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STABILITY AND HOPF BIFURCATION ANALYSIS OF A DELAYED ECO-EPIDEMIOLOGICAL MODEL OF IYSV DISEASE DYNAMICS IN ONION PLANTS WITH NONLINEAR SATURATED INCIDENCE AND LOGISTIC GROWTH

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Abstract. Iris Yellow Spot Disease (IYSD), caused by Iris Yellow Spot Virus (IYSV) is a destructive and fast-spreading virus disease of onion plants worldwide. It is mainly transmitted by an insect vector called thrips tabaci in a persistent and propagative manner and as such, there is a significant latent time after acquisition of the virus by the vector and an incubation time is needed for the appearance of disease symptoms on plants. In this paper, we formulate and analyze a non-linear mathematical model to explore the dynamics of IYSV disease in onion plants using system of delay differential equations by incorporating incubation and latent periods as time delays factors. The delays are introduced by adding an exposed population for onion plants representing the plants that are infected but not yet infective and by taking into account that there is fraction of the newly exposed onion plants that do not die during incubation period before becoming infective. It is assumed that the onion plant grows

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logistically in the farm so that the total onion plant population is taken as variable. The local stability of the disease-free equilibrium in the presence of delays is investigated using Descartes's rule of signs. We establish the sufficient conditions for the stability of the endemic equilibrium in presence of delays and we investigate the occurrence of Hopf bifurcation when certain conditions are satisfied by considering the two delays as bifurcation parameters. We compute the critical values of the delays which preserve the local asymptotic stability of the endemic equilibrium and the model shows an oscillatory behavior beyond these critical values. Finally, numerical simulations are performed and displayed graphically to support the analytical results, and the eco-epidemiological implications of the key outcomes are briefly discussed.

Keywords: IYSV; time delays; Holling type II; Hopf bifurcation; stability analysis; logistic growth.

2010 AMS Subject Classification: 92C80.

1. INTRODUCTION

Onion (Allium cepa L.) is one of the most consumed vegetable worldwide due to its richness in minerals [1]. It serves as cash crop to farmers and it is grown across Africa especially in tropical regions [2]. In Kenya, biotic and abiotic factors are responsible of many losses in onion farms and as consequence onions production can not meet the local demand [3, 4]. Among biotic factors, the most limiting causing as much as 60% production losses is Iris Yellow Spot Disease (IYSD) [2, 4]. It is caused by Iris Yellow Spot Virus (IYSV; family Bunyaviridae, genus Tospovirus), transmitted by the vector thrips tabaci and is the most destructive disease of onions in Africa including Kenya [5, 6]. The first report of Iris yellow spot virus (IYSV) in Kenya was observed in 2009 and since then, the incidence of the disease has increased dramatically in major onion production farms of the country currently ranging from 27 to 72% [4]. The disease is now endemic in the country and the virus is transmitted to onion plants via the bites of infected thrips vectors. After virus latent period of 4–5 days, an infected thrips vector can transmit the virus for the rest of its lifetime [7]. Upon bitten by an infected thrips, how long an infected onion plant take to show symptoms called incubation period is not clear in onions because lesion symptoms are influenced by host cultivar, how many thrips were feeding, how long they fed and climate factors [8]. However for persistent and propagative transmitted plant diseases, both incubation and latent periods are greatly long and therefore significant. In recent years, mathematical modeling technique has gained much interest because it gives a better understanding of the disease dynamics from which appropriate decisions can be taken for disease control purposes[9]. Many biological processes such as gestation and infection take time to complete and in the case of persistently transmitted diseases, it takes time for a virus to invade a cell and spread throughout the host and the process is called time delays [10]. In order to investigate the effect of time delays in the dynamics of the disease, one of the most strong mathematical tools are delay differential equations (DDEs). They have been extensively applied in various fields of biology and biotechnology to study the transmission dynamics of many animal and human diseases [11]. From disease control perspectives, time delay is essential in designing appropriate disease prevention measures and control policies [12, 13]. Models formulated using delay differential equations (DDEs) are more biological realistic as compared to ODE models at the cost of increasing complexity in the analysis [10]. In plant diseases epidemic, the first model formulated using delay differential equations was developed by Van der Plank in [14] by incorporating the single delay and has formed the basis for vector-borne plant diseases modeling. Subsequently, some few models with time delay(s) have been proposed. For example, in [15], the authors proposed a plant disease model by considering incubation period as delay factor. In [16], the authors formulated a model for the transmission dynamics of vector-borne plant disease considering incubation delay. In the above mentioned papers, the authors have modelled the incubation period by incorporating it as delay factor in the form of delayed SIR model, meanwhile they did not introduced the total number of exposed plants during the incubation period which according to [17] is more appropriate from the biological point of view. Moreover, they considered only a single delay as incubation period in plant and have neglected the latent period in vector, meaning that once a healthy vector has been infected it becomes instantly infectious which is not biologically realistic since the vector takes time to become infected after acquisition of the virus [18]. However, the authors in [19] proposed a vector-borne plant disease by including two delays standing for incubation and latent periods, but they ignored the number of exposed plants in the incubation period and did not give detailed theoretical analysis to this model. Moreover, they assumed that the total plant population is constant which is not biologically realistic as compare to logistic growth for which the intrinsic growth rate and the carrying capacity play an important role in disease dynamics [20]. The authors in [21] proposed a vector-borne plant viral disease with two delays accounting for incubation and latent periods. But, they used mass action incidence as rates of infection which is realistically limited as compare to nonlinear saturated incidence of Holling type II form although being mathematically correct [19]. They also ignored the number of exposed plants in the incubation period and the existence of Hopf bifurcation of the system was not systematically analysed, which could induce misleading conclusions about the stability of the system. To the best of our knowledge, no model has been considered to systematically analyze the dynamics of IYSV disease in onion plants with Holling type II nonlinear saturated incidence, logistic growth and time delays in terms of incubation and latent periods. Motivated by references [19, 21] and the above discussion, in this paper, we extend the model in [22] by introducing the class of exposed plants representing the number of exposed plants in the incubation period and by incorporating two delays accounting for incubation and latent periods to explore their effect on the dynamics of the disease. Here, to increase realism the onion plants have logistic growth so that the total onion plants population is not constant. The rest of the paper is organized as follows. In section 2, the mathematical model with two delays is described and formulated. In section 3, the positivity and boundedness of solutions are investigated. Section 4 deals with the computation of the threshold parameter ($\mathscr{R}_0$) and the mathematical analysis of the model including the local stability of the disease-free and endemic equilibrium points together with the existence of Hopf-bifurcation. In section 5, numerical simulations are performed to support theoretical results. The last section presents a brief discussion and conclusions of the paper.

2. DESCRIPTION AND FORMULATION OF THE MODEL

2.1. Assumptions.

The following assumptions will be used to simplify the model.

(i) In the presence of the disease, the total number of onion plants is then given by: $N_O(t) = S(t)+E(t)+I(t)$. The total number of thrips vector is given by: $N_v(t) = X(t)+W(t)$.

(ii) In the absence of the disease, the susceptible onion plant population grows logistically as
$$rS(t)\left(1-\frac{S(t)+E(t)+I(t)}{K}\right).$$

(iii) The susceptible and exposed onion plants have the same death rate (g).

(iv) After a contact with infective thrips vectors, susceptible onion plants are converted into exposed plants that become infective only after an incubation period $\tau_1 \in \mathbb{R}^+$.

(v) There is fraction of the newly exposed onion plants that do not die during incubation period τ_1 before becoming infective and as such, $e^{-g\tau_1}$ is the survival rate.

(vi) The length of latent period is $\tau_2 \in \mathbb{R}^+$. Thus $e^{-\sigma\tau_2}$ represents the survival rate of the thrips vectors during the latent period τ_2 .

2.2. The model derivation.

The model considered in this work is an extension of the model of IYSV disease dynamics in onion plants proposed in [22]. Clearly, we introduce the class of exposed plant representing the number of exposed plants in the incubation period and then, we include the time delays in terms of incubation and latent periods. Therefore, the total onion plants population is sub-divided into three sub-classes: Susceptible $S(t)$, Exposed $E(t)$, and Infected $I(t)$, in which exposed means the onion plants that have acquired the IYSV disease but are not yet infective. The total number of onion plants is then given by $N_O(t) = S(t) + E(t) + I(t)$. The total thrips vector population is sub-divided into two sub-classes: Susceptible $X(t)$, and Infected $W(t)$. The total number of thrips vector is given by $N_v(t) = X(t) + W(t)$. We do not consider recovered classes because onion plants do not recover once infected by IYSV disease and thrips vectors do not get sick from IYSV as they are just carriers. Since the death rate for the exposed class is constant (g), it is appropriate to assume that the density of exposed plants at time t with the incubation time τ_1, denoted by $E(t,\tau_1)$ satisfies the age-structured modeling approach of McKendrick-Von Foerster in [23]. At any time t, since $\dfrac{\delta_1 S(t)W(t)}{1+m_1W(t)}$ is the rate of infection, then the accumulated total number of exposed onion plants is obtained by integrating

$$\frac{\delta_1 S(t-u)W(t-u)e^{-gu}}{1+m_1W(t-u)} \tag{2.1}$$

for $u \geq 0$. The term $\frac{\delta_1 S(t-u)W(t-u)}{1+m_1W(t-u)}$ stands for the rate of infection at previous time, where e^{-gu} represents the survival rate from susceptible to exposed onion plants as time goes from $t-u$ to t. With the above observations, the accumulated total number of exposed onion plants is

$$E(t) = \int_0^{\tau_1} \frac{\delta_1 S(t-u)W(t-u)e^{-gu}}{1+m_1W(t-u)} du, \tag{2.2}$$

and with a change of variable $s = t - u$, we obtain

$$E(t) = \int_{t-\tau_1}^{t} \frac{\delta_1 S(s) W(s) e^{-g(t-s)}}{1 + m_1 W(s)} ds. \tag{2.3}$$

By differentiating both sides of equation (2.3) with respect to t using the generalized Leibnitz rule of differentiation [24], we obtain

$$\frac{dE(t)}{dt} = \frac{\delta_1 W(t) S(t)}{1 + m_1 W(t)} - \frac{\delta_1 e^{-g\tau_1} W(t-\tau_1) S(t-\tau_1)}{1 + m_1 W(t-\tau_1)} - gE(t). \tag{2.4}$$

The time τ_1 is the incubation period of onion plants. $S(t-\tau_1), W(t-\tau_1)$ denotes the numbers of susceptible onion plants population and infected thrips vector population respectively at time $t - \tau_1$ and the term $\frac{\delta_1 e^{-g\tau_1} W(t-\tau_1) S(t-\tau_1)}{1+m_1 W(t-\tau_1)}$ represents the individuals surviving in the incubation period τ_1 and entering into infectious class at time t. By applying the balance law of compartment, the rate of change in the number of infected onion plants is given by

$$\frac{dI(t)}{dt} = \frac{\delta_1 e^{-g\tau_1} W(t-\tau_1) S(t-\tau_1)}{1 + m_1 W(t-\tau_1)} - \rho I(t) - gI(t). \tag{2.5}$$

Letting τ_2 be the latent period of the thrips vector, the rate of change in the number of infected thrips vector is given by

$$\frac{dW(t)}{dt} = \frac{\delta_2 e^{-\sigma\tau_2} X(t-\tau_2) I(t-\tau_2)}{1 + m_2 I(t-\tau_2)} - \sigma W(t), \tag{2.6}$$

where $X(t-\tau_2), I(t-\tau_2)$ stands for the number of susceptible thrips vectors and infected onion plants respectively at time $t - \tau_2$. All parameters are summarized in Table 1.

Table 1. Description of parameters for model (2.7)

Parameter	Description
r	Intrinsic growth rate of onion plants
K	Carrying capacity onion farm
δ_1	Transmission rate from infected thrips to healthy onion plants
δ_2	Transmission rate from infected onion plants to healthy thrips
m_1	Saturation rate of healthy onion plants due to infected thrips
m_2	Saturation rate of healthy thrips due to infected onion plants
g	Natural mortality rate of onion plants
ρ	Death rate of onion plants due to IYSD disease
λ	Birth or immigration rate of thrips vectors
σ	Natural death rate for thrips vectors
τ_1	The length of incubation period
τ_2	The length of latent period

From the above assumptions and description, the mathematical model with incubation and latent delays takes the following form:

$$
\begin{cases}
\frac{dS(t)}{dt} = rS(t)\left(1-\frac{S(t)+E(t)+I(t)}{K}\right)-\frac{\delta_1 W(t)S(t)}{1+m_1W(t)}-gS(t),\\
\frac{dE(t)}{dt} = \frac{\delta_1 W(t)S(t)}{1+m_1W(t)}-\frac{\delta_1 e^{-g\tau_1}W(t-\tau_1)S(t-\tau_1)}{1+m_1W(t-\tau_1)}-gE(t),\\
\frac{dI(t)}{dt} = \frac{\delta_1 e^{-g\tau_1}W(t-\tau_1)S(t-\tau_1)}{1+m_1W(t-\tau_1)}-\rho I(t)-gI(t),\\
\frac{dX(t)}{dt} = \lambda-\frac{\delta_2 X(t)I(t)}{1+m_2I(t)}-\sigma X(t),\\
\frac{dW(t)}{dt} = \frac{\delta_2 e^{-\sigma\tau_2}X(t-\tau_2)I(t-\tau_2)}{1+m_2I(t-\tau_2)}-\sigma W(t).
\end{cases}
\tag{2.7}
$$

Let C be a Banach space of continuous functions $\phi : [-\tau, 0] \longrightarrow \mathbb{R}^5_+$ equipped with the sup-norm,

$$\| \phi \|_C = \sup_{-\tau\le t\le 0} \{| \phi_1(t) |, | \phi_2(t) |, | \phi_3(t) |, | \phi_4(t) |, | \phi_5(t) |\}.$$

The initial conditions for the model system (2.7) are taken as below

$$S(t) = \phi_1(t), E(t) = \phi_2(t), I(t) = \phi_3(t), X(t) = \phi_4(t), W(t) = \phi_5(t),\ -\tau \le t \le 0,$$

where $\tau = \max\{\tau_1, \tau_2\}$ and $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in C\left([-\tau, 0], \mathbb{R}^5_+\right)$, such that

$$\phi_i(t) = \phi_i(0) \geq 0, \quad (i = 1, 2, 3, 4, 5). \tag{2.8}$$

3. MODEL PROPERTIES

3.1. Nonnegativity of solutions. The following theorem establishes the positivity of the solutions of the model system (2.7).

Theorem 3.1. *All solutions of the model (2.7) with the initial conditions given in (2.8) remain non-negative for all $t \in [0, +\infty)$.*

Proof. From the first equation of the system (2.7), we have

$$\frac{dS(t)}{dt} - S(t)\left[r\left(1 - \frac{S(t)+E(t)+I(t)}{K}\right) - \frac{\delta_1 W(t)}{1+m_1 W(t)} - g\right] = 0$$

Let $H(S(t), E(t), I(t), W(t)) = r\left(1 - \frac{S(t)+E(t)+I(t)}{K}\right) - \frac{\delta_1 W(t)}{1+m_1 W(t)} - g$. Then the above equation can be written as

$$\frac{dS(t)}{dt} - S(t)H(S(t), E(t), I(t), W(t)) = 0$$

$$\Longrightarrow \left[\frac{dS(t)}{dt} - S(t)H(S(t), E(t), I(t), W(t))\right] \exp\left(-\int_0^t H(S(\xi), E(\xi), I(\xi), W(\xi))d\xi\right) = 0$$

$$\Longrightarrow \frac{d}{dt}\left[S(t)\exp\left(-\int_0^t H(S(\xi), E(\xi), I(\xi), W(\xi))d\xi\right)\right] = 0$$

$$\Longrightarrow S(t) = S(0)\exp\left(\int_0^t H(S(\xi), E(\xi), I(\xi), W(\xi))d\xi\right).$$

Since, $S(0) = \phi_1(0) > 0$, then $S(t) > 0$ for $t \geq 0$.

