Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2015, 2015:1
ISSN: 2052-2541

HOPF BIFURCATION IN A DELAYED LOGISTIC GROWTH WITH FEEDBACK CONTROL

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Abstract. In this paper, the bifurcation of the delayed logistic model with feedback control variable are investigated. By regarding the corresponding characteristic equations, the linear stability of the system is discussed and Hopf bifurcations are demonstrated, especially the stability switch is discussed in this system. By the normal form and the center manifold theory, the explicit formulae are derived to determine the stability, direction and other properties of bifurcating periodic solutions. Finally, some examples are presented to verify our main results.

Keywords: Feedback control; Stability switch; Hopf bifurcation.

2010 AMS Subject Classification: 34D23, 92B05.

1. Introduction

The single-species logistic growth model governed by delay differential (and integro-differential) equations plays an important role in population dynamics and ecology that has been investigated in-depth involving the stability, persistent, oscillations and chaotic behavior of solutions

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Received October 25, 2014

[1]-[11]. Gopalsamy and Weng [12] considered the following control system:

$$\begin{aligned}\frac{dn(t)}{dt} &= rn(t)[1-\frac{a_1n(t)+a_2n(t-\tau)}{K}-cu(t)],\\ \frac{du(t)}{dt} &= -au(t)+bn(t-\tau),\end{aligned}\tag{1.1}$$

where $r,K,c,a,b,a_1,a_2 \in (0,+\infty)$, $\tau \in [0,+\infty)$. The authors presented some sufficient conditions for the global asymptotic stability of the positive equilibrium of the system. On one hand, in Song *et al.* [13], the authors considered the Hopf bifurcation for a regulated logistic growth model which is a special case of (1.1) as follows

$$\begin{aligned}\frac{dn(t)}{dt} &= rn(t)[1-\frac{n(t-\tau)}{K}-cu(t)],\\ \frac{du(t)}{dt} &= -au(t)+bn(t-\tau),\end{aligned}\tag{1.2}$$

where $r,K,c,a,b \in (0,+\infty)$, $\tau \in [0,+\infty)$. And the authors gave the explicit algorithm determining the direction of Hopf bifurcations and the stability of the periodic solutions, while they didn't discuss the existence of stability switches of this system. On the other hands, Gopalsamy and Weng [12] investigate the following control system:

$$\begin{aligned}\frac{dn(t)}{dt} &= rn(t)[1-\frac{n(t-\tau)}{K}-cu(t)],\\ \frac{du(t)}{dt} &= -au(t)+bn(t),\end{aligned}\tag{1.3}$$

where $r,K,c,a,b \in (0,+\infty)$, $\tau \in [0,+\infty)$, the initial conditions for the system (1.3) take the form of $n(s)=\phi(s)\geq 0$; $\phi(0)>0$; $\phi\in C([-\tau,0],R_+)$; $u(0)=u_0$. It is not difficult to see that solutions of (1.3) are defined for all $t>0$ and also satisfy $n(t)>0, u(t)>0$ for $t>0$. And system (1.3) has unique positive equilibrium $(n^*,u^*)=(\frac{aK}{a+Kbc},\frac{bK}{a+Kbc})$. Then by the linear chain trick technique [12], system (1.3) can be transformed into the following equivalent system

$$\begin{aligned}\frac{dx(t)}{dt} &= -ax(t)+bn^*S(t),\\ \frac{dS(t)}{dt} &= -crx(t)-\frac{rn^*}{K}S(t-\tau),\end{aligned}\tag{1.4}$$

where $r,K,c,a,b \in (0,+\infty)$, $\tau \in [0,+\infty)$. The author obtained when the condition (H) $\frac{bc}{a}>\frac{1}{K}$ and $a>(1+\sqrt{2})r$ hold, the positive equilibrium (n^*,u^*) of (1.3) is linearly asymptotically stable irrespective of the size of the delay τ. It concludes that a delay induced switching from stability to instability cannot take place; that is, an appropriate indirect feedback control can be used to avoid the occurrence of a Hopf-type bifurcation. We are interested in the effect of delay τ on dynamics of system (1.3) when the condition (H) is not satisfied. Taking the delay τ as a parameter, we show that the stability switches and a Hopf bifurcation occurs when the delay τ passes through a critical value.

The organization of this paper is as follows. In Section 2, we study the stability and the Hopf bifurcation of system (1.3). In the next section, by the normal form method and the center manifold theory introduced by Hassard et al. [14], the direction of Hopf bifurcation and the stability of bifurcating periodic solutions are determined. The main results are illustrated by examples with numerical simulations in the last section.

2. Local stability and Hopf bifurcation

The characteristic equation associated with nonlinear system (1.4) is

$$\lambda^2+\lambda(a+\frac{r}{K}n^*e^{-\lambda\tau})+\frac{arn^*}{K}e^{-\lambda\tau}+bcrn^*=0 \tag{2.1}$$

Case 1. If $\tau=0$, the equation (2.1) becomes

$$\lambda^2+\lambda(a+\frac{r}{K}n^*)+(\frac{ar}{K}+bcr)n^*=0$$

whose roots have negative real parts. Thus the trivial solution of the linear of system (1.3) is asymptotically stable when $\tau=0$ [15].

