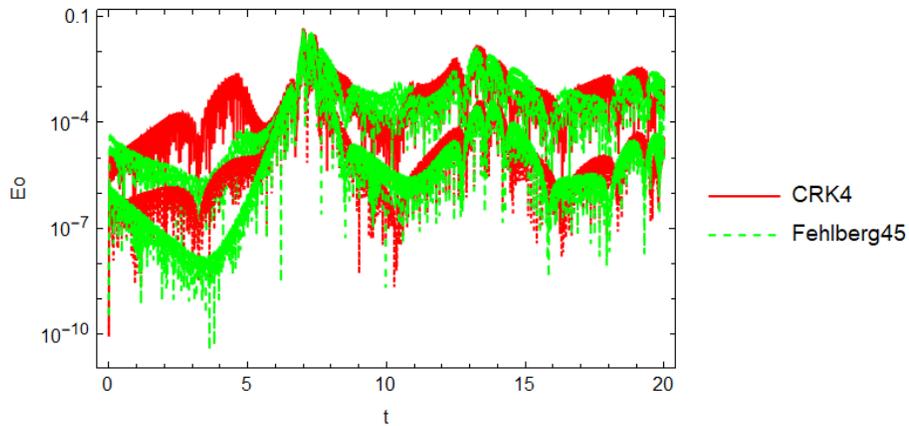
FIGURE 5. Graphical solution of RKF45 and 4-stage R-K method

As discussed, the RKF45 and 4-stage R-K method gave the same numerical values and its represented in Table 3 for objectives of the research in *case 2*.

TABLE 3. Values of $x(t), y(t), t \in [0, 10]$ between RKF45 and 4-stage R-K method

t	$x(t)$	$y(t)$
0	10	5
1	24.0458	1.2158
2	57.8003	0.335139
3	135.652	0.123702
4	300.436	0.0869564
5	593.831	0.213341
6	980.972	3.56553
7	960.452	298.988
8	109.144	398.988
9	73.4177	116.024
10	117.389	40.4762

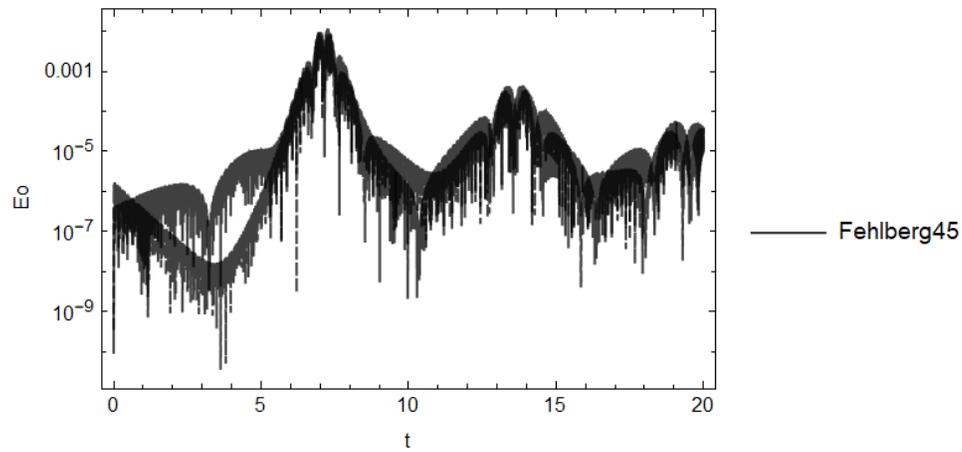
In the same way as seen in case 1, when encountered with such case, we introduce the residuals and $\log - plot$. Figure 6 shows the solution obtained through the residuals and $\log - plot$ deepest spike using the built-in algorithm of Mathematica 12.2[®] to arrive at the effective method for obtaining the approximate solution in the order of the error.