Let $G(S(t), W(t)) = \frac{W(t)S(t)}{1+m_1 W(t)}$ and $F(X(t-\tau_2), I(t-\tau_2)) = \frac{X(t-\tau_2)I(t-\tau_2)}{1+m_2 I(t-\tau_2)}$. From the fundamental theorem of differential equations [25], the solutions of the differential equations of $E(t)$ and $W(t)$ can be written as

$$E(t) = e^{-gt}\left(E(0) + \int_{t-\tau_1}^t \delta_1 G(S(t), W(t))e^{g\eta}d\eta\right) \tag{3.1}$$

$$W(t) = e^{-\sigma t}\left(W(0) + \int_0^t \delta_2 F(X(t-\tau_2), I(t-\tau_2))d\eta\right) \tag{3.2}$$

Then, the non-negativity of the solutions $X(t)$ and $I(t)$ $\forall t > 0$, guarantee the non-negativity of $W(t)$ and thus, the non-negativity of $E(t)$ since $S(t) > 0 \, \forall t > 0$. Now, we use the method of steps to prove the positivity of $X(t)$. That is let suppose that the positivity of $X(t)$ does not holds, this implies that there exists $t_0 > 0$ such that $X(t_0) = 0$, $\frac{dX(t_0)}{dt} \leq 0$ and $X(t) > 0 \forall t \in [0,t_0)$. Then, $X(t)$ must be negative from some t_0. But, $X(t)$ is non-negative in the interval $[0,t_0)$, and the derivative at t_0 is negative. Consequently, for t_0, we obtain from the fourth equation of the model (2.7) that

$$\frac{dX(t_0)}{dt} = \lambda - \frac{\delta_2 X(t_0) I(t_0)}{1+m_2 I(t_0)} - \sigma X(t_0) = \lambda > 0,$$

which is a contradiction of the fact that $\dfrac{dX(t_0)}{dt} \leq 0$. Thus, we must have $X(t) > 0$, for all $t \geq 0$. Similarly, let suppose that $I(t)$ loses its non-negativity, then there exists $t_1 > 0$ such that $I(t_1) = 0$, implying that $I(t) > 0$ for $t \in [0,t_1)$. Now, using the fundamental theory of functional differential equations [25], we solve the third equation of the system (2.7) on $[0,t_1]$ to obtain

$$I(t_1) = e^{-(\rho+g)t} \left(I(0) + \int_0^{t_1} \delta_1 P(S(t-\tau_1), W(t-\tau_1)) e^{\rho\eta} d\eta \right) \tag{3.3}$$

where $P(S(t-\tau_1), W(t-\tau_1)) = \frac{S(t-\tau_1)W(t-\tau_1)}{1+m_1 W(t-\tau_1)}$. Since $S(k), I(k) > 0\ \forall k \in [0,t_1)$, the right hand side of equality (3.3) is non-negative, which yields that $I(t_1) > 0$, contradicting the fact that $I(t_1) = 0$. Therefore, since $X(t)$ and $I(t)$ are non-negative, then $W(t)$ is non-negative and thus, $E(t)$ is non-negative. This completes the proof of the theorem, and hence the solution set $(S(t), E(t), I(t), X(t), W(t))$ of the model (2.7) is positive for all $t > 0$. This implies that the model (2.7) is meaningful and mathematically well posed.

3.2. Invariant region.

Let $N_o(t) = S(t) + E(t) + I(t)$ be the total population of onion plants. Then the rate at which the total population of onion plant changes is given by

$$\frac{dN_o(t)}{dt} = \frac{dS(t)}{dt} + \frac{dE(t)}{dt} + \frac{dI(t)}{dt} = rS(t)\left(1 - \frac{S(t)+E(t)+I(t)}{K}\right) - g(S(t)+E(t)+I(t)) - \rho I(t).$$

Since $N_o(t) = S(t) + E(t) + I(t)$ and $S(t) \leq N_o(t)$, we get

$$\frac{dN_o(t)}{dt} \leq rN_o(t)\left(1 - \frac{N_o(t)}{K}\right) - gN_o(t) \tag{3.4}$$

Solving inequation (3.4) by separation of variable principle and using the standard comparison principle, we obtain

$$\limsup_{t\to\infty} N_o(t) \leq K\left(\frac{r-g}{r}\right), \text{ when } r > g.$$

Therefore $0 \leq N_o(t) \leq K\left(\frac{r-g}{r}\right)$, when $r > g$. Define

$$\Gamma_o = \left\{(S(t), E(t), I(t)) \in \mathbb{R}^3_+ \mid 0 \leq S(t) + E(t) + I(t) \leq K\left(\frac{r-g}{r}\right)\right\}.$$

To prove that $X(t)$ and $W(t)$ are bounded, let

$$N_v(t) = X(t-\tau_2) + W(t)$$

Then,

$$\begin{aligned}\frac{dN_v(t)}{dt} &= \lambda - \sigma X(t-\tau_2) - \frac{\delta_2 X(t-\tau_2) I(t-\tau_2)}{1+m_2 I(t-\tau_2)} + \frac{\delta_2 e^{-\sigma\tau_2} X(t-\tau_2) I(t-\tau_2)}{1+m_2 I(t-\tau_2)} - \sigma W(t),\\ &\leq \lambda - \sigma X(t-\tau_2) - \frac{\delta_2 X(t-\tau_2) I(t-\tau_2)}{1+m_2 I(t-\tau_2)} + \frac{\delta_2 X(t-\tau_2) I(t-\tau_2)}{1+m_2 I(t-\tau_2)} - \sigma W(t),\\ &\leq \lambda - \sigma N_v(t).\end{aligned}$$

By standard comparison theorem, we get $\lim_{t\to\infty} \sup N_v(t) \leq \frac{\lambda}{\sigma}$. Therefore, $X(t)$ and $W(t)$ are bounded. Define $\Gamma_v = \left\{(X(t), W(t)) \in \mathbb{R}^2_+ \mid 0 \leq X(t) + W(t) \leq \frac{\lambda}{\sigma}\right\}$. Therefore, according to [26], it is sufficient to study the dynamics of the model system (2.7) in the biological plausible region Γ given by

$\Gamma = \Gamma_o \times \Gamma_v = \left\{(S(t), E(t), I(t), X(t), W(t)) \in \mathbb{R}^5_+ : 0 \leq N_o(t) \leq K\left(\frac{r-g}{r}\right); 0 \leq N_v(t) \leq \frac{\lambda}{\sigma}\right\}$.

4. MODEL ANALYSIS

4.1. Disease-free equilibrium point(DFE).

4.1. Disease-free equilibrium point(DFE). Let $E^0 = (S^0, E^0, I^0, X^0, W^0)$ be the disease-free equilibrium point (DFE) of the model (2.7). Assume that $\lim_{t\to\infty} S(t-\tau_1) = \lim_{t\to\infty} S(t)$, $\lim_{t\to\infty} W(t-\tau_1) = \lim_{t\to\infty} W(t)$, $\lim_{t\to\infty} I(t-\tau_2) = \lim_{t\to\infty} I(t)$, $\lim_{t\to\infty} X(t-\tau_2) = \lim_{t\to\infty} X(t)$. The DFE is obtained by setting the right hand side of equations in the model system (2.7) to zero and letting $E(t) = 0$, $I(t) = 0$ and $W(t) = 0$. That is

$$\frac{dS(t)}{dt} = \frac{dE(t)}{dt} = \frac{dI(t)}{dt} = \frac{dX(t)}{dt} = \frac{dW(t)}{dt} = 0. \tag{4.1}$$

Solving (4.1) and letting $E(t) = I(t) = W(t) = 0$, we get $S^0 = K\left(\frac{r-g}{r}\right)$ and $X^0 = \frac{\lambda}{\sigma}$. Therefore, the disease-free equilibrium point of the model system (2.7) is given by $E^0 = (S^0, E^0, I^0, X^0, W^0) = \left(K\left(\frac{r-g}{r}\right), 0, 0, \frac{\lambda}{\sigma}, 0\right)$.

4.2. The basic reproduction number ($\mathscr{R}_0$). As we know, the basic reproduction number, $\mathscr{R}_0$ is the expected number of secondary infections arising from one newly infected individual introduced into a healthy population that determines the dynamic behaviour of the model. There are several methods for calculating $\mathscr{R}_0$, but for this work, we follow the idea of [27]. Firstly, we rewrite the model system (2.7) considering only the equations for diseased population. That is

$$\begin{cases} \frac{dE(t)}{dt} = \frac{\delta_1 W(t)S(t)}{1+m_1W(t)} - \frac{\delta_1 e^{-g\tau_1} W(t-\tau_1)S(t-\tau_1)}{1+m_1W(t-\tau_1)} - gE(t), \\ \frac{dI(t)}{dt} = \frac{\delta_1 e^{-g\tau_1} W(t-\tau_1)S(t-\tau_1)}{1+m_1W(t-\tau_1)} - \rho I(t) - gI(t), \\ \frac{dW(t)}{dt} = \frac{\delta_2 e^{-\sigma\tau_2} X(t-\tau_2)I(t-\tau_2)}{1+m_2I(t-\tau_2)} - \sigma W(t). \end{cases} \tag{4.2}$$

Next, we linearise system (4.2) at the disease-free equilibrium $E_0 = \left(K\left(\frac{r-g}{r}\right),\ 0,\ 0,\ \frac{\lambda}{\sigma},\ 0\right)$ to obtain

$$\begin{pmatrix} \frac{dE(t)}{dt} \\ \frac{dI(t)}{dt} \\ \frac{dW(t)}{dt} \end{pmatrix} = \begin{pmatrix} -g & 0 & \delta_1 K\left(\frac{r-g}{r}\right) \\ 0 & -(\rho+g) & 0 \\ 0 & 0 & -\sigma \end{pmatrix} \begin{pmatrix} E(t) \\ I(t) \\ W(t) \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & -\delta_1 e^{-g\tau_1} K\left(\frac{r-g}{r}\right) \\ 0 & 0 & \delta_1 e^{-g\tau_1} K\left(\frac{r-g}{r}\right) \\ 0 & \frac{\delta_2 e^{-\sigma\tau_2}\lambda}{\sigma} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ I(t-\tau_2) \\ W(t-\tau_1) \end{pmatrix} \tag{4.3}$$

That is

$$\begin{cases} \frac{dE(t)}{dt} = \delta_1 K\left(\frac{r-g}{r}\right) W(t) - \delta_1 e^{-g\tau_1} K\left(\frac{r-g}{r}\right) W(t-\tau_1) - gE(t), \\ \frac{dI(t)}{dt} = \delta_1 e^{-g\tau_1} K\left(\frac{r-g}{r}\right) W(t-\tau_1) - (\rho+g)I(t), \\ \frac{dW(t)}{dt} = \frac{\delta_2 e^{-\sigma\tau_2}\lambda}{\sigma} I(t-\tau_2) - \sigma W(t). \end{cases} \tag{4.4}$$

Let $E(0)$, $I(0)$ and $W(0)$ be the numbers of each diseased class at $t = 0$, and $E(t)$, $I(t)$ and $W(t)$ be the remaining population of each class at time t, respectively. We obtain

$I(t)=I(0)e^{-(g+\rho)t}$, $W(t)=W(0)e^{-\sigma t}$. Then, from system (4.4), the total number of newly infected in diseased populations E, I and W are

$$\overline{E}=\int_0^\infty \delta_1 K\left(\frac{r-g}{r}\right)W(t)dt=\int_0^\infty \delta_1 K\left(\frac{r-g}{r}\right)W(0)e^{-\sigma t}dt=\frac{\delta_1 K\left(\frac{r-g}{r}\right)}{\sigma}W(0),$$

$$\overline{I}=\int_{\tau_1}^\infty \delta_1 e^{-g\tau_1} K\left(\frac{r-g}{r}\right)W(t-\tau_1)dt=\frac{\delta_1 e^{-g\tau_1}K\left(\frac{r-g}{r}\right)}{\sigma}W(0),$$

$$\overline{W}=\int_{\tau_2}^\infty \frac{\delta_2 e^{-\sigma\tau_2}\lambda}{\sigma}I(t-\tau_2)dt=\int_{\tau_2}^\infty \frac{\delta_2 e^{-\sigma\tau_2}\lambda}{\sigma}I(0)e^{-(g+\rho)(t-\tau_2)}dt=\frac{\delta_2 e^{-\sigma\tau_2}\lambda}{\sigma(g+\rho)}I(0).$$

Then, it follows that

$$\begin{pmatrix}\overline{E}\\ \overline{I}\\ \overline{W}\end{pmatrix}=\begin{pmatrix}0 & 0 & \frac{\delta_1 K\left(\frac{r-g}{r}\right)}{\sigma}\\ 0 & 0 & \frac{\delta_1 e^{-g\tau_1}K\left(\frac{r-g}{r}\right)}{\sigma}\\ 0 & \frac{\delta_2 e^{-\sigma\tau_2}\lambda}{\sigma(g+\rho)} & 0\end{pmatrix}\begin{pmatrix}E(0)\\ I(0)\\ W(0)\end{pmatrix}$$

The following 3×3 matrix denoted by

$$M=\begin{pmatrix}0 & 0 & \frac{\delta_1 K\left(\frac{r-g}{r}\right)}{\sigma}\\ 0 & 0 & \frac{\delta_1 e^{-g\tau_1}K\left(\frac{r-g}{r}\right)}{\sigma}\\ 0 & \dfrac{\delta_2 e^{-\sigma\tau_2}\lambda}{\sigma(g+\rho)} & 0\end{pmatrix}$$

is the next infection operator and the basic reproduction number $\mathscr{R}_0$ is the spectral radius of the matrix M. Let ξ be the eigenvalue of the matrix M, then we have $-\xi\left(\xi^2-\dfrac{\delta_1\,\delta_2\,\lambda\,K\,(r-g)}{r\,\sigma^2\,(\rho+g)}e^{-(g\tau_1+\sigma\tau_2)}\right)=0$

Hence, the basic reproduction number $\mathscr{R}_0$ of the system (2.7) is given by

$$\mathscr{R}_0=\sqrt{\frac{\lambda\,\delta_1\,\delta_2\,K\,(r-g)}{r\,\sigma^2\,(\rho+g)}e^{-(g\tau_1+\sigma\tau_2)}} \tag{4.5}$$

The above expression of $\mathscr{R}_0$ can be rewritten as

$$\mathscr{R}_0=\sqrt{\mathscr{R}_{vo}\times\mathscr{R}_{ov}} \tag{4.6}$$

where $\mathscr{R}_{vo}=\dfrac{\lambda\,\delta_2}{\sigma^2}e^{-\sigma\tau_2}$ and $\mathscr{R}_{ov}=\dfrac{\delta_1\,K\,(r-g)}{r(\rho+g)}e^{-g\tau_1}$.

4.3. Local stability of disease-free equilibrium point. The local stability of the disease-free equilibrium point E^0 is investigated by adopting the same technique as in [28].

Theorem 4.1. *When $g < r$, the disease-free equilibrium point E_0 of the system (2.7) is locally asymptotically stable if $\mathscr{R}_0 < 1$ and unstable if $\mathscr{R}_0 > 1$.*

Proof. The linearization of the Jacobian matrix of the model system (2.7) around the disease-free equilibrium point E^0 is given by

$$J(E_0) = \begin{pmatrix} -(r-g) & -(r-g) & -(r-g) & 0 & -\delta_1 K\left(\frac{r-g}{r}\right) \\ 0 & -g & 0 & 0 & \delta_1 K\left(\frac{r-g}{r}\right)(1-e^{-g\tau_1}) \\ 0 & 0 & -(\rho+g) & 0 & \delta_1 K\left(\frac{r-g}{r}\right)e^{-g\tau_1} \\ 0 & 0 & -\frac{\delta_2\lambda}{\sigma} & -\sigma & 0 \\ 0 & 0 & \frac{\delta_2\lambda}{\sigma}e^{-\sigma\tau_2} & 0 & -\sigma \end{pmatrix} \tag{4.7}$$

The characteristic equation of the Jacobian matrix (4.7) is found to be

$$(g-r-\xi)(-\sigma-\xi)(-g-\xi)\begin{vmatrix} -(\rho+g)-\xi & \delta_1 K\left(\frac{r-g}{r}\right)e^{-g\tau_1} \\ \frac{\delta_2\lambda}{\sigma}e^{-\sigma\tau_2} & -\sigma-\xi \end{vmatrix} = 0 \tag{4.8}$$

From the characteristic equation (4.8), the eigenvalues of the Jacobian matrix of system (2.7) evaluated at E^0, are given as the roots of the following polynomial

$$(g-r-\xi)(-\sigma-\xi)(-g-\xi)\left[\xi^2+(\rho+g+\sigma)\xi+\left(\sigma(g+\rho)-\frac{\lambda\,\delta_1\,\delta_2\,K(r-g)}{r\sigma}e^{-(g\tau_1+\sigma\tau_2)}\right)\right]=0. \tag{4.9}$$

Hence, the first three eigenvalues are $\xi_1 = g-r < 0$, (since $r > g$), $\xi_2 = -\sigma < 0$, and $\xi_3 = -g < 0$ while the others eigenvalues are the roots of the quadratic equation

$$\xi^2+(\rho+g+\sigma)\xi+\left(\sigma(g+\rho)-\frac{\lambda\,\delta_1\,\delta_2\,K(r-g)}{r\sigma}e^{-(g\tau_1+\sigma\tau_2)}\right)=0. \tag{4.10}$$

Next, since $\mathscr{R}_0 < 1$ then the coefficients of the equation

$$\xi^2 + (\rho + g + \sigma)\xi + \sigma(g + \rho)\left(1 - \mathscr{R}_0^2\right) = 0, \tag{4.11}$$

are positives. Therefore, since there is no sign change between its terms, we conclude by Descartes's rule of signs that equation (4.11) does not have positive roots. Replacing ξ by $-\xi$ in equation (4.11), we get

$$\xi^2 - (\rho + g + \sigma)\xi + \sigma(g + \rho)\left(1 - \mathscr{R}_0^2\right) = 0. \tag{4.12}$$

Thus, if $\mathscr{R}_0 < 1$, equation (4.12) has two sign changes in its terms, and we conclude by Descartes's rule of signs that equation (4.12) has exactly two negative roots. Consequently, the disease-free equilibrium $E^0 = \left(K\left(\dfrac{r-g}{r}\right), 0, 0, \frac{\lambda}{\sigma}, 0\right)$ is locally asymptotically stable if $\mathscr{R}_0 < 1$.