Case 2. If $\tau>0$, we assume $\lambda=i\omega$ $(\omega>0)$ is a purely imaginary root of (2.1), then we can obtained

$$-\omega^2+\frac{\omega rn^*}{K}sin\omega\tau+\frac{arn^*}{K}cos\omega\tau+bcrn^*+i(\omega a+\frac{\omega rn^*}{K}cos\omega\tau-\frac{arn^*}{K}sin\omega\tau)=0. \tag{2.2}$$

Separating the real and imaginary parts of (2.2),

$$\begin{aligned} -bcrn^*+\omega^2 &= \frac{\omega rn^*}{K}sin\omega\tau+\frac{arn^*}{K}cos\omega\tau, \\ -\omega a &= \frac{\omega rn^*}{K}cos\omega\tau-\frac{arn^*}{K}sin\omega\tau. \end{aligned} \tag{2.3}$$

then

$$cos\omega\tau = \frac{-abcK}{\omega^2+a^2}, \quad sin\omega\tau = \frac{K\omega(\omega^2+a^2-bcrn^*)}{(\omega^2+a^2)rn^*}. \tag{2.4}$$

Since $sin^2\omega\tau + cos^2\omega\tau = 1$, therefore,

$$\omega^4 + w^2[a^2 - 2bcrn^* - (\frac{rn^*}{K})^2] + (bcrn^*)^2 - (\frac{arn^*}{K})^2 = 0. \tag{2.5}$$

We know that $n^* = \dfrac{aK}{a+Kbc}$, so

$$n^* < K, \quad bcn^* < a. \tag{2.6}$$

Let $F(M) = M^2 + M[a^2 - 2bcrn^* - (\frac{rn^*}{K})^2] + (bcrn^*)^2 - (\frac{arn^*}{K})^2$, then it follows from (2.5) that F(M)=0, where $M = \omega^2$.

(1) If the condition $\dfrac{bc}{a} < \dfrac{1}{K}$ holds, then $(bcrn^*)^2 - (\frac{arn^*}{K})^2 < 0$ is satisfied. So, the equation (2.5) has a solution $\omega_0 > 0$, since the first equation of the (2.4),

$$\tau_n = \frac{1}{\omega_0} arccos \frac{-abcK}{\omega_0^2+a^2} + \frac{2n\pi}{\omega_0}, \ n = 0, 1, 2, 3, ...$$

then, $\pm i\omega_0$ is the purely imaginary root of (2.1), according to Rorche theorem, the positive equilibrium of (1.3) will be locally asymptotically stability at $\tau \in [0, \tau_0)$.

We compute $d\lambda/d\tau$ from equation (2.1),

$$\frac{d\lambda}{d\tau} = \frac{(\lambda+a)\lambda rn^* e^{-\lambda\tau}}{(2\lambda+a)K + rn^*(1-\lambda\tau - a\tau)e^{-\lambda\tau}}$$

The sign of the real part of $\dfrac{d\lambda}{d\tau}$ and $(\dfrac{d\lambda}{d\tau})^{-1}$ are identical. So,

$$\begin{aligned}(\frac{d\lambda}{d\tau})^{-1} &= \frac{(2\lambda+a)K + rn^*(1-\lambda\tau - a\tau)e^{-\lambda\tau}}{(\lambda+a)\lambda rn^* e^{-\lambda\tau}} \\ &= \frac{(2\lambda+a)K + rn^* e^{-\lambda\tau}}{(\lambda+a)\lambda rn^* e^{-\lambda\tau}} - \frac{\tau}{\lambda}\end{aligned}$$

When $\lambda = i\omega$,

$$\begin{aligned}
(\frac{d\lambda}{d\tau})^{-1} &= \frac{(2i\omega+a)K+rn^*(cos\omega_0\tau - isin\omega_0\tau)}{i\omega_0(i\omega_0+a)rn^*(cos\omega_0\tau - isin\omega_0\tau)} - \frac{\tau}{i\omega_0} \\
&= \frac{aK+rn^*cos\omega_0\tau + i(2K\omega_0 - rn^*sin\omega_0\tau)}{-rn^*\omega_0^2cos\omega_0\tau + irn^*\omega_0^2sin\omega_0\tau + irn^*a\omega_0cos\omega_0\tau + rn^*a\omega_0sin\omega_0\tau} - \frac{\tau}{i\omega_0} \\
&= \frac{M+iN}{C+iD} - \frac{\tau}{i\omega_0} \\
&= \frac{MC+ND}{C^2+D^2} - i\frac{MD-NC}{C^2+D^2} - \frac{\tau}{i\omega_0},
\end{aligned}$$

where

$$M = aK + rn^*cos\omega_0\tau,\ N = 2K\omega_0 - rn^*sin\omega_0\tau$$

and

$$C = rn^*a\omega_0sin\omega_0\tau - rn^*\omega_0^2cos\omega_0\tau,\ D = rn^*a\omega_0cos\omega_0\tau + rn^*\omega_0^2sin\omega_0\tau.$$