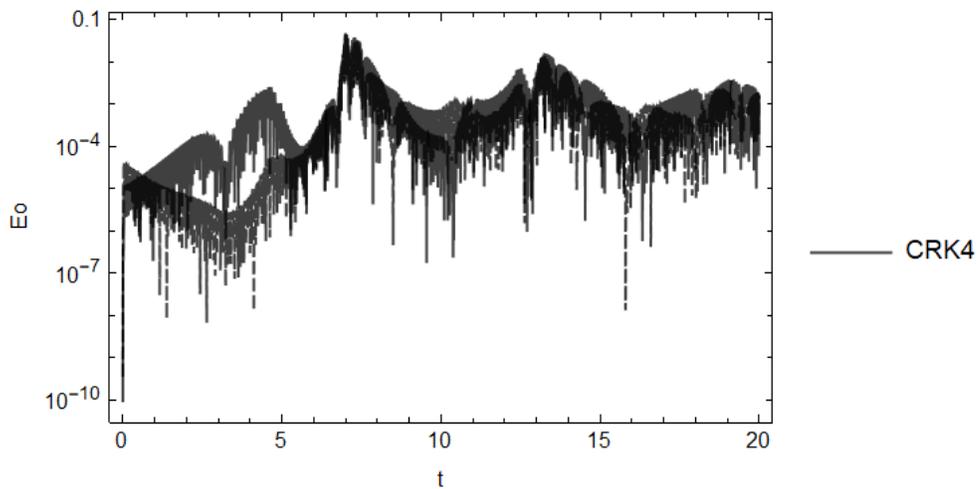
(a)

FIGURE 6. Residual solution and comparison between RKF45 and 4-stage R-K method

The Figure 7 shows an independent investigative plot of the *log – plot*, the deepest spike, graphical approximation of each population and the order of error for the Runge-Kutta Fehlberg method and 4-stage R-K method of solution. From both graphs in Figure 7, its observed that the RKF45 has an error order of 10^{-11} and the CRK4 is of order 10^{-10} .



(a)



(b)

FIGURE 7. Residual solution: (a) RKF45 plot of $x(t), y(t)$, (b) 4-stage R-K $x(t), y(t)$

5. DISCUSSION AND CONCLUSION

Two cases were considered in the paper. Case 2 was considered in the presence of graphs and numerical values are identifiable by using the numerical methods. This procedure was investigated on the predator-prey model, and the problem encountered depicted both numerical methods obtained the same approximate solution and its represented on the graphs in Figure 5 and numerical values in Table 3. By introducing the residual and *log – plot* to identify the effective numerical methods, through the built-in function of Mathematica 12.2[®], from Figure 7 in case 2 we can conclude that the RKF45 is efficient and effective in obtaining a better approximate solution to the model with no closed form solution, indistinguishable graphs and numerical values. RKF can be seen to have an error order 10^{-11} and 4 stage R-K method has an error order 10^{-10} . Hence, the RKF method gives a better approximation.

Case 1 details the approximate solution whose graphs are identical but conclusive inference can't be drawn from the numerical values as seen in Table 2. From Figure 3 and Figure 4, the log-plot was used to investigate the prey-predator-scavenger model with residual to obtain the approximate solution through the Runge-Kutta Fehlberg and 4 stage R-K method built-in function algorithm. From Figure 3, RKF has an error order 10^{-22} while the 4 stage R-K method used in investigating the model by the residual has an error order 10^{-28} . From the graphical representation of the residuals, we can conclude that the 4 stage R-K method gives better approximation to the prey-predator-scavenger model when compared to the Runge-Kutta Fehlberg method. Ideally, the RKF45 method should give better approximations but from our investigation of the prey-predator-scavenger model, there was an exception despite the graphical presentations in Figure 1 and numerical values in Table 2. The results are not strong enough to conclude, hence the reason for residual and *log – plot* as seen in Figure 2. The reason for this procedure and its importance is to investigate how to obtain an effective, and efficiency of a numerical technique when encountered with a system of ODEs with no closed form solution and the experimental procedure obtained as the same outcome (graphically and numerically).

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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