The basic reproduction number with no delays $(\tau_1 = 0, \tau_2 = 0)$ is given by $\mathscr{R}_0 = \sqrt{\dfrac{\lambda\,\delta_1\,\delta_2\,K\,(r-g)}{r\,\sigma^2\,(\rho+g)}}$.

Similarly as above, it can be shown that with no delays $(\tau_1 = 0, \tau_2 = 0)$, the disease-free equilibrium point E_0 is locally asymptotically stable if $\mathscr{R}_0 < 1$ and unstable otherwise. In order to graphically illustrate the local stability of the disease-free equilibrium point with no delays, the parameters values in Table 2 are used with corresponding estimates: $S(0) = 800, E(0) = 0, I(0) = 0, X(0) = 200, W(0) = 0$. Biologically, this implies that the susceptible populations will be stable in the long run and IYSV disease will be under control.

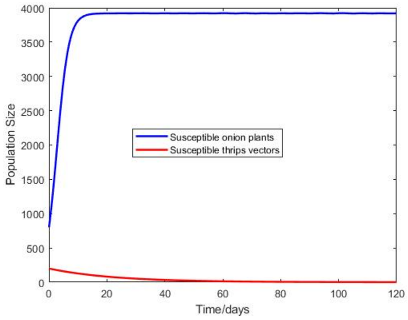


FIGURE 1. Simulation showing the local stability of DFE with no delays $(\tau_1 = \tau_2 = 0)$.

4.4. Existence of endemic equilibrium point (E^*)**.** Let $E^* = (S^*, E^*, I^*, X^*, W^*)$ be the endemic equilibrium point of the model system (2.7). As in [29], the delay–dependency must vanish so that $\lim_{t\to\infty} S(t-\tau_1) = \lim_{t\to\infty} S(t) = S^*$, $\lim_{t\to\infty} W(t-\tau_1) = \lim_{t\to\infty} W(t) = W^*$, $\lim_{t\to\infty} I(t-\tau_2) = \lim_{t\to\infty} I(t) = I^*$, $\lim_{t\to\infty} X(t-\tau_2) = \lim_{t\to\infty} X(t) = X^*$, such that at equilibrium, we obtain

$$\begin{cases} rS^*\left(1-\frac{S^*+E^*+I^*}{K}\right) - \frac{\delta_1 W^* S^*}{1+m_1 W^*} - gS^* = 0, \\ \frac{\delta_1 W^* S^*}{1+m_1 W^*} - \frac{\delta_1 W^* S^* e^{-g\tau_1}}{1+m_1 W^*} - gE^* = 0, \\ \frac{\delta_1 W^* S^* e^{-g\tau_1}}{1+m_1 W^*} - \rho I^* - gI^* = 0, \\ \lambda - \frac{\delta_2 X^* I^*}{1+m_2 I^*} - \sigma X^* = 0, \\ \frac{\delta_2 X^* I^* e^{-\sigma\tau_2}}{1+m_2 I^*} - \sigma W^* = 0. \end{cases} \tag{4.13}$$

From the fourth equation of system (4.13), we isolate X^* as

$$X^* = \frac{\lambda(1+m_2 I^*)}{\sigma + (\delta_2 + \sigma m_2) I^*} \tag{4.14}$$

From the fifth equation of system (4.13), we obtain W^* as

$$W^* = \frac{\delta_2 \lambda\, e^{-\sigma\tau_2} I^*}{\sigma^2 + I^*(\sigma\delta_2 + \sigma^2 m_2)}$$

From the third equation of system (4.13), S^* is found to be

$$S^* = \frac{(\rho+g)\sigma^2 + (\rho+g)\left(\sigma\delta_2 + \sigma^2 m_2 + \lambda\delta_2 m_1\, e^{-\sigma\tau_2}\right) I^*}{\lambda\delta_1\delta_2\, e^{-(g\tau_1+\sigma\tau_2)}} \tag{4.15}$$

Substituting the expression of W^* and S^* into the second equation of (4.13), we obtain E^* as

$$E^* = \frac{(\rho+g)(e^{g\tau_1}-1)I^*}{g} \tag{4.16}$$

Hence, we summarize X^*, W^*, S^*, E^* as

$$\text{(4.17)} \qquad \begin{cases} X^* & = \dfrac{\lambda(1+m_2 I^*)}{\sigma+(\delta_2+\sigma m_2)I^*} \\ W^* & = \dfrac{\delta_2 \lambda\, e^{-\sigma\tau_2} I^*}{\sigma^2 + I^*(\sigma\delta_2+\sigma^2 m_2)} \\ S^* & = \dfrac{(\rho+g)\sigma^2 + (\rho+g)\left(\sigma\delta_2+\sigma^2 m_2 + \lambda\,\delta_2 m_1\, e^{-\sigma\tau_2}\right) I^*}{\lambda\,\delta_1\,\delta_2\, e^{-(g\tau_1+\sigma\tau_2)}} \\ E^* & = \dfrac{(\rho+g)(e^{g\tau_1}-1)I^*}{g} \text{ and} \end{cases}$$

I^* satisfies the following quadratic polynomial describing the existence of the possible endemic equilibrium:

$$\text{(4.18)} \qquad H(I^*) = c_2 {I^*}^2 + c_1 I^* + c_0 = 0,$$

with

$$\begin{aligned} c_2 &= \Big[-r(\rho+g)(g\sigma\delta_2 + g\sigma^2 m_2 + \lambda\delta_2 e^{-\sigma\tau_2}(g m_1+\delta_1)) + r\,\lambda\,\sigma^2\,\rho\,\delta_1\,\delta_2\, e^{-(g\tau_1+\sigma\tau_2)}\Big] \\ &\quad \times (\sigma\delta_2+\sigma^2 m_2 + m_1\lambda\delta_2 e^{-\sigma\tau_2}), \\ c_1 &= \left(g\,\lambda\,\delta_1\,\delta_2\, e^{-(g\tau_1+\sigma\tau_2)}\right)\left[K(r-g)(\sigma\delta_2+\sigma^2 m_2+m_1\lambda\delta_2 e^{-\sigma\tau_2}) - \lambda\delta_1\delta_2 K e^{-\sigma\tau_2}\right] \\ &\quad - r(\rho+g)\sigma^2\left[2g(\sigma\delta_2+\sigma^2 m_2+m_1\lambda\delta_2 e^{-\sigma\tau_2}) + \lambda\delta_1\delta_2 e^{-\sigma\tau_2}\right] \\ &\quad + r\,\lambda\,\sigma^2\,\rho\,\delta_1\,\delta_2\, e^{-(g\tau_1+\sigma\tau_2)}, \\ c_0 &= -r\,g\,(\rho+g)\,\sigma^4\left[1-\mathscr{R}_0^2\right]. \end{aligned}$$

4.5. Stability of the Endemic Equilibrium and Hopf- Bifurcation. Under this section, we investigate the local stability of the endemic equilibrium E^* and existence of the local Hopf-bifurcation occurring at E^* by choosing τ_1, τ_2 and $\tau = \max\{\tau_1, \tau_2\}$ as bifurcation parameters. To this end, we follow the Hopf-bifurcation theory in [30] by first computing Transcendental Equation. The expression for the transcendental equation is obtained by linearizing the model system (2.7) around the endemic equilibrium point $E^* = (S^*, E^*, I^*, X^*, W^*)$. We get the following system

$$\text{(4.19)} \qquad \frac{dY(t)}{dt} = FY(t) + GY(t-\tau_1) + QY(t-\tau_2),$$

where F, G, H are 5×5 matrix given by

(4.20)

$$F = [f_{ij}] = \begin{pmatrix} r\left(1 - \frac{2S^*+I^*+E^*}{K}\right) - \frac{\delta_1 W^*}{1+m_1 W^*} - g & -\frac{rS^*}{K} & -\frac{rS^*}{K} & 0 & -\frac{\delta_1 S^*}{(1+m_1 W^*)^2} \\ \frac{\delta_1 W^*}{1+m_1 W^*} & -g & 0 & 0 & \frac{\delta_1 S^*}{(1+m_1 W^*)^2} \\ 0 & 0 & -(\rho+g) & 0 & 0 \\ 0 & 0 & -\frac{\delta_2 X^*}{(1+m_2 I^*)^2} & -\frac{\delta_2 I^*}{1+m_2 I^*} - \sigma & 0 \\ 0 & 0 & 0 & 0 & -\sigma \end{pmatrix},$$

$$G = [g_{ij}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -\frac{\delta_1 e^{-g\tau_1} W^*}{1+m_1 W^*} & 0 & 0 & 0 & -\frac{\delta_1 e^{-g\tau_1} S^*}{(1+m_1 W^*)^2} \\ \frac{\delta_1 e^{-g\tau_1} W^*}{1+m_1 W^*} & 0 & 0 & 0 & \frac{\delta_1 e^{-g\tau_1} S^*}{(1+m_1 W^*)^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.21}$$

$$Q = [q_{ij}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\delta_2 e^{-\sigma\tau_2} X^*}{(1+m_2 I^*)^2} & \frac{\delta_2 e^{-\sigma\tau_2} I^*}{1+m_2 I^*} & 0 \end{pmatrix}. \tag{4.22}$$

Then, the resulting Jacobian matrix is found to be

(4.23)

$$J = F + Ge^{-\xi\tau_1} + Qe^{-\xi\tau_2} = \begin{pmatrix} b_1 & b_2 & b_3 & 0 & b_4 \\ b_5(1-e^{-(g+\xi)\tau_1}) & b_6 & 0 & 0 & b_7(1-e^{-(g+\xi)\tau_1}) \\ b_8 e^{-(g+\xi)\tau_1} & 0 & b_9 & 0 & b_{10} e^{-(g+\xi)\tau_1} \\ 0 & 0 & b_{11} & b_{12} & 0 \\ 0 & 0 & b_{13} e^{-(\sigma+\xi)\tau_2} & b_{14} e^{-(\sigma+\xi)\tau_2} & b_{15} \end{pmatrix},$$

with

$$b_1 = r\left(1 - \frac{2S^*+I^*+E^*}{K}\right) - \frac{\delta_1 W^*}{1+m_1 W^*} - g,\ b_2 = -\frac{rS^*}{K},\ b_3 = -\frac{rS^*}{K},\ b_4 = -\frac{\delta_1 S^*}{(1+m_1 W^*)^2},$$

$$b_5 = \frac{\delta_1 W^*}{1+m_1 W^*},\ b_6 = -g,\ b_7 = \frac{\delta_1 S^*}{(1+m_1 W^*)^2},\ b_8 = \frac{\delta_1 W^*}{1+m_1 W^*},\ b_9 = -(g+\rho),\ b_{10} = \frac{\delta_1 S^*}{(1+m_1 W^*)^2},$$

$$b_{11} = -\frac{\delta_2 X^*}{(1+m_2 I^*)^2},\ b_{12} = -\frac{\delta_2 I^*}{1+m_2 I^*} - \sigma,\ b_{13} = \frac{\delta_2 X^*}{(1+m_2 I^*)^2},\ b_{14} = \frac{\delta_2 I^*}{1+m_2 I^*},\ b_{15} = -\sigma.$$

Hence, the transcendental equation of system (2.7) is explicitly written as

$$\begin{aligned}\phi(\xi,\tau_1,\tau_2) &= \xi^5+a_4\xi^4+a_3\xi^3+a_2\xi^2+a_1\xi+a_0\\ &\quad+\left(p_3\xi^3+p_2\xi^2+p_1\xi+p_0\right)e^{-\xi(\tau_1+\tau_2)}e^{-(g\tau_1+\sigma\tau_2)}\\ &\quad+\left(q_3\xi^3+q_2\xi^2+q_1\xi+q_0\right)e^{-(g+\xi)\tau_1}\\ &\quad+(h_1\xi+h_0)\,e^{-2(g+\xi)\tau_1}e^{-(\sigma+\xi)\tau_2}=0, \end{aligned}\tag{4.24}$$

where

$$a_4=-(b_{15}+b_{12}+b_9+b_6+b_1),$$

$$a_3=(b_{12}+b_{15})(b_1+b_6+b_9)+b_1(b_6+b_9)+b_6b_9+b_{15}b_{12}-b_5b_2,$$

$$a_2=(b_9+b_{12}+b_{15})(b_2b_5-b_1b_6)-b_9(b_{12}+b_{15})(b_1+b_6)-b_{12}b_{15}(b_1+b_6+b_9),$$

$$a_1=(b_9b_{12}+b_9b_{15}+b_{12}b_{15})(b_1b_6-b_2b_5)+b_9b_{12}b_{15}(b_1+b_6),$$

$$a_0=b_9b_{12}b_{15}(b_2b_5-b_1b_6),$$

$$p_3=-b_{10}b_{13},$$

$$p_2=b_{10}b_{13}(b_1+b_6+b_{12})-(b_4b_8b_{13}+b_{14}b_{11}b_{10}),$$

$$p_1=(b_{11}b_{14}-b_{12}b_{13})(b_{10}(b_1+b_6)-b_4b_8)+b_{13}(b_8(b_4b_6-b_2b_7)-b_{10}(b_1b_6-b_2b_5)),$$

$$p_0=(b_{12}b_{13}-b_{11}b_{14})(b_1b_6b_{10}+b_2b_8b_7-b_2b_{10}b_5-b_4b_6b_8),$$

$$q_3=b_5b_2-b_8b_3,$$

$$q_2=b_3b_8(b_6+b_{12}+b_{15})-b_2b_5(b_9+b_{12}+b_{15}),$$

$$q_1=-b_3b_8(b_6b_{12}+b_{15}(b_6+b_{12}))+b_2b_5(b_9b_{12}+b_{15}(b_9+b_{12})),$$

$$q_0=b_{12}b_{15}(b_3b_6b_8-b_2b_9b_5),$$

$$h_1=b_2b_{13}(b_8b_7-b_{10}b_5),$$

$$h_0=(b_{12}b_{13}-b_{11}b_{14})(b_2b_{10}b_5-b_2b_8b_7).$$

Now, we investigate the stability of the endemic equilibrium E^* by considering four different cases which are: no delays, delay only for incubation period τ_1 with $\tau_2=0$, delay only for latent period τ_2 with $\tau_1 = 0$ and both delays for incubation period τ_1 and latent period τ_2 that are non-zero.

Case 1: Delay–free system ($\tau_1 = \tau_2 = 0$)

Here, we consider a case where there are no delays during the latent period ($\tau_1 = 0$) and during incubation period ($\tau_2 = 0$). Then, by letting $\tau_1 = \tau_2 = 0$, equation (4.24) reduces to

$$\phi(\xi) = \xi^5 + A_4\xi^4 + A_3\xi^3 + A_2\xi^2 + A_1\xi + A_0, \tag{4.25}$$

(4.26)

$$A_4 = a_4, A_3 = a_3 + p_3 + q_3, A_2 = a_2 + p_2 + q_2, A_1 = a_1 + p_1 + q_1 + h_1, A_0 = a_0 + p_0 + q_0 + h_0.$$

Therefore, the local stability of the endemic equilibrium $E^* = (S^*, E^*, I^*, X^*, W^*)$, of the model system (2.7) without delays is given by the following theorem.