Therefore, it follows from (2.4) that

$$\begin{aligned}
MC+ND &= a^2rK\omega_0n^*sin\omega_0\tau + Karn^*\omega_0^2cos\omega_0\tau + 2K\omega_0^3rn^*sin\omega_0\tau - r^2(n^*)^2\omega_0^2 \\
&= Karn^*\omega_0^2cos\omega_0\tau + Kr\omega_0n^*(a^2+2\omega_0^2)sin\omega_0\tau - r^2(n^*)^2\omega_0^2 \\
&= Karn^*\frac{-Kabc}{\omega_0^2+a^2} + Kr\omega_0n^*(a^2+2\omega_0^2)\frac{K\omega_0(\omega_0^2+a^2-bcrn^*)}{(\omega_0^2+a^2)rn^*} - r^2(n^*)^2\omega_0^2 \\
&= \frac{K^2\omega_0^2[2\omega_0^2(a^2+\omega_0^2) + \big(a^2-2bcrn^*-(\frac{rn^*}{K})^2\big)(a^2+\omega_0^2)]}{\omega_0^2+a^2} \\
&= K^2\omega_0^2\big(2\omega_0^2+a^2-2bcrn^*-(\frac{rn^*}{K})^2\big).
\end{aligned}$$

Since $2\omega_0^2 = 2bcrn^* + (\frac{rn^*}{K})^2 - a^2 + \sqrt{\triangle}$, where, $\triangle = [a^2-2bcrn^*-(\frac{rn^*}{K})^2]^2 - 4\big((bcrn^*)^2 - (\frac{arn^*}{K})^2\big)$, we find that

$$MC+ND = K^2\omega_0^2\sqrt{\triangle} > 0.$$

Thus,

$$sgn\Big\{[Re(\frac{d\lambda}{d\tau})]_{\lambda=i\omega_0}\Big\} = sgn\Big\{[Re(\frac{d\lambda}{d\tau})^{-1}]_{\lambda=i\omega_0}\Big\} = sgn\Big\{\frac{MC+ND}{C^2+D^2}\Big\} > 0.$$

It follows from Transversal condition that the positive equilibrium of (1.3) occur Hopf bifurcation when $\tau = \tau_0$. Consequently, the positive equilibrium of (1.3) will be locally asymptotically stability at $\tau \in [0, \tau_0)$, and will occur Hopf bifurcation when $\tau = \tau_n, n = 0, 1, 2....$

(2) If $\frac{bc}{a} > \frac{1}{K}$ and $\frac{r}{a+Kbc} > \sqrt{2}-1$ are satisfied, we can obtain $a^2 - 2bcrn^* - (\frac{rn^*}{K})^2 < 0$ and $(bcrn^*)^2 - (\frac{arn^*}{K})^2 > 0$. Then, the equation (2.1) has two imaginary solutions, $\lambda_\pm = i\omega_\pm$, with $\omega_+ > \omega_- > 0$ in this case. It follows from (1) that

$$2\omega_\pm^2 = 2bcrn^* + (\frac{rn^*}{K})^2 - a^2 \pm \sqrt{\triangle},$$

where $\triangle = [a^2 - 2bcrn^* - (\frac{rn^*}{K})^2]^2 - 4\big((bcrn^*)^2 - (\frac{arn^*}{K})^2\big)$. It is obvious that all purely imaginary roots are simple (unless a=r=0).

The quantity of interest is again the sign of the derivative of $Re\lambda$ with respect to τ at the points where λ is purely imaginary. Similar to (1),

$$sgn\Big\{[Re(\frac{d\lambda}{d\tau})]_{\lambda=i\omega_+}\Big\} = sgn\Big\{[Re(\frac{d\lambda}{d\tau})^{-1}]_{\lambda=i\omega_+}\Big\} = sgn\Big\{\frac{K^2\omega_0^2\sqrt{\triangle}}{C^2+D^2}\Big\} > 0,$$

$$sgn\Big\{[Re(\frac{d\lambda}{d\tau})]_{\lambda=i\omega_-}\Big\} = sgn\Big\{[Re(\frac{d\lambda}{d\tau})^{-1}]_{\lambda=i\omega_-}\Big\} = sgn\Big\{\frac{-K^2\omega_0^2\sqrt{\triangle}}{C^2+D^2}\Big\} < 0.$$

Therefore, crossing of the imaginary axis from left to right with increasing τ occurs whenever τ assumes a value corresponding to ω_+, and crossing from right to left occurs for values of the τ corresponding to ω_-. Using the first equation of the (2.4), then

$$\begin{aligned} \tau_n^+ &= \frac{1}{\omega_+} arccos \frac{-abcK}{\omega_+^2 + a^2} + \frac{2n\pi}{\omega_+} \\ \tau_n^- &= \frac{1}{\omega_-} arccos \frac{-abcK}{\omega_-^2 + a^2} + \frac{2n\pi}{\omega_-} \quad (n = 0, 1, 2, ...) \end{aligned}.$$

Since the zero solution of system (1.3) is stable for $\tau = 0$, it is obvious that $\tau_0^+ < \tau_0^-$. If $0 < \tau_0^+ < \tau_0^- < \tau_1^+$, since $\tau_{n+1}^+ - \tau_n^+ = \frac{2\pi}{\omega_+} < \frac{2\pi}{\omega_-} = \tau_{n+1}^- - \tau_n^-$, there can be only a finite number of switches between stability and instability, while eventually it becomes unstable. If $0 < \tau_0^+ < \tau_1^+ < \tau_0^-$, it's obviously that there have two roots with positive real parts at $\tau \in [\tau_0^+, \tau_1^+]$, so the zero solution of system (1.3) is unstable eventually.