Theorem 4.2. *In the absence of delays $\tau_1 = \tau_2 = 0$, the endemic equilibrium E^* is locally asymptotically stable for $\mathscr{R}_0 > 1$ iff the following Routh–Hurwitz conditions are satisfied,* $A_4 = -tr(J) > 0, A_4A_3 - A_2 > 0, A_4A_3A_2 + A_4A_0 - A_4^2A_1 - A_2^2 > 0$, $(A_2A_1 - A_3A_0)(A_4A_3 - A_2) - (A_4A_1 - A_0)^2 > 0$ *and* $A_1\left[(A_2A_1 - A_3A_0)(A_4A_3 - A_2) - (A_4A_1 - A_0)^2\right] > 0$, *with A_4, A_3, A_2, A_1 and A_0 defined in (4.26) and unstable otherwise.*

Case 2: Delay only for incubation period ($\tau_1 > 0$, $\tau_2 = 0$)

In this case the transcendental equation (4.24) becomes

(4.27)

$$\phi(\xi,\tau_1) = \xi^5 + a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0 + \left(B_3\xi^3 + B_2\xi^2 + B_1\xi + B_0\right)e^{-(g+\xi)\tau_1} + (h_1\xi + h_0)e^{-2(g+\xi)\tau_1} = 0,$$

$$B_3 = p_3 + q_3, B_2 = p_2 + q_2, B_1 = p_1 + q_1, B_0 = p_0 + q_0. \tag{4.28}$$

Multiplying both sides of equation (4.27) by $e^{(g+\xi)\tau_1}$ and letting $\alpha_1 = e^{g\tau_1}$ and $\alpha_2 = e^{-g\tau_1}$, its reduces to

(4.29)

$$\phi(\xi,\tau_1) = B_3\xi^3 + B_2\xi^2 + B_1\xi + B_0 + \left(C_5\xi^5 + C_4\xi^4 + C_3\xi^3 + C_2\xi^2 + C_1\xi + C_0\right)e^{\xi\tau_1} + (D_1\xi + D_0)e^{-\xi\tau_1} = 0,$$

where

$$\begin{aligned} &C_5=\alpha_1,\, C_4=a_4\alpha_1,\, C_3=a_3\alpha_1,\, C_2=a_2\alpha_1,\\ &C_1=a_1\alpha_1,\, C_0=a_0\alpha_1,\, D_1=\alpha_2 h_1,\, D_0=\alpha_2 h_0. \end{aligned} \tag{4.30}$$

Equation (4.29) is a transcendental equation in ξ having infinitely many roots. Possible Hopf-bifurcation occurs if purely imaginary roots of the transcendental equation (4.29) exist. Let $\xi = iw_1\,(w_1>0)$ be a purely imaginary root of the equation (4.29). Substituting $\xi = iw_1$ into (4.29) and expressing the exponential in terms of trigonometric ratios, we obtain

$$\begin{aligned} &-B_3w_1^3i-B_2w_1^2+B_1w_1i+B_0\\ &+\Big(C_5w_1^5i+C_4w_1^4-C_3w_1^3i-C_2w_1^2+C_1w_1i+C_0\Big)\left(\cos w_1\tau_1+i\sin w_1\tau_1\right)\\ &+\left(D_1w_1i+D_0\right)\left(\cos w_1\tau_1-i\sin w_1\tau_1\right)\\ &=0. \end{aligned} \tag{4.31}$$

Separating real and imaginary parts, we obtain the following systems of equations:

$$\begin{cases} \big(C_5w_1^5-C_3w_1^3+w_1(C_1+D_1)\big)\cos w_1\tau_1+\big(C_4w_1^4-C_2w_1^2+C_0-D_0\big)\sin w_1\tau_1=B_3w_1^3-B_1w_1,\\ \big(C_4w_1^4-C_2w_1^2+C_0+D_0\big)\cos w_1\tau_1-\big(C_5w_1^5-C_3w_1^3+w_1(C_1-D_1)\big)\sin w_1\tau_1=B_2w_1^2-B_0. \end{cases} \tag{4.32}$$

Solving system (4.32), we get

$$\begin{aligned} \sin w_1\tau_1 &= \frac{f_7w_1^7+f_5w_1^5+f_3w_1^3+f_1w_1}{g_{10}w_1^{10}+g_8w_1^8+g_6w_1^6+g_4w_1^4+g_2w_1^2+g_0},\\ \cos w_1\tau_1 &= \frac{f_8w_1^8+f_6w_1^6+f_4w_1^4+f_2w_1^2+f_0}{g_{10}w_1^{10}+g_8w_1^8+g_6w_1^6+g_4w_1^4+g_2w_1^2+g_0}, \end{aligned}$$

where

$$f_0=-B_0(C_0-D_0),\; f_1=-B_1(C_0+D_0)+B_0(C_1+D_1),\; f_2=B_0C_2+B_2(C_0-D_0)-B_1(C_1-D_1),$$

$$f_3=C_2B_1+(C_0+D_0)B_3-B_0C_3-B_2(C_1+D_1),\; f_4=-B_0C_4-C_2B_2+B_1C_3+B_3(C_1-D_1),$$

$$f_5=-B_1C_4-C_2B_3+B_0C_5+C_3B_2,\; f_6=C_4B_2-B_1C_5-C_3B_3,\; f_7=C_4B_3-C_5B_2,\; f_8=C_5B_3,$$

$$g_0=C_0^2-D_0^2,\; g_2=-2C_2C_0+C_1^2-D_1^2,\; g_4=2C_0C_4+C_2^2-2C_1C_3,\; g_6=-2C_2C_4+C_3^2+2C_1C_5,$$

$$g_8=C_4^2-2C_3C_5,\;\; g_{10}=C_5^2.$$

Using the fundamental Pythagorean trigonometric identity, we obtain

$$M_{10}w_1^{20}+M_9w_1^{18}+M_8w_1^{16}+M_7w_1^{14}+M_6w_1^{12}+M_5w_1^{10}+M_4w_1^8+M_3w_1^6+M_2w_1^4+M_1w_1^2+M_0=0, \tag{4.33}$$

where

$$M_{10}=g_{10}^2,\ M_9=2g_{10}g_8,\ M_8=g_8^2+2g_{10}g_6-f_8^2,\ M_7=2g_{10}g_4+2g_6g_8-f_7^2-2f_8f_6,$$

$$M_6=2g_{10}g_2+g_6^2+2g_8g_4-2f_4f_8-f_6^2-2f_7f_5,$$

$$M_5=2g_0g_{10}+2g_8g_2+2g_6g_4-2f_8f_2-2f_6f_4-2f_7f_3-f_5^2,$$

$$M_4=g_4^2+2g_0g_8+2g_6g_2-f_4^2-2f_0f_8-2f_6f_2-2f_7f_1-2f_5f_3,$$

$$M_3=2g_6g_0+2g_4g_2-2f_4f_2-2f_0f_6-2f_5f_1-f_3^2,$$

$$M_2=g_2^2+2g_4g_0-f_2^2-2f_0f_4-2f_1f_3,\ M_1=2g_0g_2-f_1^2-2f_0f_2,\ M_0=g_0^2-f_0^2.$$

Let $w_1^2=\eta_1$, then equation (4.33) becomes

$$p(\eta_1)=M_{10}\eta_1^{10}+M_9\eta_1^9+M_8\eta_1^8+M_7\eta_1^7+M_6\eta_1^6+M_5\eta_1^5+M_4\eta_1^4+M_3\eta_1^3+M_2\eta_1^2+M_1\eta_1+M_0=0, \tag{4.34}$$

which is the Hopf frequency. Next, we make the following assumption for the Hopf frequency equation.

(H_1): Equation (4.34) has at least one positive root.

Now, if condition (H_1) holds, equation (4.34) has a positive root η_{1_0} and then equation (4.33) will definitely have a pair of purely imaginary roots $\pm iw_{1_0}=\pm i\sqrt{\eta_{1_0}}$. Consequently the corresponding critical value of τ_1 at which a Hopf bifurcation occurs is found to be

$$\tau_{1_k}=\frac{1}{w_{1_k}}\arccos\left[\frac{f_8w_{1_k}^8+f_6w_{1_k}^6+f_4w_{1_k}^4+f_2w_{1_k}^2+f_0}{g_{10}w_{1_k}^{10}+g_8w_{1_k}^8+g_6w_{1_k}^6+g_4w_{1_k}^4+g_2w_{1_k}^2+g_0}\right]+\frac{2k\pi}{w_{1_k}},\ (k=0,1,2,3,...). \tag{4.35}$$

Hence, (w_{1_0},τ_{1_k}) is the solution of the equation (4.29), meaning that there exists a pair of purely imaginary roots $\xi=\pm iw_{1_0}$ for equation (4.29) when $\tau_1=\tau_{1_k}$. Especially, for $k=0$, we obtain

$$\tau_{1_0}=\frac{1}{w_{1_0}}\arccos\left[\frac{f_8w_{1_0}^8+f_6w_{1_0}^6+f_4w_{1_0}^4+f_2w_{1_0}^2+f_0}{g_{10}w_{1_0}^{10}+g_8w_{1_0}^8+g_6w_{1_0}^6+g_4w_{1_0}^4+g_2w_{1_0}^2+g_0}\right]. \tag{4.36}$$

Differentiating both sides of equation (4.29) with respect to τ_1 and after some calculations, we get

(4.37)
$$\left[\frac{d\xi}{d\tau_1}\right]^{-1} = \frac{3B_3\xi^2+2B_2\xi+B_1+D_1e^{-\xi\tau_1}+(5C_5\xi^4+4C_4\xi^3+3C_3\xi^2+2C_2\xi+C_1)e^{\xi\tau_1}}{(D_1\xi^2+D_0\xi)e^{-\xi\tau_1}-(C_5\xi^6+C_4\xi^5+C_3\xi^4+C_2\xi^3+C_1\xi^2+C_0\xi)e^{\xi\tau_1}} - \frac{\tau_1}{\xi}$$

Now, taking the real component of $\left[\frac{d\xi}{d\tau_1}\right]^{-1}$ at $\tau_1 = \tau_{1_0}$, with $\xi = iw_{1_0}$, we obtain

$$R_e\left[\frac{d\xi}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{1_0}} = R_e\left[\frac{d\xi}{d\tau_1}\right]^{-1}_{\xi=iw_0} = \frac{V_1N_1+V_2N_2}{N_1^2+N_2^2}, \tag{4.38}$$

where

$$N_1 = (C_5w_{1_0}^6 - C_3w_{1_0}^4 + C_1w_{1_0}^2 - D_1w_{1_0}^2)\cos w_{1_0}\tau_{1_0} + (D_0w_{1_0} - C_2w_{1_0}^3 + C_4w_{1_0}^5 + C_0w_{1_0})\sin w_{1_0}\tau_{1_0},$$

$$N_2 = (C_2w_{1_0}^3 - C_4w_{1_0}^5 - C_0w_{1_0} + D_0w_{1_0})\cos w_{1_0}\tau_{1_0} + (C_5w_{1_0}^6 - C_3w_{1_0}^4 + C_1w_{1_0}^2 + D_1w_{1_0}^2)\sin w_{1_0}\tau_{1_0},$$

$$V_1 = (5C_5w_{1_0}^4 - 3C_3w_{1_0}^2 + C_1 + D_1)\cos w_{1_0}\tau_{1_0} - (2C_2w_{1_0} - 4C_4w_{1_0}^3)\sin w_{1_0}\tau_{1_0} - 3B_3w_{1_0}^2 + B_1,$$

$$V_2 = (2C_2w_{1_0} - 4C_4w_{1_0}^3)\cos w_{1_0}\tau_{1_0} + (5C_5w_{1_0}^4 - 3C_3w_{1_0}^2 + C_1 - D_1)\sin w_{1_0}\tau_{1_0} + 2B_2w_{1_0}.$$

Obviously, we observe that if the condition $(H_2) : V_1N_1 + V_2N_2 \neq 0$ holds, then $R_e\left[\frac{d\xi}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{1_0}} \neq 0$. Therefore, following the above analysis and the Hopf bifurcation theory in [30], we have the theorem below.

Theorem 4.3. *Suppose that the conditions* $(H_1)-(H_2)$ *hold, then the endemic equilibrium* $E^* = (S^*,E^*,I^*,X^*,W^*)$ *of the system (2.7) is locally asymptotically stable when* $\tau_1 \in [0,\tau_{1_0})$ *and the system (2.7) undergoes a Hopf bifurcation at* $E^* = (S^*,E^*,I^*,X^*,W^*)$ *when* $\tau_1 = \tau_{1_0}$.

Case 3: Delay only in latent period ($\tau_1 = 0$, $\tau_2 > 0$)

In this case the transcendental equation (4.24) becomes

(4.39)
$$\phi(\xi,\tau_2) = \xi^5 + E_4\xi^4 + E_3\xi^3 + E_2\xi^2 + E_1\xi + E_0 + \left(T_3\xi^3 + T_2\xi^2 + T_1\xi + T_0\right)e^{-\xi\tau_2} = 0.$$

where

(4.40)
$$E_4 = a_4,\ E_3 = a_3+q_3,\ E_2 = a_2+q_2,\ E_1 = a_1+q_1,\ E_0 = a_0+q_0,$$
$$\beta = e^{-\sigma\tau_2},\ T_3 = \beta p_3,\ T_2 = \beta p_2,\ T_1 = \beta(p_1+h_1),\ T_0 = \beta(p_0+h_0).$$

Let $\xi = iw_2\,(w_2 > 0)$ be a purely imaginary root of the equation (4.39). Substituting $\xi = iw_2$ into (4.39) and expressing the exponential $e^{-iw_2\tau_2}$ in terms of trigonometric ratios, we obtain

$$w_2^5 i + E_4 w_2^4 - E_3 w_2^3 i - E_2 w_2^2 + E_1 wi + E_0 + \left(-T_3 w_2^3 i - T_2 w_2^2 + T_1 w_2 i + T_0\right)\left(\cos w_2\tau_2 - i\sin w_2\tau_2\right) = 0. \tag{4.41}$$

Separating real and imaginary parts, we obtain the following systems of equations:

$$\begin{cases} \left(T_0 - T_2 w_2^2\right)\cos w_2\tau_2 + \left(T_1 w_2 - T_3 w_2^3\right)\sin w_2\tau_2 & = E_2 w_2^2 - E_4 w_2^4 - E_0, \\ \left(T_1 w_2 - T_3 w_2^3\right)\cos w_2\tau_2 - \left(T_0 - T_2 w_2^2\right)\sin w_2\tau_2 & = E_3 w_2^3 - w_2^5 - E_1 w_2. \end{cases} \tag{4.42}$$

Solving system (4.42), we get

$$\cos w_2\tau_2 = \frac{\left(T_0 - T_2 w_2^2\right)\left(E_2 w_2^2 - E_4 w_2^4 - E_0\right) + \left(T_1 w_2 - T_3 w_2^3\right)\left(E_3 w_2^3 - w_2^5 - E_1 w_2\right)}{\left(T_0 - T_2 w_2^2\right)^2 + \left(T_1 w_2 - T_3 w_2^3\right)^2} \tag{4.43}$$

Eliminating τ_2 from equation (4.42), by squaring and adding these two equations, we get

$$w_2^{10} + R_4 w_2^8 + R_3 w_2^6 + R_2 w_2^4 + R_1 w_2^2 + R_0 = 0, \tag{4.44}$$

where

$$R_4 = E_4^2 - 2E_3,\ R_3 = 2E_1 + E_3^2 - 2E_2E_4 - T_3^2,\ R_2 = 2E_0E_4 - 2E_1E_3 + E_2^2 + 2T_1T_3 - T_2^2,$$

$$R_1 = E_1^2 - 2E_0E_2 + 2T_0T_2 - T_1^2,\ R_0 = E_0^2 - T_0^2.$$

Taking $w_2^2 = \theta$, the Hopf frequency is found to be

$$f(\theta) = \theta^5 + R_4\theta^4 + R_3\theta^3 + R_2\theta^2 + R_1\theta + R_0 = 0. \tag{4.45}$$

Next, if the coefficients in the above equation (4.45) satisfy the Routh Hurvitz criterion, then we may not have any non-negative value of w_2 satisfying the equation (4.44). As in [31], the two results about stability and critical delay may be written in the form of following lemmas.

Lemma 1. *If the coefficients $R_i (i = 0, 1, 2, 3, 4)$ satisfy the conditions of the Routh–Hurwitz criteria, then all eigenvalues of the equation (4.45) have negative real parts for all delay $\tau_2 > 0$. Then the endemic equilibrium E^* of system (2.7) is locally asymptotically stable whenever $\tau_2 > 0$, provided the endemic steady state is stable in the absence of the latent period delay.*

Lemma 2. *If the coefficients* $R_i(i = 0, 1, 2, 3, 4)$ *do not satisfy the conditions of the Routh–Hurwitz criteria, then the assumption for the existence of a positive root of the equation (4.45) is* $R_0 < 0$*, that is*

$$E_0^2 - T_0^2 < 0. \tag{4.46}$$

Hence, if condition (4.46) holds, the equation (4.45) has a positive root θ_0 *and then equation (4.44) has a pair of purely imaginary roots* $\pm i w_{2_0} = \pm i\sqrt{\theta_0}$.

Consequently the corresponding critical value of τ_2 at which a Hopf bifurcation occurs is found to be

(4.47)

$$\tau_{2_0} = \frac{1}{w_{2_0}} \arccos \left[\frac{\left(T_0 - T_2 w_{2_0}^2\right)\left(E_2 w_{2_0}^2 - E_4 w_{2_0}^4 - E_0\right) + \left(T_1 w_{2_0} - T_3 w_{2_0}^3\right)\left(E_3 w_{2_0}^3 - w_{2_0}^5 - E_1 w_{2_0}\right)}{\left(T_0 - T_2 w_{2_0}^2\right)^2 + \left(T_1 w_{2_0} - T_3 w_{2_0}^3\right)^2} \right]$$

Next, to investigate whether there is a phenomenon of Hopf bifurcation as τ_2 increases through τ_{2_0}, we need to verify the following transversality condition.

Lemma 3. *The following transversality condition is satisfied:*

$$sign \left[\frac{d(R_e(\xi))}{d\tau_2} \right]_{\tau_2 = \tau_{2_0}} > 0,$$

provided that condition (4.46) holds.

Proof. (Transversality condition for Hopf–bifurcation)

Differentiating both sides of equation(4.39) with respect to τ_2 and after some calculations, we get

$$\left[\frac{d\xi}{d\tau_2}\right]^{-1} = \frac{5\xi^4 + 4E_4\xi^3 + 3E_3\xi^2 + 2E_2\xi + E_1}{\xi(T_3\xi^3 + T_2\xi^2 + T_1\xi + T_0)e^{-\xi\tau_2}} + \frac{3T_3\xi^2 + 2T_2\xi + T_1}{\xi(T_3\xi^3 + T_2\xi^2 + T_1\xi + T_0)} - \frac{\tau_2}{\xi} \tag{4.48}$$

Now, taking the sign of the real component of $\left[\frac{d\xi}{d\tau_2}\right]^{-1}$ at $\xi = i w_{2_0}$, when $\tau_2 = \tau_{2_0}$, we obtain

$$\begin{aligned} \text{sign}\left[\frac{d(R_e\xi)}{d\tau_2}\right]_{\tau_2 = \tau_{2_0}} &= \text{sign}\left[R_e\left(\frac{d\xi}{d\tau_2}\right)^{-1}\right]_{\xi = i w_{2_0}} \\ &= \text{sign}\left(R_e\left[\frac{5\xi^4 + 4E_4\xi^3 + 3E_3\xi^2 + 2E_2\xi + E_1}{-\xi(\xi^5 + E_4\xi^4 + E_3\xi^3 + E_2\xi^2 + E_1\xi + E_0)}\right]_{\xi = i w_{2_0}}\right. \end{aligned}$$

$$
\left. + R_e\left[\frac{3T_3\xi^2+2T_2\xi+T_1}{\xi(T_3\xi^3+T_2\xi^2+T_1\xi+T_0)}\right]_{\xi=iw_{2_0}}\right)
$$

$$
\begin{aligned}
&= \operatorname{sign}\left(\frac{w_{2_0}^2\left[5w_{2_0}^8+(4E_4^2-8E_3)w_{2_0}^6+(6E_1+3E_3^2-6E_2E_4)w_{2_0}^4\right]}{(w_{2_0}^6-E_3w_{2_0}^4+E_1w_{2_0}^2)^2+(E_4w_{2_0}^5-E_2w_{2_0}^3+E_0w_{2_0})^2}\right.\\
&\quad+\frac{w_{2_0}^2\left[(2E_2^2-4E_1E_3+4E_0E_4)w_{2_0}^2+E_1^2-2E_0E_2\right]}{(w_{2_0}^6-E_3w_{2_0}^4+E_1w_{2_0}^2)^2+(E_4w_{2_0}^5-E_2w_{2_0}^3+E_0w_{2_0})^2}\\
&\quad\left.+\frac{(4T_1T_3-2T_2^2)w_{2_0}^4-3T_3^2w_{2_0}^6+(2T_0T_2-T_1^2)w_{2_0}^2}{(T_3w_{2_0}^4-T_1w_{2_0}^2)^2+(T_2w_{2_0}^3-T_0w_{2_0})^2}\right)\\
&= \operatorname{sign}\left(\frac{5w_{2_0}^8+(4E_4^2-8E_3)w_{2_0}^6+(6E_1+3E_3^2-6E_2E_4)w_{2_0}^4}{(w_{2_0}^5-E_3w_{2_0}^3+E_1w_{2_0})^2+(E_4w_{2_0}^4-E_2w_{2_0}^2+E_0)^2}\right.\\
&\quad+\frac{(2E_2^2-4E_1E_3+4E_0E_4)w_{2_0}^2+E_1^2-2E_0E_2}{(w_{2_0}^5-E_3w_{2_0}^3+E_1w_{2_0})^2+(E_4w_{2_0}^4-E_2w_{2_0}^2+E_0)^2}\\
&\quad\left.+\frac{(4T_1T_3-2T_2^2)w_{2_0}^2-3T_3^2w_{2_0}^4+(2T_0T_2-T_1^2)}{(T_3w_{2_0}^3-T_1w_{2_0})^2+(T_2w_{2_0}^2-T_0)^2}\right)
\end{aligned}
$$

After submitting $\xi = iw_{2_0}$ into equation (4.39), we observe that

(4.49)

$$
E_4w_{2_0}^4-E_2w_{2_0}^2+E_0+i(w_{2_0}^5-E_3w_{2_0}^3+E_1w_{2_0})+(T_0-T_2w_{2_0}^2+i(T_1w_{2_0}-T_3w_{2_0}^3))e^{-iw_{2_0}\tau_2}=0.
$$

Since $e^{-iw_{2_0}\tau_2}=\cos w_{2_0}\tau_2-i\sin w_{2_0}\tau_2$, then $\mid e^{-iw_{2_0}\tau_2}\mid=1$ and we obtain from (4.49) that:

$$
\mid E_4w_{2_0}^4-E_2w_{2_0}^2+E_0+i(w_{2_0}^5-E_3w_{2_0}^3+E_1w_{2_0})\mid=\mid T_2w_{2_0}^2-T_0+i(T_3w_{2_0}^3-T_1w_{2_0})\mid. \tag{4.50}
$$

Consequently

(4.51)

$$
(E_4w_{2_0}^4-E_2w_{2_0}^2+E_0)^2+(w_{2_0}^5-E_3w_{2_0}^3+E_1w_{2_0})^2=(T_2w_{2_0}^2-T_0)^2+(T_3w_{2_0}^3-T_1w_{2_0})^2.
$$

Finally, from (4.51) we get

$$
\begin{aligned}
\operatorname{sign}\left[\frac{d(R_e\xi)}{d\tau_2}\right]_{\tau_2=\tau_{2_0}} &= \operatorname{sign}\left(\frac{5w_{2_0}^8+4(E_4^2-2E_3)w_{2_0}^6+3(2E_1+E_3^2-2E_2E_4-T_3^2)w_{2_0}^4}{(T_2w_{2_0}^2-T_0)^2+(T_3w_{2_0}^3-T_1w_{2_0})^2}\right.\\
&\quad\left.+\frac{2(2E_0E_4-2E_1E_3+E_2^2+2T_1T_3-T_2^2)w_{2_0}^2+E_1^2-2E_0E_2+2T_0T_2-T_1^2}{(T_2w_{2_0}^2-T_0)^2+(T_3w_{2_0}^3-T_1w_{2_0})^2}\right)
\end{aligned}
$$

$$= \operatorname{sign}\left(\frac{5w_{2_0}^8+4R_4w_{2_0}^6+3R_3w_{2_0}^4+2R_2w_{2_0}^2+R_1}{(T_2w_{2_0}^2-T_0)^2+(T_3w_{2_0}^3-T_1w_{2_0})^2}\right),$$

$$= \operatorname{sign}\left(\frac{f'(w_{2_0}^2)}{(T_2w_{2_0}^2-T_0)^2+(T_3w_{2_0}^3-T_1w_{2_0})^2}\right).$$

Obviously, it can be noted that $f'(w_{2_0}^2) > 0$ if the condition (4.46) is satisfied. Thus the sign $\left[\frac{d(R_e\xi)}{d\tau_2}\right]_{\tau_2=\tau_{2_0}} > 0$, and the transversality condition of lemma 3 holds. Based on the above discussions and Hopf bifurcation theory, we have the following theorem.

Theorem 4.4. *If $\mathscr{R}_0 > 1$ and condition (4.46) hold, then the endemic equilibrium $E^* = (S^*,E^*,I^*,X^*,W^*)$ is locally asymptotically stable for $\tau_2 \in [0,\tau_{2_0})$ and system (2.7) undergoes a Hopf bifurcation at $E^* = (S^*,E^*,I^*,X^*,W^*)$ when $\tau_2 = \tau_{2_0}$.*

Case 4: Delay in latent and incubation periods $(\tau_1 = \tau_2 = \tau > 0)$

In this case the transcendental equation (4.24) becomes

$$\begin{aligned}\phi(\xi,\tau) = {} & \xi^5+a_4\xi^4+a_3\xi^3+a_2\xi^2+a_1\xi+a_0\\ & +\left(\gamma_1p_3\xi^3+\gamma_1p_2\xi^2+\gamma_1p_1\xi+\gamma_1p_0\right)e^{-2\xi\tau}\\ & +\left(\gamma_2q_3\xi^3+\gamma_2q_2\xi^2+\gamma_2q_1\xi+\gamma_2q_0\right)e^{-\xi\tau}\\ & +(\gamma_3h_1\xi+\gamma_3h_0)e^{-3\xi\tau}=0,\end{aligned} \tag{4.52}$$

$$\gamma_1 = e^{-\tau(g+\sigma)},\ \gamma_2 = e^{-g\tau},\ \gamma_3 = e^{-\tau(\sigma+2g)}. \tag{4.53}$$

Multiplying both sides of equation (4.52) by $e^{\xi\tau}$, we obtain

$$\begin{aligned}\phi(\xi,\tau) = {} & \left(K_3\xi^3+K_2\xi^2+K_1\xi+K_0\right)\\ & +(\xi^5+a_4\xi^4+a_3\xi^3+a_2\xi^2+a_1\xi+a_0)e^{\xi\tau}\\ & +\left(U_3\xi^3+U_2\xi^2+U_1\xi+U_0\right)e^{-\xi\tau}\\ & +(J_1\xi+J_0)e^{-2\xi\tau}=0,\end{aligned} \tag{4.54}$$

where

$$\begin{aligned}& K_3=\gamma_2q_3,\ K_2=\gamma_2q_2,\ K_1=\gamma_2q_1,\ K_0=\gamma_2q_0,\\ & U_3=\gamma_1p_3,\ U_2=\gamma_1p_2,\ U_1=\gamma_1p_1,\ U_0=\gamma_1p_0, J_1=\gamma_3h_1,\ J_0=\gamma_3h_0.\end{aligned} \tag{4.55}$$

Let $\xi = iw (w>0)$ be a purely imaginary root of the equation (4.54). Substituting $\xi = iw$ into (4.54) and separating real and imaginary parts, we obtain the following systems of equations:

$$\begin{cases} (G_1+F_1)\cos 2w\tau-(G_2-F_2)\sin 2w\tau+K_0-K_2w^2=-J_1w\sin 2w\tau-J_0\cos 2w\tau, \\ (G_2+F_2)\cos 2w\tau+(G_1-F_1)\sin 2w\tau+K_1w-K_3w^3=J_0\sin 2w\tau-J_1w\cos 2w\tau, \end{cases} \tag{4.56}$$

where $G_1=a_4w^4-a_2w^2+a_0,\ G_2=w^5-a_3w^3+a_1w,\ F_1=U_0-U_2w^2,\ F_2=U_1w-U_3w^3.$ By using trigonometric identity and after some algebraic calculations, we get

$$(Q_1\cos w\tau-Q_2\sin w\tau+Q_3)^2+(Q_4\cos w\tau+Q_5\sin w\tau+Q_6)^2=L_1^2+L_2^2, \tag{4.57}$$

$$\begin{aligned} &Q_1=G_1+F_1,\ Q_2=G_2-F_2,\ Q_3=K_0-K_2w^2,\ Q_4=G_2+F_2, \\ &Q_5=G_1-F_1,\ Q_6=K_1w-K_3w^3,\ L_1=J_1w,\ L_2=J_0. \end{aligned} \tag{4.58}$$

We know that $\sin^2 w\tau+\cos^2 w\tau=1$. Then, the following two cases are considered.

Case (i): $\sin w\tau=\sqrt{1-\cos^2 w\tau}$, and substituting $\sin w\tau$ into equation (4.57), we get

$$(Q_1\cos w\tau-Q_2\sqrt{1-\cos^2 w\tau}+Q_3)^2+(Q_4\cos w\tau+Q_5\sqrt{1-\cos^2 w\tau}+Q_6)^2=L_1^2+L_2^2, \tag{4.59}$$

which after some algebraic calculations is equivalent to

$$\mu_4\cos^4 w\tau+\mu_3\cos^3 w\tau+\mu_2\cos^2 w\tau+\mu_1\cos w\tau+\mu_0=0, \tag{4.60}$$

where

$$\begin{aligned} \mu_4 &= (Q_1^2+Q_4^2-Q_2^2-Q_5^2)^2+4(Q_4Q_5-Q_1Q_2)^2, \\ \mu_3 &= 4(Q_1^2+Q_4^2-Q_2^2-Q_5^2)(Q_1Q_3+Q_6Q_4)+8(Q_4Q_5-Q_1Q_2)(Q_5Q_6-Q_2Q_3), \\ \mu_2 &= 2(Q_1^2+Q_4^2-Q_2^2-Q_5^2)(Q_2^2+Q_5^2+Q_3^2+Q_6^2-L_1^2-L_2^2)+4(Q_1Q_3+Q_6Q_4)^2+4(Q_5Q_6-Q_2Q_3)^2 \\ &\quad -4(Q_4Q_5-Q_1Q_2)^2, \\ \mu_1 &= 4(Q_1Q_3+Q_6Q_4)(Q_2^2+Q_5^2+Q_3^2+Q_6^2-L_1^2-L_2^2)-8(Q_4Q_5-Q_1Q_2)(Q_5Q_6-Q_2Q_3), \\ \mu_0 &= (Q_2^2+Q_5^2+Q_3^2+Q_6^2-L_1^2-L_2^2)^2-4(Q_5Q_6-Q_2Q_3)^2. \end{aligned}$$

We obtain the expression of $\cos w\tau$ from equation (4.60), and denote it by $p_1(w) = \cos w\tau$. Further, we can obtain the expression of $\sin w\tau$ when $\sin w\tau = \sqrt{1-\cos^2 w\tau}$ and we denote it by $p_2(w) = \sin w\tau$. Thus

$$p_1^2(w) + p_2^2(w) = 1 \tag{4.61}$$

Therefore, in order to give another result of this work, we make the following assumption.

(H_3): Equation (4.61) has at least one positive root.

If condition (H_3) holds, equation (4.61) has one positive root w_{0_*} such that equation (4.54) has a pair of purely imaginary roots $\pm i w_{0_*}$. Thus, the corresponding critical value of τ at which bifurcation occurs is

$$\tau_{0_*} = \frac{1}{w_{0_*}} \arccos p_1(w_{0_*}) \tag{4.62}$$

Case (ii): $\sin w\tau = -\sqrt{1-\cos^2 w\tau}$, and substituting $\sin w\tau$ into equation (4.57), we get

$$(Q_1 \cos w\tau + Q_2\sqrt{1-\cos^2 w\tau} + Q_3)^2 + (Q_4 \cos w\tau - Q_5\sqrt{1-\cos^2 w\tau} + Q_6)^2 = L_1^2 + L_2^2. \tag{4.63}$$

Similarly as in case (i), we can also get the expression of $\cos w\tau$ and $\sin w\tau$ when $\sin w\tau = -\sqrt{1-\cos^2 w\tau}$ and we denote by $p_3(w) = \cos w\tau$ and $p_4(w) = \sin w\tau$. Thus the function with respect to w is

$$p_3^2(w) + p_4^2(w) = 1. \tag{4.64}$$

If equation (4.64) has one positive root $w_{0_{**}}$ such that equation (4.54) has a pair of purely imaginary roots $\pm i w_{0_{**}}$, then the corresponding critical value of the delay τ at which bifurcation occurs is found to be

$$\tau_{0_{**}} = \frac{1}{w_{0_{**}}} \arccos p_3(w_{0_{**}}). \tag{4.65}$$

Let $\tau_0 = \min\{\tau_{0_*}, \tau_{0_{**}}\}$, then equation (4.54) has a pair of purely imaginary roots $\pm i w_0$ when $\tau = \tau_0$. Next, we verify the transversality condition. Taking the derivative of ξ with respect to τ in equation (4.54) gives

$$\left[\frac{d\xi}{d\tau}\right]^{-1} = \frac{y_1(\xi) + y_2(\xi)e^{\xi\tau} + y_3(\xi)e^{-\xi\tau} + y_4(\xi)e^{-2\xi\tau}}{y_5(\xi) - y_6(\xi)e^{\xi\tau}} - \frac{\tau}{\xi}, \tag{4.66}$$

$y_1(\xi) = 3K_3\xi^2 + 2K_2\xi + K_1$, $y_2(\xi) = 5\xi^4 + 4a_4\xi^3 + 3a_3\xi^2 + 2a_2\xi + a_1$,

$y_3(\xi) = 3U_3\xi^2 + 2U_2\xi + U_1, y_4(\xi) = J_1$, $y_5(\xi) = U_3\xi^4 + U_2\xi^3 + (U_1 + 2J_1)\xi^2 + (U_0 + 2D_0)\xi$,

$y_6(\xi) = \xi^6 + a_4\xi^5 + a_3\xi^4 + a_2\xi^3 + a_1\xi^2 + a_0\xi$.