Then we have the follow results.

Theorem 2.1. *Assume that condition* $\frac{bc}{a} < \frac{1}{K}$ *hold, then the unique interior positive equilibrium* $M^* = (n^*, u^*)$ *of system (1.3) is locally asymptotically stable for* $0 \le \tau < \tau_0$ *and unstable for* $\tau > \tau_0$. *Furthermore, system (1.3) undergoes Hopf bifurcation at* $M^* = (n^*, u^*)$ *when* $\tau = \tau_n, n = 0, 1, 2, \ldots$.

Theorem 2.2. *Assume that condition* $\dfrac{bc}{a} > \dfrac{1}{K}$ *and* $\dfrac{r}{a+Kbc} > \sqrt{2}-1$ *hold, then the stability of the unique interior positive equilibrium* $P^* = (n^*, u^*)$ *of system (1.3) can change a finite number of times, at most, as* τ *is increased, and eventually it becomes unstable.*

3. Direction and the stability of Hopf bifurcation

In this section we study the direction of the Hopf bifurcation and the stability of the bifurcation periodic solutions when $\tau = \tau_j$ under the condition $\dfrac{bc}{a} < \dfrac{1}{K}$ and using techniques from normal form and center manifold theory by Hassard et al.

Let $x_1(t) = n(\tau t) - n^*$, $x_2(t) = u(\tau t) - u^*$, $\tau = \tau_j + \mu$. Then system (1.3) can be written as a functional differential equation in $C = C([-1,0], R^2)$.

$$x^{'}(t) = L_\mu(x_t) + f(\mu, x_t), \tag{3.1}$$

where $x(t) = (x_1(t), x_2(t))^T \in R^2$ and $L_\mu : C \to R^2, f : R \times C \to R^2$ are given, respectively, by

$$L_\mu(\phi) = (\tau_j+\mu)\begin{pmatrix} 0 & -crn^* \\ b & -a \end{pmatrix}\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} + (\tau_j+\mu)\begin{pmatrix} -\dfrac{rn^*}{K} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix} \tag{3.2}$$

and

$$f(\mu, \phi) = (\tau_j+\mu)\begin{pmatrix} Q \\ 0 \end{pmatrix}, \tag{3.3}$$

where, $Q = -\dfrac{r}{K}\phi_1(0)\phi_1(-1) - cr\phi_1(0)\phi_2(0)$, $\phi = (\phi_1, \phi_2) \in C$. By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1,0]$ such that

$$L_\mu(\theta) = \int_{-1}^{0} \mathrm{d}\eta(\theta,\mu)\phi(\theta), \phi \in C. \tag{3.4}$$

In fact, we can choose

$$\eta(\theta,\mu) = (\tau_j+\mu)\begin{pmatrix} 0 & -crn^* \\ b & -a \end{pmatrix}\delta(\theta) - (\tau_j+\mu)\begin{pmatrix} -\dfrac{rn^*}{K} & 0 \\ 0 & 0 \end{pmatrix}\delta(\theta+1), \tag{3.5}$$

where $\delta(\theta)=\begin{cases} 0,\ \theta\neq 0, \\ 1,\ \theta=0. \end{cases}$. That (3.2) is satisfied. For $\phi\in C^1([-1,0],R^2)$, define

$$A(\mu)\phi=\begin{cases} \dfrac{d\phi(\theta)}{d\theta},\ \theta\in[-1,0), \\ \int_{-1}^{0}\mathrm{d}\eta(s,\mu)\phi(s),\ \theta=0 \end{cases}$$

and

$$R(\mu)\phi=\begin{cases} 0,\ \theta\in[-1,0), \\ f(\mu,\phi),\ \theta=0. \end{cases}$$

Then system (3.1) is equivalent to

$$x_t^{'}=A(\mu)x_t+R(\mu)x_t, \tag{3.6}$$

where $x_t(\theta)=x(t+\theta)$ for $\theta\in[-1,0]$. For $\psi\in C^1([-1,0],(R^2)^*)$, define

$$A^*\psi(s)=\begin{cases} -\dfrac{d\psi(\theta)}{d\theta},\ s\in(0,1], \\ \int_{-1}^{0}\mathrm{d}\eta(s,\mu)\phi(s),\ s=0 \end{cases}$$

and a bilinear inner product

$$<\psi(s),\phi(\theta)>=\bar{\psi}(0)\phi(0)-\int_{-1}^{0}\int_{\xi=0}^{\theta}\bar{\psi}(\xi-\theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi, \tag{3.7}$$

where $\eta(\theta)=\eta(\theta,0)$. Then $A(0)$ and A^* are adjoint operators. By the discussions in Section 2, we know that $\pm i\omega_0\tau_j$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. We first need to compute the eigenvector of $A(0)$ and A^* corresponding to $i\omega_0\tau_j$ and $-i\omega_0\tau_j$, respectively.