Therefore

$$\left[\frac{d\xi}{d\tau}\right]^{-1}_{\xi=iw_0} = \frac{\Phi_1 + \Phi_2 i}{\Psi_1 + \Psi_2 i}, \text{ with } R_e\left[\frac{d\xi}{d\tau}\right]^{-1}_{\xi=iw_0} = \frac{\Phi_1\Psi_1 + \Phi_2\Psi_2}{\Psi_1^2 + \Psi_2^2},$$

where

$$\begin{aligned}\Phi_1 &= (5w_0^4 - (3a_3 + 3U_3)w_0^2 + a_1 + U_1)\cos w_0\tau_0 + (4a_4w_0^3 + (2U_2 - 2a_2)w_0)\sin w_0\tau_0\\ &\quad + J_1\cos 2w_0\tau_0 + K_1,\\ \Phi_2 &= (5w_0^4 + (3U_3 - 3a_3) + a_1 - U_1)\sin w_0\tau_0 - (4a_4w_0^3 - (2U_2 + 2a_2)w_0)\cos w_0\tau_0\\ &\quad - J_1\sin 2w_0\tau_0 + 2K_2w_0,\\ \Psi_1 &= (w_0^6 - a_3w_0^4 + a_1w_0^2)\cos w_0\tau_0 - (a_2w_0^3 - a_4w_0^5 - a_0w_0)\sin w_0\tau_0 + U_3w_0^4 - (U_1 + 2J_1)w_0^2,\\ \Psi_2 &= (w_0^6 - a_3w_0^4 + a_1w_0^2)\sin w_0\tau_0 + (a_2w_0^3 - a_4w_0^5 - a_0w_0)\cos w_0\tau_0 - U_2w_0^3 + (U_0 + 2D_0)w_0.\end{aligned}$$

Clearly, if condition $(H_4): \Phi_1\Psi_1 + \Phi_2\Psi_2 \neq 0$ holds, then $R_e\left[\frac{d\xi}{d\tau}\right]^{-1}_{\xi=iw_0} \neq 0$. Therefore, following the above analysis and the Hopf bifurcation theory in [30], we have the following theorem.

Theorem 4.5. *Suppose that the conditions* $(H_3) - (H_4)$ *hold. Then, the endemic equilibrium* $E^* = (S^*, E^*, I^*, X^*, W^*)$ *is locally asymptotically stable for* $\tau \in [0, \tau_0)$ *and system (2.7) undergoes a Hopf bifurcation at* $E^* = (S^*, E^*, I^*, X^*, W^*)$ *when* $\tau = \tau_0$.

Table 2. Parameter values for the model (2.7).

Parameter	**Value**	**Reference**
r	$0.5\ day^{-1}$	[22]
K	4000	[22]
g	$0.01\ day^{-1}$	[22]
δ_1	$0.05\ \text{vector}^{-1}\text{day}^{-1}$	[32]
δ_2	$0.25\ \text{plant}^{-1}\text{day}^{-1}$	[32]
m_1	0.01	[22]
m_2	0.02	[22]
ρ	$0.4\ day^{-1}$	[22]
λ	$0.003\ \text{day}^{-1}$	[22]
σ	$0.045\ day^{-1}$	[22]
τ_1	$7\ days$	Assumed
τ_2	$4-5\ days$	[33]

5. NUMERICAL SIMULATION AND RESULTS

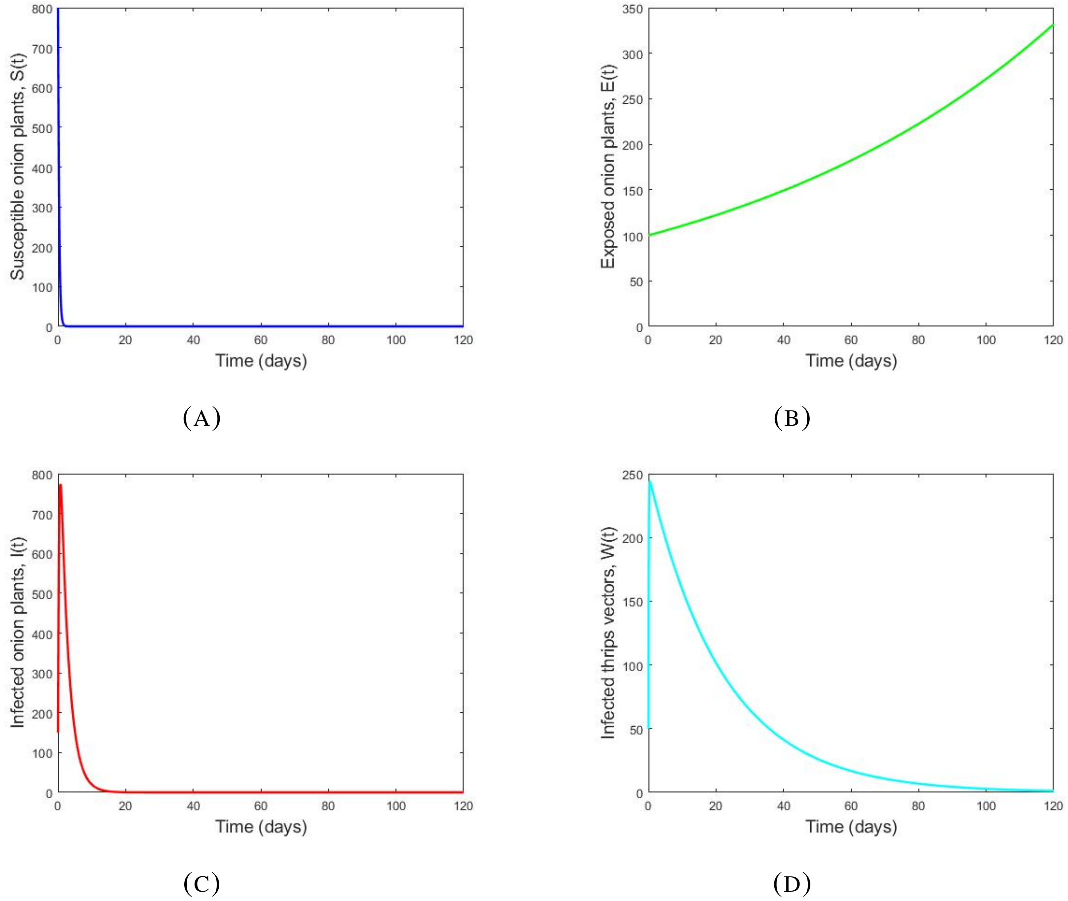
In this section, we carried out some numerical simulations of the model (2.7) in order to support some theoretical results obtained above. The set of parameters values given in Table 2 are used. We set the initial conditions of the model (2.7) as $S(0) = 800, E(0) = 100, I(0) = 150, X(0) = 200, W(0) = 50$. We set the final time as $t_f = 120$ days because onion plants need around 120 days from planting to harvest. All simulations are done using MATLAB dde23. To this end, the following case of system (2.7) is obtained:

$$
\left\{
\begin{aligned}
\frac{dS(t)}{dt} &= 0.5S(t)\left(1-\frac{S(t)+E(t)+I(t)}{4000}\right)-\frac{0.05W(t)S(t)}{1+0.01W(t)}-0.01S(t),\\
\frac{dE(t)}{dt} &= \frac{0.05W(t)S(t)}{1+0.01W(t)}-\frac{0.05e^{-0.3}W(t-\tau_1)S(t-\tau_1)}{1+0.01W(t-\tau_1)}-0.01E(t),\\
\frac{dI(t)}{dt} &= \frac{0.05e^{-0.3}W(t-\tau_1)S(t-\tau_1)}{1+0.01W(t-\tau_1)}-0.4I(t)-0.01I(t),\\
\frac{dX(t)}{dt} &= 0.003-\frac{0.25X(t)I(t)}{1+0.02I(t)}-0.045X(t),\\
\frac{dW(t)}{dt} &= \frac{0.25e^{-0.225}X(t-\tau_2)I(t-\tau_2)}{1+0.02I(t-\tau_2)}-0.045W(t).
\end{aligned}
\right.
\tag{5.1}
$$

By a simple computation, we get $\mathscr{R}_0 = 10.2341 > 1$ and the unique positive root of the quadratic polynomial (4.18) is found to be $I^* = 0.2489$. Further, using Maple software, the unique positive endemic equilibrium is obtained as $E^* = (56.8647, 4.3354, 0.2489, 0.4390, 0.1812)$.

5.1. Simulations for the local asymptotic stability of the endemic equilibrium with no delays $(\tau_1 = 0, \tau_2 = 0)$.

Figure 2 shows that with no delays $(\tau_1 = 0, \tau_2 = 0)$, the endemic equilibrium E^* of system (2.7) is locally asymptotically stable when the basic reproduction number $\mathscr{R}_0$ is greater than unity. In fact, from Figure 2(A) we observe that the susceptible onion plants decrease and converges to a steady state to acquire endemic equilibrium level while in Figure 2(B), there is an increase in the number of exposed onion plants to a certain endemic level. From Figure 2 (C), we observe that the infected onion plants increase to a certain maximum point before a decrease to acquire endemic equilibrium level. Similarly, we observe from Figure 2 (D) that the infected thrips vectors increase to a certain maximum point before a decrease to acquire endemic equilibrium level. The biological meaning is that as long as $\mathscr{R}_0 > 1$, the IYSD will persist in the onion plants population. The set parameter values in Table 2 leads to $\mathscr{R}_0 = 10.2341$ which is greater than unity. A simple computation by means of Maple 18 software shows that Routh-Hurwitz conditions in Theorem 4.2 are satisfied. Therefore, Theorem 4.2 holds.



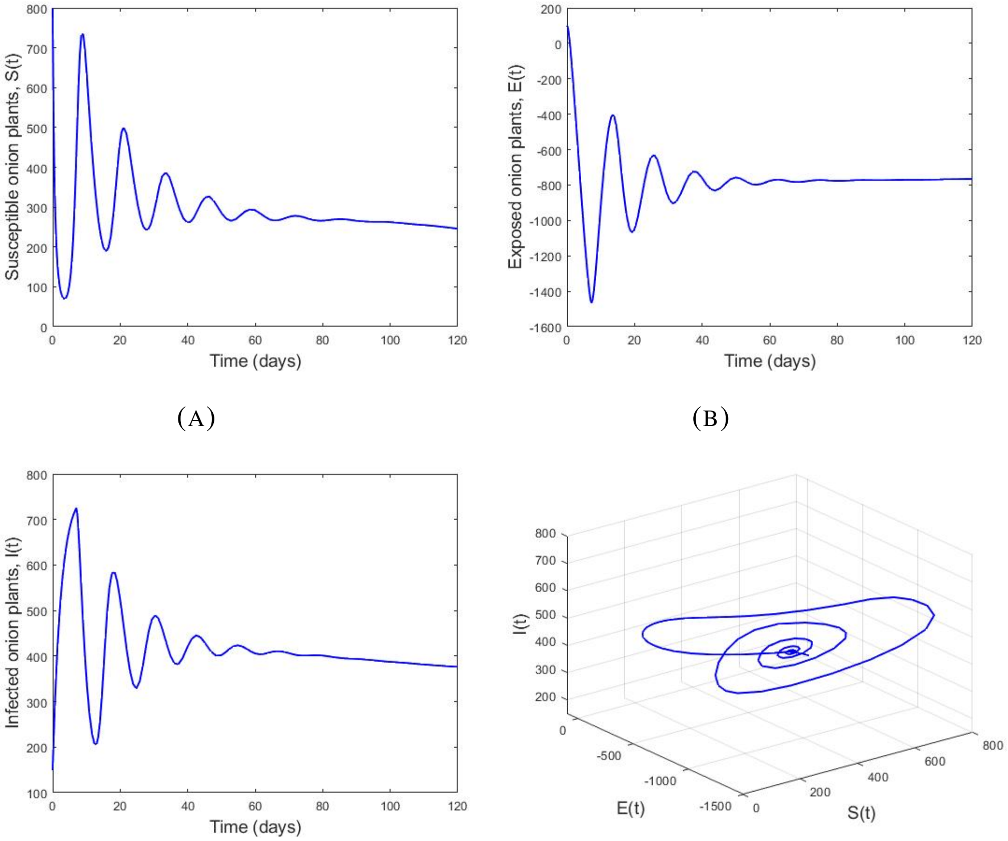
(A) (B)

(C) (D)

FIGURE 2. Graph of solutions of the model (2.7) where the endemic equilibrium point E^* is locally asymptotically stable with no delays $(\tau_1 = 0,\, \tau_2 = 0)$

5.2. Simulations for the local asymptotic stability of the endemic equilibrium and existence of Hopf-bifurcation with delay only for incubation period $(\tau_1 > 0,\, \tau_2 = 0)$.

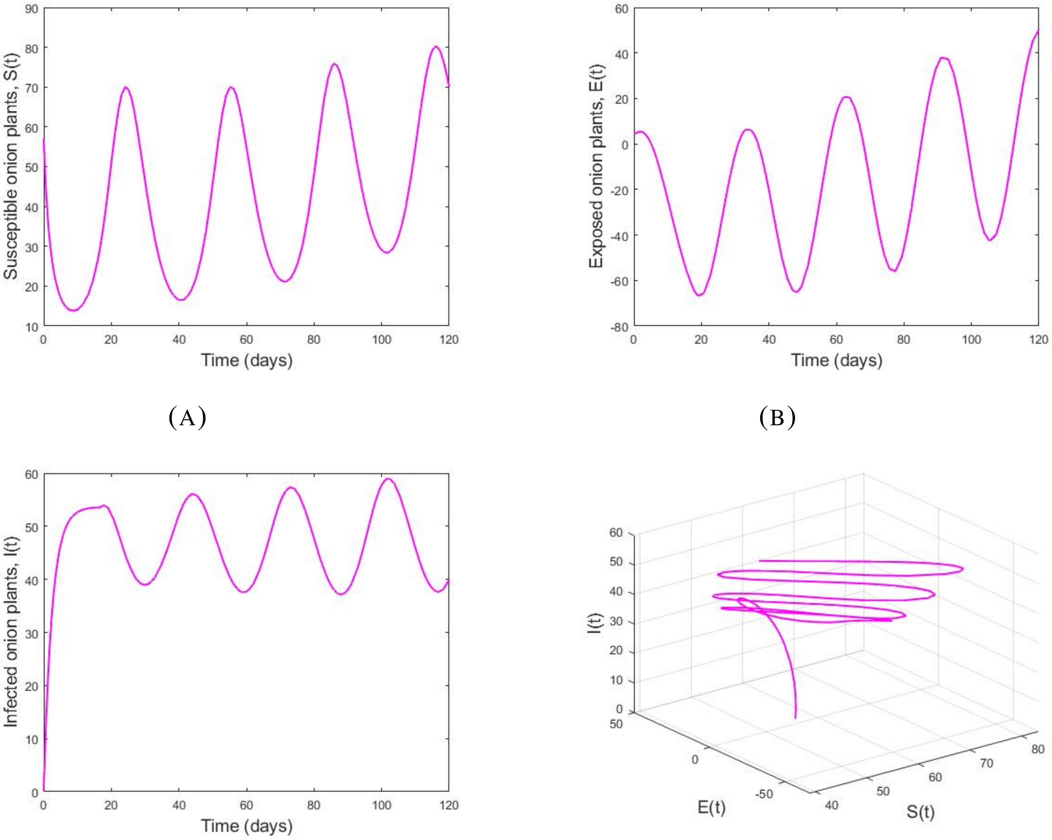
For $\tau_1 > 0$, $\tau_2 = 0$, a computation by means of Maple 18 software gives $w_{1_0} = 0.7139$ and $\tau_{1_0} = 9.8210$. Hence, according to Theorem 4.3, the positive endemic equilibrium E^* is locally asymptotically stable when $\tau_1 = 7 < 9.8210 = \tau_{1_0}$ and this result is illustrated is Figure 3. However, when the incubation delay $\tau_1 = 7$ passes through the critical value $\tau_{1_0} = 9.8210$, then the positive endemic equilibrium E^* loses its stability and model (2.7) undergoes a Hopf-bifurcation at E^*. That is, the family of periodic solutions of the model (2.7) bifurcate from the positive endemic equilibrium E^* and this property is illustrated in Figure 4.