Suppose that $q(\theta)=(1,q_1)^Te^{i\omega_0\tau_j\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau_j$, then $A(0)q(\theta)=i\omega_0\tau_jq(\theta)$. It follows from the definition of $A(0)$, (3.4) and (3.5) that

$$\tau_j\begin{pmatrix} i\omega_0+\dfrac{rn^*}{K}e^{-i\omega_0\tau_j} & crn^* \\ -b & i\omega_0+a \end{pmatrix}q(0)=0.$$

Thus we can easily get

$$q(0)=(1,q_1)^T=(1,\frac{b}{i\omega_0+a})^T.$$

On the other hand, suppose that $q^*(s) = D(1, q_1^*)e^{i\omega_0\tau_j s}$ is the eigenvector of A^* corresponding to $-i\omega_0\tau_j$. By the definition of A^*, (3.4) and (3.5), we obtain

$$\tau_j \begin{pmatrix} i\omega_0 - \dfrac{rn^*}{K}e^{i\omega_0\tau_j} & b \\ -crn^* & i\omega_0 - a \end{pmatrix} q^*(0)^T = 0,$$

which implies

$$q^*(0) = D(1, q_1^*) = D(1, \frac{rn^*e^{i\omega_0\tau_j} - iK\omega_0}{bK}).$$

In order to assure $< q^*(s), q(\theta) >= 1$, we need to determine the value of the D. From (3.7), we have

$$\begin{aligned} < q^*(s), q(\theta) > &= \bar{q}^*(0)q(0) - \textstyle\int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{q}^*(\xi-\theta)\mathrm{d}\eta(\theta)q(\xi)\mathrm{d}\xi \\ &= \bar{D}(1, \bar{q}_1^*)(1, q_1)^T - \textstyle\int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{q}_1^*)e^{-i\omega_0\tau_j(\xi-\theta)}\mathrm{d}\eta(\theta)(1, q_1)^T e^{i\omega_0\tau_j\xi}\mathrm{d}\xi \\ &= \bar{D}\big(1 + \bar{q}_1^* q_1 - \textstyle\int_{-1}^{0}(1, \bar{q}_1^*)\theta e^{i\omega_0\tau_j\theta}\mathrm{d}\eta(\theta)(1, q_1)^T\big) \\ &= \bar{D}\big(1 + \bar{q}_1^* q_1 + \tau_j e^{-i\omega_0\tau_j}(1, \bar{q}_1^*)\begin{pmatrix} -\dfrac{rn^*}{K} & 0 \\ 0 & 0 \end{pmatrix}(1, q_1)^T\big) \\ &= \bar{D}\big(1 + \bar{q}_1^* q_1 - \frac{rn^*}{K}\tau_j e^{-i\omega_0\tau_j}\big). \end{aligned}$$

Thus, we can obtain

$$D = \frac{1}{1 + q_1^*\bar{q}_1 - \dfrac{rn^*}{K}\tau_j e^{i\omega_0\tau_j}}.$$

In the following, we use the ideas in Hassard *et al.* [14] to compute the coordinates describing center manifold C_0 at $\mu = 0$. Define

$$z(t) =< q^*, x_t >, \; W(t, \theta) = x_t - 2Re z(t)q(\theta). \tag{3.8}$$

On the center manifold C_0 we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots,$$

where z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if x_t is real. We consider only real solutions. For the solution $x_t \in C_0$ of (3.6), since $\mu=0$, we have

$$\begin{aligned} z^{'}(t) &= i\omega_0\tau_j z+\bar{q}^*(0)f(0,W(z(t),\bar{z}(t),\theta)+2Rezq(0)) \\ &\overset{def}{=} i\omega_0\tau_j z+\bar{q}^*(0)f_0(z,\bar{z}) \\ &= i\omega_0\tau_j z+g(z,\bar{z}), \end{aligned}$$

where

$$g(z,\bar{z})=\bar{q}^*(0)f_0(z,\bar{z})=g_{20}\frac{z^2}{2}+g_{11}z\bar{z}+g_{02}\frac{\bar{z}^2}{2}+g_{21}\frac{z^2\bar{z}}{2}\cdots \tag{3.9}$$

Notice that $x_t(\theta)=(x_{1t}(\theta),x_{2t}(\theta))=W(t,\theta)+zq(\theta)+\bar{z}\bar{q}(\theta)$ and $q(\theta)=(1,q_1)^Te^{i\omega_0\tau_j\theta}$, then we obtain

$$\begin{aligned} &x_{1t}(0)=W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\bar{z}+W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}+z+\bar{z}+O(|(z,\bar{z})|^3) \\ &x_{1t}(-1)=W_{20}^{(1)}(-1)\frac{z^2}{2}+W_{11}^{(1)}(-1)z\bar{z}+W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2}+e^{-i\omega_0\tau_j}z+e^{i\omega_0\tau_j}\bar{z}+O(|(z,\bar{z})|^3) \\ &x_{2t}(0)=W_{20}^{(2)}(0)\frac{z^2}{2}+W_{11}^{(2)}(0)z\bar{z}+W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}+zq_1+\bar{z}\bar{q}_1+O(|(z,\bar{z})|^3). \end{aligned}$$

On the other hand, in terms of the definition of $f(\mu,x_t)$, we have

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^*(0)f_0(z,\bar{z}) \\ &= \bar{D}\tau_j\Big\{(-\frac{2r}{K}q_1e^{-i\omega_0\tau_j}-2crq_1)\frac{z^2}{2} \\ &+\Big[-\frac{r}{K}(\bar{q}_1e^{i\omega_0\tau_j}+q_1e^{-i\omega_0\tau_j})-cr(\bar{q}_1+q_1)\Big]z\bar{z}+(-\frac{2r}{K}\bar{q}_1e^{i\omega_0\tau_j}-2cr\bar{q}_1)\frac{\bar{z}^2}{2} \\ &+\Big[-\frac{r}{K}\big(W_{20}^{(1)}(0)\bar{q}_1e^{i\omega_0\tau_j}+2W_{11}^{(1)}(0)q_1e^{-i\omega_0\tau_j}+2W_{11}^{(1)}(-1)+W_{20}^{(1)}(-1)\big) \\ &-cr\big(W_{20}^{(1)}(0)\bar{q}_1+2W_{11}^{(1)}(0)q_1+2W_{11}^{(2)}(0)+W_{20}^{(2)}(0)\big)\Big]\frac{z^2\bar{z}}{2}+\cdots\Big\}. \end{aligned}$$

Comparing the coefficients with (3.9), we have

$$\begin{aligned} g_{20} &= -2(\frac{r}{K}e^{-i\omega_0\tau_j}+crq_1)\bar{D}\tau_j \\ g_{11} &= -\big[\frac{r}{K}(e^{i\omega_0\tau_j}+e^{-i\omega_0\tau_j})+cr(\bar{q}_1+q_1)\big]\bar{D}\tau_j \\ g_{02} &= -2(\frac{r}{K}e^{i\omega_0\tau_j}+cr\bar{q}_1)\bar{D}\tau_j \\ g_{21} &= \big[W_{20}^{(1)}(0)(-\frac{r}{K}e^{i\omega_0\tau_j}-cr\bar{q}_1)+W_{11}^{(1)}(0)(-\frac{2r}{K}e^{-i\omega_0\tau_j}-2crq_1) \\ &\quad -\frac{2r}{K}W_{11}^{(1)}(-1)-\frac{r}{K}W_{20}^{(1)}(-1)-2crW_{11}^{(2)}(0)-crW_{20}^{(2)}(0)\big]\bar{D}\tau_j. \end{aligned} \tag{3.10}$$

Since $W_{20}(\theta)$ and $W_{11}(\theta)$ for $\theta \in [-1,0]$ appear in g_{21}, we still need to compute them. From (3.6) and (3.8), we have

$$
\begin{aligned}
W' &= x_t' - 2Re z'(t)q(\theta) \\
&= A(0)x_t + R(0)x_t - 2Re\{[i\omega_0\tau_j z + \bar{q}_0^* f_0(z,\bar{z})]q(\theta)\} \\
&= A(0)W(t,\theta) + 2Re\{z(t)A(0)q(\theta)\} + R(0)x_t - 2Re\{i\omega_0\tau_j q(\theta)z(t)\} \\
&\quad -2Re\{\bar{q}_0^* f_0(z,\bar{z})q(\theta)\} \\
&= A(0)W(t,\theta) + R(0)x_t - 2Re\{\bar{q}_0^* f_0(z,\bar{z})q(\theta)\} \\
&= \begin{cases} A(0)W(t,\theta) - 2Re\{\bar{q}_0^* f_0(z,\bar{z})q(\theta)\}, \ \theta \in [-1,0) \\ A(0)W(t,\theta) - 2Re\{\bar{q}_0^* f_0(z,\bar{z})q(\theta)\} + f_0, \ \theta = 0 \end{cases} \\
&\overset{def}{=} A(0)W + H(z,\bar{z},\theta),
\end{aligned}
\tag{3.11}
$$

where

$$
H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}\frac{\bar{z}^2}{2} + \cdots \tag{3.12}
$$