(A) (B)

(C) (D)

FIGURE 3. Graph of model (2.7) where the endemic equilibrium point E^* is locally asymptotically stable and the corresponding phase portrait of the states S, E, I when $\tau_1 = 7 < 9.8210 = \tau_{1_0}$.

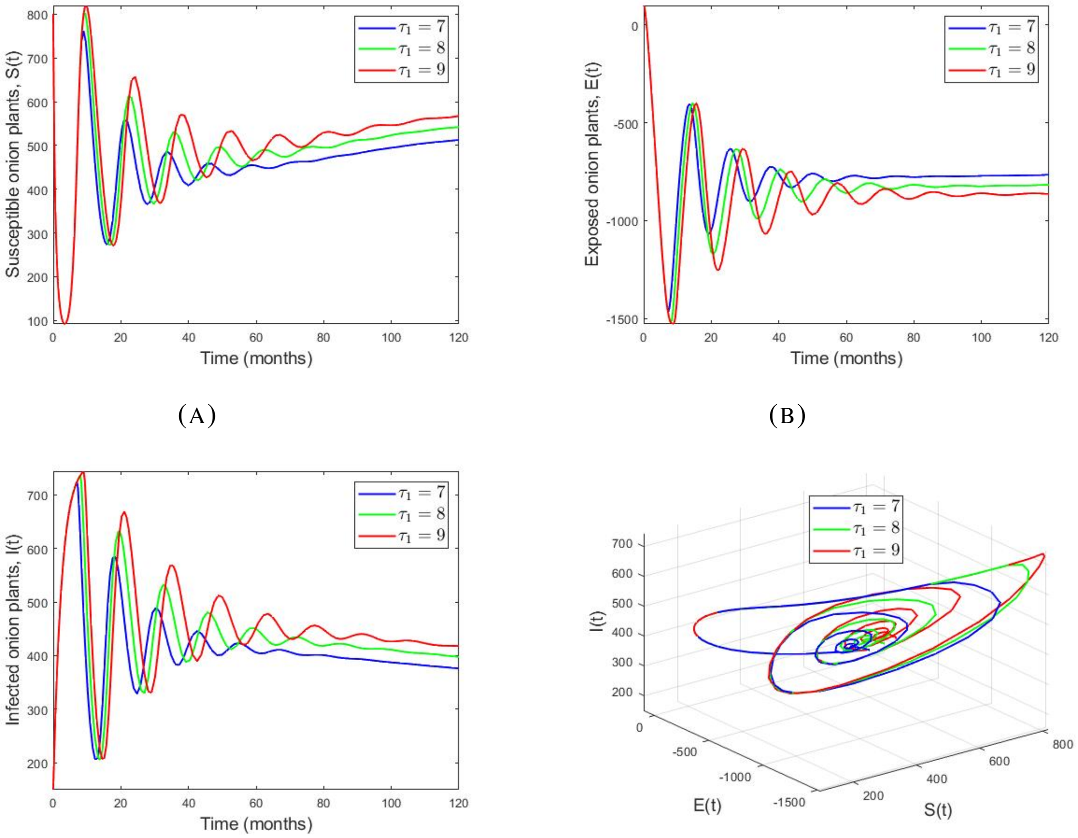


(A) (B)

(C) (D)

FIGURE 4. Graph of model (2.7) showing the occurrence of Hopf-bifurcation at endemic equilibrium point E^* and the corresponding phase portrait of the states S, E, I when $\tau_1 = 16 > 9.8210 = \tau_{1_0}$.

5.3. Simulations of the effects of incubation delay τ_1 on the dynamics of the disease ($\tau_1 > 0$, $\tau_2 = 0$). In Figure 5 (A) and 5 (C), we observe that the magnitude of the amplitude of oscillations for susceptible and infected onion population increases when the incubation delay τ_1 increases with the latent delay $\tau_2 = 0$. We observe from Figure 5 (B) that as incubation delay τ_1 increases, the number of exposed onion plants increase, which is consistent with reality from biological point of view. These scenarios mean that if no intervention is done to reduce the incubation delay, then the IYSV disease will persist in the onion plants population. Moreover, the decrease of the incubation delay will guarantee the asymptotic stability of the endemic equilibrium which means that the disease can be eradicated from the onion population.

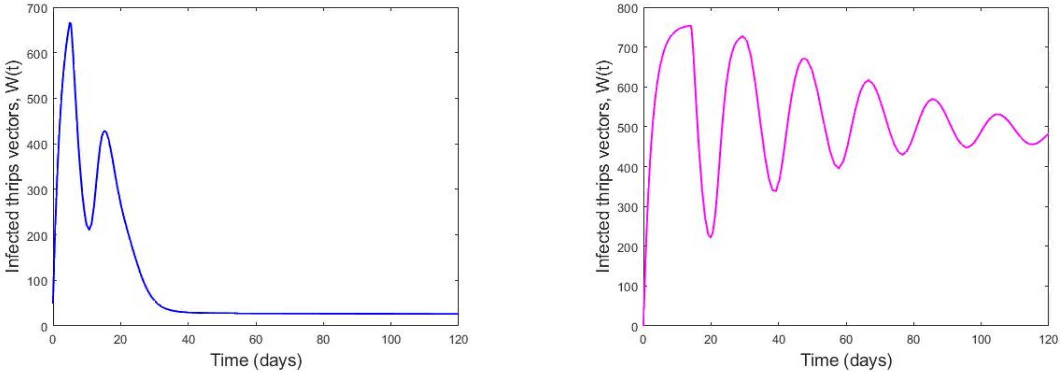


(A) (B)

(C) (D)

FIGURE 5. Graph of solutions of the model (2.7) showing the effects of varying the incubation delay τ_1 on the dynamics of the disease and the corresponding phase portrait of the states S, E, I. The delay τ_1 are chosen as $\tau_1 = 7$, 8, 9.

5.4. Simulations for the local asymptotic stability of the endemic equilibrium and existence of Hopf-bifurcation with delay only in latent period $(\tau_1 = 0,\ \tau_2 > 0)$. For $\tau_1 = 0,\ \tau_2 > 0$, a computation by means of Maple 18 software gives $w_{2_0} = 1.9618$ and $\tau_{2_0} = 8.7624$. Hence, according to Theorem 4.4, the positive endemic equilibrium E^* is locally asymptotically stable when $\tau_2 = 5 < 8.7624 = \tau_{2_0}$ and this result is illustrated is Figure 6 (A). However, when the latent delay $\tau_2 = 5$ passes through the critical value $\tau_{2_0} = 8.7624$, then the positive endemic equilibrium E^* loses its stability and model (2.7) undergoes a Hopf-bifurcation at E^*. That is, the family of periodic solutions bifurcate from the positive endemic equilibrium E^* and this property is illustrated in Figure 6 (B).



(A) (B)

FIGURE 6. Figure 6 (A) is the graph of solutions of the model (2.7) where the endemic equilibrium point E^* is locally asymptotically stable when $\tau_2 = 5 < 8.7624 = \tau_{2_0}$. Figure 6 (B) shows the occurrence of Hopf-bifurcation at the endemic equilibrium point E^* when $\tau_2 = 10 > 8.7624 = \tau_{2_0}$.

5.5. Simulations of the effects of latent delay τ_2 on the dynamics of the disease $(\tau_1 = 0, \tau_2 > 0)$. In Figure 7 , we observe that the magnitude of the amplitude of oscillations for infected thrips vectors increases when the latent delay τ_2 increases with the incubation delay $\tau_1 = 0$. Hence if no intervention is done to kill the thrips vectors who are carriers of IYSV, then the disease will persist in the onion plants population.

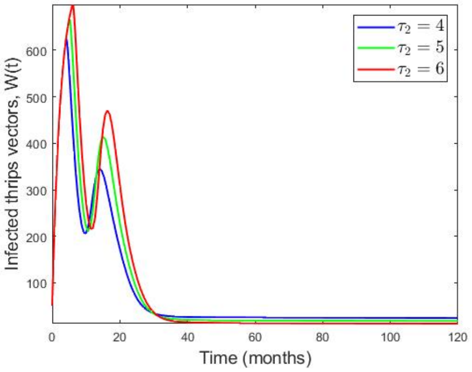
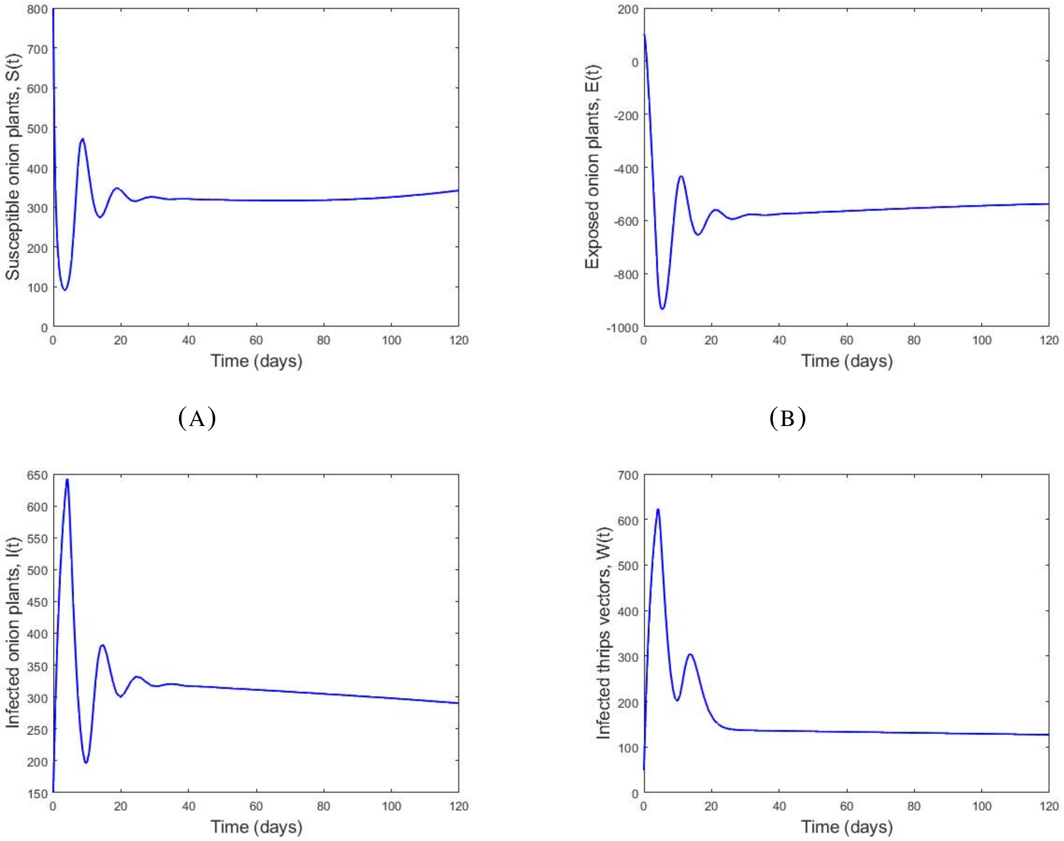


FIGURE 7. Graph of solutions of the model (2.7) showing the effects of varying the latent delay τ_2 on the dynamics of the disease. The delay τ_2 are chosen as $\tau_2 = 4, 5, 6$.

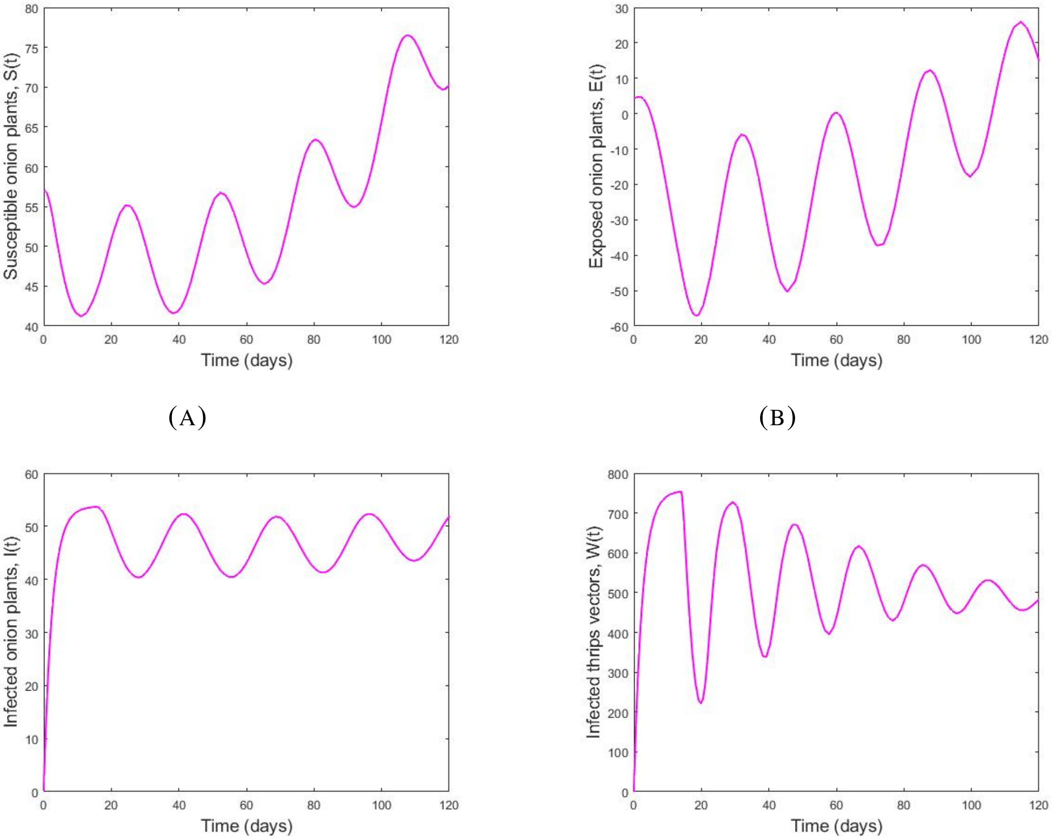
5.6. Simulations for the local asymptotic stability of the endemic equilibrium and existence of Hopf-bifurcation with both incubation and latent delays $(\tau_1 = \tau_2 = \tau > 0)$. For $\tau_1 = \tau_2 = \tau > 0$, a computation by means of Maple 18 software gives $w_0 = 0.2793$ and $\tau_0 = 6.5217$. Hence, according to Theorem 4.5, the positive endemic equilibrium E^* is locally asymptotically stable when $\tau = 4 < 6.5217 = \tau_0$ and this result is illustrated is Figure 8. However, when the time delays $\tau_1 = \tau_2 = \tau = 4$ passes through the critical value $\tau_0 = 6.5217$, then the positive endemic equilibrium E^* loses its stability and model (2.7) undergoes a Hopf-bifurcation at E^*. This implies that the family of periodic solutions of the model (2.7) bifurcate from the positive endemic equilibrium E^* and this is illustrated in Figure 9.



(A) (B)

(C) (D)

FIGURE 8. Graph of solutions of the model (2.7) where the positive endemic equilibrium point E^* is locally asymptotically stable when $\tau = 4 < 6.5217 = \tau_0$.



(A) (B)

(C) (D)

FIGURE 9. Graph of solutions of the model (2.7) showing the occurrence of Hopf-bifurcation at the positive endemic equilibrium point E^* when $\tau = 14 > 6.5217 = \tau_0$.

6. DISCUSSIONS AND CONCLUSIONS

In this paper, we have formulated a delayed SEIXW model of IYSV disease in onion plant extending the model proposed in [22], by introducing the class of exposed plants representing the number of exposed plants in the incubation period and by incorporating two delays in terms of incubation and latent periods. Compared with the IYSV disease model in [22], we mainly explore the effect of the time delays on its dynamic nature. Compared with other plant disease models, we assumed logistic growth for the onion population and incorporated the incubation by introducing the class of exposed plant which appears to be consistent with the reality. From the numerical simulations, we have shown that when the value of the incubation delay τ_1 and latent time τ_2 are below the corresponding critical values τ_{1_0} and τ_{2_0} respectively, the positive endemic equilibrium $E^* = (56.8647, 4.3354, 0.2489, 0.4390, 0.1812)$ of model (2.7) is locally asymptotically stable under some certain conditions. The result in this case implies that the spread of IYSV disease in onion plants can be controlled with ease. Conversely, once the value of the delays τ_1 and τ_2 passes through the critical values τ_{1_0} and τ_{2_0} respectively, the positive endemic equilibrium E^* loses its stability and a Hopf–bifurcation occurs with a family of periodic solutions bifurcating from $E^* = (56.8647, 4.3354, 0.2489, 0.4390, 0.1812)$. The result in this case means that IYSV disease will persist in the onion population and will be out of control. The Hopf bifurcation phenomenon in Figure 4 manifests that the onion plants population will remain in an oscillatory behavior as long as the incubation period is greater that its corresponding critical value. Then onion farmers must be alert to the IYSV even if they notice that only fewer plants are becoming infected. We can observe from equation (4.5) of the basic reproduction number $\mathscr{R}_0$ that both delays τ_1, τ_2 are present, so delayed models are more realistic and mathematically, the delays affect the dynamics and the stability of disease-free equilibrium. Thus, if the incubation or latent period can be reduced (for example by using beneficial insects and chemicals pesticides) to an level for which $\mathscr{R}_0 < 1$, the IYSV disease can be eliminated from the onion farms. Some authors considered that in vector-borne plant diseases control, taking into account the incubation and latent periods is crucial to design appropriate policy towards controlling and preventing the disease. Hence, in the near future we will come with a paper to derive the optimal control problem and cost-effective strategies for our proposed delayed model.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

Appendix

A. Computed values of: the basic reproduction number $\mathscr{R}_0$, coefficient terms in the quadratic equation (4.18), the endemic equilibrium points, the coefficient terms in the Jacobian matrix (4.23), the coefficient terms in the transcendental equations (4.24) and (4.29), the critical oscillation frequency w_{1_0} as well as the critical delay τ_{1_0} by using Maple 18 software with parameters from Table 2.