Note that on the center manifold C_0 near the origin

$$
\begin{aligned}
W_t' &= W_z z_t' + W_{\bar{z}}\bar{z}_t' \\
&= (W_{20}(\theta)z + W_{11}(\theta)\bar{z} + \cdots)(i\omega_0\tau_j z + g(z,\bar{z})) \\
&\quad +(W_{11}(\theta)z + W_{02}(\theta)\bar{z} + \cdots)(-i\omega_0\tau_j\bar{z} + \bar{g}(z,\bar{z})) \\
&= i\omega_0\tau_j W_{20}(\theta)z^2 - i\omega_0\tau_j W_{02}(\theta)\bar{z}^2 + \cdots
\end{aligned}
\tag{3.13}
$$

and

$$
A(0)W(t,\theta) = A(0)W_{20}(\theta)\frac{z^2}{2} + A(0)W_{11}(\theta)z\bar{z} + A(0)W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{3.14}
$$

It follows from (3.12)-(3.14), we can get

$$
\begin{aligned}
&2i\omega_0\tau_j W_{20}(\theta) - A(0)W_{20}(\theta) = H_{20}(\theta) \\
&-A(0)W_{11}(\theta) = H_{11}(\theta).
\end{aligned}
\tag{3.15}
$$

From (3.11), for $\theta \in [-1,0)$

$$
H(z,\bar{z},\theta) = -\bar{q}_0^* f_0(z,\bar{z})q(\theta) - q_0^*\bar{f}_0(z,\bar{z})\bar{q}(\theta).
$$

Comparing the coefficients with (3.12) yields

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) &= -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{aligned} \tag{3.16}$$

It follows from (3.15),(3.16) and the definition of A that

$$W_{20}'(\theta) = 2i\omega_0\tau_j W_{20} + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Note that $q(\theta) = q(0)e^{i\omega_0\tau_j\theta}$, then

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_j}q(0)e^{i\omega_0\tau_j\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_j}\bar{q}(0)e^{-i\omega_0\tau_j\theta} + E_1 e^{2i\omega_0\tau_j\theta}, \tag{3.17}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}) \in R^2$ is a constant vector. In the sequel, the appropriate E_1 will be determined. Similarly, from (3.15) and (3.16), we obtain

$$W_{11}'(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta).$$

Then

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_j}q(0)e^{i\omega_0\tau_j\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_j}\bar{q}(0)e^{-i\omega_0\tau_j\theta} + E_2, \tag{3.18}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}) \in R^2$ is a constant vector. In what follows, we shall seek appropriate E_1 and E_2, respectively. By the definition of $A(0)$ and (3.15), we can obtain

$$\begin{aligned} &\textstyle\int_{-1}^{0} \mathrm{d}\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_j W_{20}(\theta) - H_{20}(\theta), \\ &\textstyle\int_{-1}^{0} \mathrm{d}\eta(\theta)W_{11}(\theta) = -H_{11}(0)(\theta), \end{aligned} \tag{3.19}$$

where $\eta(\theta) = \eta(0,\theta)$. It follows from (3.11) and (3.12) that

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) - 2\tau_j \begin{pmatrix} \frac{r}{K}e^{-i\omega_0\tau_j} + crq_1 \\ 0 \end{pmatrix} \\ H_{11}(0) &= -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) - \tau_j \begin{pmatrix} \frac{r}{K}(e^{i\omega_0\tau_j} + e^{-i\omega_0\tau_j}) + cr(\bar{q}_1 + q_1) \\ 0 \end{pmatrix}. \end{aligned} \tag{3.20}$$

Note that $q(\theta)$ is the eigenvector of $A(0)$ and from (3.17) and the definition of $A(0)$, we know that

$$\begin{aligned}\int_{-1}^{0} \mathrm{d}\eta(\theta)W_{20}(\theta) &= \frac{ig_{20}}{\omega_0\tau_j}\int_{-1}^{0}\mathrm{d}\eta(\theta)q(\theta)+\frac{i\bar{g}_{02}}{3\omega_0\tau_j}\int_{-1}^{0}\mathrm{d}\eta(\theta)\bar{q}(\theta)+E_1\int_{-1}^{0}\mathrm{d}\eta(\theta)e^{2i\omega_0\tau_j\theta}\\ &= \frac{ig_{20}}{\omega_0\tau_j}A(0)q(0)+\frac{i\bar{g}_{02}}{3\omega_0\tau_j}A(0)\bar{q}(0)+E_1\int_{-1}^{0}\mathrm{d}\eta(\theta)e^{2i\omega_0\tau_j\theta}\\ &= -g_{20}q(0)+\frac{1}{3}\bar{g}_{02}\bar{q}(0)+E_1\int_{-1}^{0}\mathrm{d}\eta(\theta)e^{2i\omega_0\tau_j\theta}\end{aligned} \tag{3.21}$$

and

$$2i\omega_0\tau_jW_{20}(\theta)=-2g_{20}q(0)-\frac{2}{3}\bar{g}_{02}\bar{q}(0)+2i\omega_0\tau_jE_1. \tag{3.22}$$

Then, we can obtain

$$\left(\int_{-1}^{0}\mathrm{d}\eta(\theta)e^{2i\omega_0\tau_j\theta}-2i\omega_0\tau_jI\right)E_1=2\tau_j\begin{pmatrix}\frac{r}{K}e^{-i\omega_0\tau_j}+crq_1\\ 0\end{pmatrix},$$

which yields

$$\begin{pmatrix}-2i\omega_0-\frac{rn^*}{K}e^{-2i\omega_0\tau_j} & -crn^*\\ b & -2i\omega_0-a\end{pmatrix}E_1=2\begin{pmatrix}\frac{r}{K}e^{-i\omega_0\tau_j}+crq_1\\ 0\end{pmatrix}.$$