A.1 Basic reproduction number, coefficient terms in the quadratic equation (4.18) and the endemic equilibrium points.

$$\mathscr{R}_0 = 10.2341,\ c_2 = -3.3084\times 10^{-7},\ c_1 = -0.000003420,\ c_0 = 8.7204\times 10^{-7},$$

$$S^* = 56.8647,\ E^* = 4.3354,\ I^* = 0.2489,\ X^* = 0.4390,\ W^* = 0.1812.$$

A.2 Coefficient terms in the Jacobian matrix (4.23)

$$b_1 = 0.4746,\ b_2 = -0.0071,\ b_3 = -0.0071, b_4 = -2.8422,\ b_5 = 0.0009048,$$

$$b_6 = -0.01,\ b_7 = 2.8422,\ b_8 = 0.0009048,\ b_9 = -0.41,\ b_{10} = 2.8422,$$

$$b_{11} = -0.0108,\ b_{12} = -0.1069,\ b_{13} = 0.0108,\ b_{14} = 0.0619,\ b_{15} = -0.045.$$

A.3 Coefficient terms in the transcendental equation (4.24)

$$a_4 = 0.0972,\ a_3 = -0.1987,\ a_2 = -0.0318,\ a_1 = -0.0012,\ a_0 = -0.000009350,$$

$$p_3 = -0.0308,\ p_2 = -0.009165,\ p_1 = 0.0007939,\ p_0 = 0.000006609,\ q_3 = 0,$$

$$q_2 = 0.000007453,\ q_1 = -0.000006390,\ q_0 = -1.2377\times 10^{-8},\ h_1 = 0,\ h_0 = 0.$$

A.4 Coefficient values in the transcendental equation (4.29)and the coefficient value of $\sin w_1\tau_1$ and $\cos w_1\tau_1$ obtained from system (4.32)

$$C_5 = 1.3498,\ C_4 = 0.1312,\ C_3 = -0.2682,\ C_2 = -0.0429,\ C_1 = -0.0016,\ C_0 = -0.00001262,$$

$$D_1 = 0,\ D_0 = 0\ f_0 = 8.3241\times 10^{-11},\ f_1 = -6.1535\times 10^{-10},\ f_2 = 1.092\times 10^{-6},\ f_3 = -4.627\times 10^{-5},$$

$$f_4 = 5.555\times 10^{-4},\ f_5 = 0.001039,\ f_6 = -0.0105,\ f_7 = 0.008316,\ f_8 = -0.0416,\ g_0 = 1.5926\times 10^{-10},$$

$$g_2 = 1.477\times 10^{-6},\ g_4 = 9.788\times 10^{-4},\ g_6 = 0.0788,\ g_8 = 0.7412,\ g_{10} = 1.8219.$$

A.5 Coefficient values in the Hopf frequency equation (4.34)

$M_{10} = 3.3195$, $M_9 = 2.7010$, $M_8 = 0.8351$, $M_7 = 0.1195$, $M_6 = 0.0075$, $M_5 = 1.446 \times 10^{-4}$,

$M_4 = 10^{-6}$,$M_3 = 1.9926 \times 10^{-9}$, $M_2 = 1.3357 \times 10^{-12}$, $M_1 = 2.8825 \times 10^{-16}$,

$M_0 = -1.8435 \times 10^{-20}$.

Hence, we obtain the critical oscillation frequency $w_{1_0} = 0.7139$ from equation (4.34), and the critical value of delay τ_1 as $\tau_{1_0} = 9.8210$ from equation (4.36).

B. Computed values of the coefficients terms of the transcendental equations (4.39) and the coefficients terms of the Hopf frequency equation (4.45) needed to compute the critical oscillation frequency w_{2_0} and the corresponding critical delay τ_{2_0} by using Maple 18 software with parameters from Table 2.

B.1 Coefficient terms in the transcendental equation (4.39).

$$E_4 = 0.0972,\ E_3 = -0.1987,\ E_2 = -0.0317,\ E_1 = -0.001206,\ E_0 = -1.2377 \times 10^{-8},$$

$$\beta = 0.7985,\ T_3 = -0.0245,\ T_2 = -0.007318,\ T_1 = 0.0006339,\ T_0 = 0.00000527.$$

B.2 Coefficient terms in the Hopf frequency equation (4.45).

$$R_4 = 0.4068,\ R_3 = 0.0425,\ R_2 = 0.0001913,\ R_1 = -0.02354,\ R_0 = -0.01233.$$

Hence, we obtain the critical oscillation frequency $w_{2_0} = 1.9618$ from equation (4.45), and the critical value of delay τ_2 as $\tau_{2_0} = 8.7624$ from equation (4.47).

C. Derivation of the expression of $\cos w\tau$ from equation (4.60) when $\sin w\tau = \pm\sqrt{1-\cos^2 w\tau}$.

Equation (4.60) is given by

$$\mu_4 \cos^4 w\tau + \mu_3 \cos^3 w\tau + \mu_2 \cos^2 w\tau + \mu_1 \cos w\tau + \mu_0 = 0,$$

Let $\cos w\tau = \alpha$, then the above equation becomes

$$\mu_4 \alpha^4 + \mu_3 \alpha^3 + \mu_2 \alpha^2 + \mu_1 \alpha + \mu_0 = 0,$$

Denote

$$g(\alpha) = \alpha^4 + \frac{\mu_3}{\mu_4}\alpha^3 + \frac{\mu_2}{\mu_4}\alpha^2 + \frac{\mu_1}{\mu_4}\alpha + \frac{\mu_0}{\mu_4},$$

Thus

$$g'(\alpha)=4\alpha^3+3\frac{\mu_3}{\mu_4}\alpha^2+2\frac{\mu_2}{\mu_4}\alpha+\frac{\mu_1}{\mu_4},$$

We set

$$g'(\alpha)=4\alpha^3+3\frac{\mu_3}{\mu_4}\alpha^2+2\frac{\mu_2}{\mu_4}\alpha+\frac{\mu_1}{\mu_4}=0.$$

Let $x=\alpha+\frac{\mu_3}{4\mu_4}\Longrightarrow\alpha=x-\frac{\mu_3}{4\mu_4}$. Then we get

$$x^3+\rho_1x+\rho_0=0,$$

where

$$\rho_1=\frac{\mu_2}{2\mu_4}-\frac{3\mu_3^2}{16\mu_4^2},\ \rho_0=\frac{\mu_3^3}{32\mu_4^3}-\frac{\mu_2\mu_3}{8\mu_4^2}+\frac{\mu_1}{4\mu_4}$$

We define

$$\varepsilon_1=\left(\frac{\rho_0}{2}\right)^2+\left(\frac{\rho_1}{3}\right)^3,\ \varepsilon_2=\frac{-1+\sqrt{3}i}{2}$$

$$x_1=\sqrt[3]{-\frac{\rho_0}{2}+\sqrt{\varepsilon_1}}+\sqrt[3]{-\frac{\rho_0}{2}-\sqrt{\varepsilon_1}},$$

$$x_2=\sqrt[3]{-\frac{\rho_0}{2}+\sqrt{\varepsilon_1}\varepsilon_2}+\sqrt[3]{-\frac{\rho_0}{2}-\sqrt{\varepsilon_1}\varepsilon_2^2},$$

$$x_3=\sqrt[3]{-\frac{\rho_0}{2}+\sqrt{\varepsilon_1}\varepsilon_2^2}+\sqrt[3]{-\frac{\rho_0}{2}-\sqrt{\varepsilon_1}\varepsilon_2}.$$

$$\alpha_i=x_i-\frac{\mu_3}{4\mu_4},\ i=1,2,3$$

We have calculated the roots of equations (4.61) and (4.64) (functions with respect to w) by means of Maple 18 using the parameters in Table 2 and we discussed about the roots similarly to that in [21]. To this end, we obtained the critical oscillation frequency as $w_0=0.2793$ and the critical value of τ as $\tau_0=6.5217$, from equations (4.62) and (4.65).

REFERENCES

[1] G. Griffiths, L. Trueman, T. Crowther, et al. Onions–A global benefit to health, Phytother. Res. 16 (2002), 603–615. https://doi.org/10.1002/ptr.1222.

[2] R.K. Birithia, S. Subramanian, D.K. Kuria, Farmers' preference for onion varieties and implications of knowledge of Iris yellow spot disease in Kenya, Afr. Crop Sci. J. 29 (1970), 229–239. https://doi.org/10.4314/acsj.v29i2.4.

[3] R.K. Birithia, S. Subramanian, J.W. Muthomi, R.D. Narla, Seasonal dynamics and alternate hosts of thrips transmitted Iris yellow spot virus in Kenya, Afr. Crop Sci. J. 26 (2018), 365-376. https://doi.org/10.4314/acsj.v26i3.3.

[4] R. Birithia, S. Subramanian, H.R. Pappu, et al. First Report of Iris yellow spot virus Infecting Onion in Kenya and Uganda, Plant Dis. 95 (2011), 1195–1195. https://doi.org/10.1094/pdis-01-11-0057.

[5] H.R. Pappu, R.A.C. Jones, R.K. Jain, Global status of tospovirus epidemics in diverse cropping systems: Successes achieved and challenges ahead, Virus Res. 141 (2009), 219–236. https://doi.org/10.1016/j.virusres.2009.01.009.

[6] B. Mandal, R.K. Jain, M. Krishnareddy, et al. Emerging problems of tospoviruses (Bunyaviridae) and their management in the Indian subcontinent, Plant Dis. 96 (2012), 468–479. https://doi.org/10.1094/pdis-06-11-0520.

[7] D. Persley, C. Wilson, J. Thomas, et al. IXth international symposium on thysanoptera and tospoviruses, J. Insect Sci. 10 (2010), 66. https://doi.org/10.1673/031.010.14126.

[8] S. Weilner, G. Bedlan, et al. Detection of Iris yellow spot virus (IYSV) in selected Allium species and overwintering hosts in Austrian onion-producing areas, J. Kulturpflanzen. 65 (2013), 60–67.

[9] M.J. Jeger, J. Holt, F. Van Den Bosch, et al. Epidemiology of insect-transmitted plant viruses: modelling disease dynamics and control interventions, Physiol. Entomol. 29 (2004), 291–304. https://doi.org/10.1111/j.0307-6962.2004.00394.x.

[10] M. Jackson, Modeling Plant Virus Propagation and an Optimal Control, PhD thesis, The University of Texas at Arlington, 2018. http://hdl.handle.net/10106/27601.

[11] F.A. Basir, S. Ray, E. Venturino, Role of media coverage and delay in controlling infectious diseases: A mathematical model, Appl. Math. Comput. 337 (2018), 372–385. https://doi.org/10.1016/j.amc.2018.05.042.

[12] F. Al Basir, P.K. Roy, Dynamics of mosaic disease with roguing and delay in Jatropha curcas plantations, J. Appl. Math. Comput. 58 (2017), 1–31. https://doi.org/10.1007/s12190-017-1131-2.

[13] M. Leclerc, T. Doré, C.A. Gilligan, et al. Estimating the delay between host infection and disease (incubation period) and assessing its significance to the epidemiology of plant diseases, PLoS ONE. 9 (2014), e86568. https://doi.org/10.1371/journal.pone.0086568.

[14] J.E. Van der Plank, Plant diseases: epidemics and control, Elsevier, 2013.

[15] T. Zhang, X. Meng, Y. Song, et al. Dynamical analysis of delayed plant disease models with continuous or impulsive cultural control strategies, Abstr. Appl. Anal. 2012 (2012), 1–25. https://doi.org/10.1155/2012/428453.

[16] S. Ray, F.A. Basir, Impact of incubation delay in plant–vector interaction, Math. Computers Simul. 170 (2020), 16–31. https://doi.org/10.1016/j.matcom.2019.09.001.

[17] A. Kaddar, A. Abta, H.T. Alaoui, A comparison of delayed SIR and SEIR epidemic models, Nonlinear Anal.: Model. Control. 16 (2011), 181–190. https://doi.org/10.15388/na.16.2.14104.

[18] L.M. Cai, X.Z. Li, B. Fang, et al. Global properties of vector–host disease models with time delays, J. Math. Biol. 74 (2016), 1397–1423. https://doi.org/10.1007/s00285-016-1047-8.

[19] M. Jackson, B.M. Chen-Charpentier, Modeling plant virus propagation with delays, J. Comput. Appl. Math. 309 (2017), 611–621. https://doi.org/10.1016/j.cam.2016.04.024.

[20] A.K. Shaw, A. Peace, A.G. Power, et al. Vector population growth and condition-dependent movement drive the spread of plant pathogens, Ecology. 98 (2017), 2145–2157. https://doi.org/10.1002/ecy.1907.

[21] F. Al Basir, S. Adhurya, M. Banerjee, et al. Modelling the effect of incubation and latent periods on the dynamics of vector-borne plant viral diseases, Bull. Math. Biol. 82 (2020), 94. https://doi.org/10.1007/s11538-020-00767-2.

[22] P.C. Kawe, O.J. Abonyo, D.M. Malonza, et al. Modeling and stability analysis of an eco-epidemiological model of IYSV disease dynamics in onion plants with nonlinear saturated incidence rate and logistic growth, Glob. J. Pure Appl. Math. 18 (2022), 171–204.

[23] B.L. Keyfitz, N. Keyfitz, The McKendrick partial differential equation and its uses in epidemiology and population study, Math. Computer Model. 26 (1997), 1–9. https://doi.org/10.1016/s0895-7177(97)00165-9.

[24] C.M. Kribs-Zaleta, J.X. Velasco-Hernández, A simple vaccination model with multiple endemic states, Math. Biosci. 164 (2000), 183–201. https://doi.org/10.1016/s0025-5564(00)00003-1.

[25] J.K. Hale, Topics in dynamic bifurcation theory, American Mathematical Society, 1981.

[26] H.W. Hethcote, The mathematics of infectious diseases, SIAM Rev. 42 (2000), 599–653. https://doi.org/10.1137/s0036144500371907.

[27] Z. Xu, X.Q. Zhao, A vector-bias malaria model with incubation period and diffusion, Discrete Contin. Dyn. Syst. - B. 17 (2012), 2615–2634. https://doi.org/10.3934/dcdsb.2012.17.2615.

[28]]A.J. Arenas, G. González-Parra, J.J. Naranjo, et al. Mathematical analysis and numerical solution of a model of HIV with a discrete time delay, Mathematics. 9 (2021), 257. https://doi.org/10.3390/math9030257.

[29] F.K. Mbabazi, J.Y.T. Mugisha, M. Kimathi, Hopf-bifurcation analysis of pneumococcal pneumonia with time delays, Abstr. Appl. Anal. 2019 (2019), 3757036. https://doi.org/10.1155/2019/3757036.

[30] B.D. Hassard, N.D. Kazarinoff, Y.H. Wan, Theory and applications of Hopf bifurcation, Cambridge University Press, 1981.

[31] F. Zhang, S. Gao, H. Cao, et al. Dynamical analysis of a schistosomiasis japonicum model with time delay, J. Appl. Math. Phys. 7 (2019), 948–967. https://doi.org/10.4236/jamp.2019.74064.

[32] M.J. Jeger, F. van den Bosch, N. McRoberts, Modelling transmission characteristics and epidemic development of the tospovirus–thrip interaction, Arthropod-Plant Interact. 9 (2015), 107–120. https://doi.org/10.1007/s11829-015-9363-2.

[33] C.L. Hsu, C.A. Hoepting, M. Fuchs, et al. Temporal dynamics of iris yellow spot virus and its vector, thrips tabaci (thysanoptera: thripidae), in seeded and transplanted onion fields, Environ. Entomol. 39 (2010), 266–277. https://doi.org/10.1603/en09165.