It follows that

$$E_1=\frac{\frac{r}{K}e^{-i\omega_0\tau_j}+crq_1}{B_1}\begin{pmatrix}-a-2i\omega_0\\ -b\end{pmatrix},$$

where

$$B_1=\begin{vmatrix}-2i\omega_0-\frac{rn^*}{K}e^{-2i\omega_0\tau_j} & -crn^*\\ b & -2i\omega_0-a\end{vmatrix}.$$

Similarly, from (3.20)-(3.22), we get E_2 defined by

$$E_2=\frac{\frac{r}{K}(e^{i\omega_0\tau_j}+e^{-i\omega_0\tau_j})+cr(\bar{q}_1+q_1)}{B_2}\begin{pmatrix}-a\\ -b\end{pmatrix},$$

where

$$B_2=\begin{vmatrix}-\frac{rn^*}{K} & -crn^*\\ b & -a\end{vmatrix}.$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (3.17) and (3.18). Furthermore, g_{21} can be expressed explicitly. Thus, we can compute the following values:

$$\begin{aligned}
c_1(0) &= \frac{i}{2\omega_0\tau_j}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\
\mu_2 &= -\frac{Re(c_1(0))}{Re_{\lambda'(\tau_j)}}, \\
\beta_2 &= 2Re(c_1(0)), \\
T_2 &= -\frac{Im(c_1(0)) + \mu_2 Im(\lambda'(\tau_j))}{\omega_0\tau_j},
\end{aligned} \tag{3.23}$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value τ_j. Particularly, μ_2 determines the directions of the Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_0(\tau < \tau_0)$; β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0(\beta_2 > 0)$; and T_2 determines the period of the bifurcating periodic solutions: the period increase (decrease) if $T_2 > 0(T_2 < 0)$.

4. Examples

Example 1. We consider the following system

$$\begin{aligned}
\frac{dn(t)}{dt} &= n(t)[1 - n(t-\tau) - \frac{1}{2}u(t)], \\
\frac{du(t)}{dt} &= -2u(t) + 2n(t).
\end{aligned}$$

Thus, the coefficient of this system satisfy the condition $\frac{bc}{a} < \frac{1}{K}$ and $n^* = u^* = \frac{2}{3}$, when $\tau = 2.8 < \tau_0 = 2.8991$, the unique interior equilibrium of system (1.3) will be locally asymptotically stable. while when $\tau = 2.9 > \tau_0 = 2.8991$, the unique interior equilibrium M^* losses its stability and a Hopf bifurcation occurs. The periodic oscillations bifurcating from M^* are depicted.

Example 2. We consider the following system

$$\begin{aligned}
\frac{dn(t)}{dt} &= 4n(t)[1 - n(t-\tau) - 3u(t)], \\
\frac{du(t)}{dt} &= -2u(t) + n(t).
\end{aligned}$$

Thus, the coefficient of this system satisfy the condition of theorem 2.1 and $n^* = \frac{2}{5}, u^* = \frac{1}{5}$, when $\tau = 0.5 < \tau_0^+ = 0.89964225684$, the unique interior equilibrium of system (1.3) will be locally asymptotically stable. when $\tau_0^+ = 0.89964225684 < \tau = 1.5 < \tau_0^- = 2.02269358533$, the unique interior equilibrium P^* losses its stability and a Hopf bifurcation occurs, the periodic oscillations bifurcating from P^* are depicted. while when $\tau_0^- = 2.02269358533 < \tau = 2.5 < \tau_1^+ = 3.4559750717$, the unique interior equilibrium P^* of system (1.3) will be locally asymptotically stable. Due to $\tau_1^+ = 3.4559750717 < \tau_2^+ = 6.012307886569 < \tau_1^- = 6.33924936076$, we can know that When $\tau > \tau_1^+ = 3.4559750717$, the unique interior equilibrium P^* will be unstable.

5. Conclusion

Gopalsamy [12] illustrated that an appropriate indirect feedback control can be used to avoid the occurrence of a Hopf bifurcation, while our works show that with the same range of the feedback control variable, i.e, the condition $\frac{bc}{a} > \frac{1}{K}$ holds, the system (1.3) will occur the stability switches when $\frac{r}{a+Kbc} > \sqrt{2}-1$ holds (Theorem 2.2). And we also illustrate that if the condition $\frac{bc}{a} < \frac{1}{K}$ holds, the system (1.3) undergoes Hopf bifurcation at positive equilibrium $M^* = (n^*, u^*)$ when $\tau = \tau_n, n = 0, 1, 2, \ldots$ (Theorem 2.1). And then, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying normal form theory and the center manifold theorem.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

This work was supported by the Natural Science Foundation of Fujian Province (2013J01011, 2013J01010) and the Foundation of Fujian Edication Bureau (JA13361).

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