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COMPOSITE REFINEMENT TECHNIQUES FOR SOLVING LINEAR SYSTEMS

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Abstract. A composite refinement approach for stationary iterative methods is introduced. Two new formulas (RJGS and RGSJ) are compared with the classical forms. Rates of convergence of the introduced composite formulas (RJGS and RGSJ) are well established. The efficient performance of the new forms is established theoretically and confirmed through numerical examples. The decrease in the required number of iterations for convergence is established through the calculation of the spectral radius of the iteration matrices. The algorithmic structure of the new formulas is announced. Three numerical examples with different convergent properties are considered. The calculations are performed with the help of computer algebra software Mathematica.

Keywords: Jacobi; Gauss-Seidel; RJGS; RGSJ method; refinement techniques.

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1. Introduction

The developments and efficiency of computer algebra systems like Mathematica, Maple or MATLAB have great consequences in computational mathematical subjects. We have used Mathematica in the developments of iterative techniques for solving large linear systems of

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algebraic equations. As we will see from the spectral radius values below a clear decision about the rate of convergence can be seen easily from the spectral analysis of the iteration matrices. The question of solving a large system of algebraic equations is a fundamental question in most modern modeling issues, [1, 2, 3, 4]. Any linear system of algebraic equations can be written in matrix form as:

$$Ax = b, \tag{1}$$

where, $A \in R^{nxn}$ is a coefficient nonsingular matrix (in this work), $b \in R^n$ is a known column of constants and x is the unknown vector. Theoretically, $x = A^{-1}b$ is known as the exact solution of the system (1). Efficient direct methods for solving such systems requires about $(n^3/3)$ operations which is not suitable for large sparse systems, [2, 5, 6, 7] so iterative methods seem to be the appropriate choice especially when the convergence of the method up to the required accuracy is achieved within n steps. It is known that, the evaluation process in each step of an iterative technique is equivalent to a matrix vector multiplications. Usually, coefficient matrices contains many zeros (sparse matrices), in iterative methods unlike direct methods zeros do not affect the computational work (there is no fill in attitudes). It is well known that any splitting of the coefficient matrix A, $A = M - N$, with nonsingular matrix M, defines an iterative technique, [1, 2, 3, 4, 5, 6, 7]

$$Mx^{[k+1]} = Nx^{[k]} + b,\ k = 0, 1, 2, \ldots \tag{2}$$

The spectral radius of the iteration matrix $\rho(M^{-1}N)$ is taken as the measure of the rate of convergence of the iterative technique, the method with smaller spectral radius of its iteration matrix is known as asymptotically faster. Also, the splitting, $A = D - L - U$, where D is the diagonal part of the matrix A, and $-L$, $-U$ are the strictly lower and upper triangular parts of A, respectively [2, 3, 4, 7] is used in the matrix reformulation of the standard stationary iterative techniques. We are interested in this work with two of the classical iterative methods:
Jacobi method [1, 2, 4, 5, 6, 7, 8]

$$x^{[k+1]} = D^{-1}(L+U)x^{[k]} + D^{-1}b = T_J x^{[k]} + C_J. \tag{3}$$

Gauss-Seidel method [1, 2, 4, 5, 6, 7, 8]

$$x^{[k+1]} = (D-L)^{-1}Ux^{[k]} + (D-L)^{-1}b = T_{GS}x^{[k]} + C_{GS}. \tag{4}$$

2. JACOBI GAUSS-SEIDEL AND GAUSS-SEIDEL JACOBI COMPOSITE REFINEMENT

Stationary iterative techniques are characterized by the fixed construction of their iteration matrices during the evaluation process. In general, in stationary iterative methods the iteration matrix is calculated only in the first step and used in the consecutive iterations, so the computational overheads are of order at most n^2 per iteration for fully dense matrices. Refinement techniques of the same iterative methods are considered in many publications [8, 9]. For a convergent iterative method, the speed of convergence is doubled with refinement treatment when the same iterative technique is used. We introduce the composite refinement approach in which two different iterative techniques are considered consecutively. The achievement in the speed of convergence of the refinement treatments dominates the increase in computational costs appears in the first step. The basic idea in the refinement treatment is the use of a virtual step ($x^{[vir]}$) like the case of double sweep methods or the symmetric and unsymmetric techniques, [2, 7] but without reversing the ordering of the equations.

The general iterative technique (2) can be written as

$$x^{[vir]} = M^{-1}Nx^{[k]} + M^{-1}b \tag{5}$$

and this virtual calculated data is used in a subsequent iteration as

$$x^{[k+1]} = M^{-1}Nx^{[vir]} + M^{-1}b,\ k = 0,1,2,... \tag{6}$$

Which can be rearranged in the form

$$x^{[k+1]} = (M^{-1}N)^2x^{[k]} + (I + M^{-1}N)M^{-1}b,\ k = 0,1,2,... \tag{7}$$

In the composite refinement different iterative techniques in the consecutive sweeps are considered. We apply this concept on the two of the simple iterative methods Jacobi and Gauss-Seidel methods.

2.1. Jacobi Gauss-Seidel (RJGS) Composite Refinement. The iterative formulation of the RJGS method can be written in the form

$$x^{[k+1]} = T_{RJGS}x^{[k]} + C_{RJGS};$$
$$x^{[vir]} = (D-L)^{-1}Ux^{[k]} + (D-L)^{-1}b,$$

where

$$T_{RJGS} = D^{-1}(L+U)(D-L)^{-1}U,$$
$$C_{RJGS} = D^{-1}[I+(L+U)(D-L)^{-1}]b, \tag{8}$$

and this can be obtained by direct application of formulas (3) and (4)

Remark 2.1. *From (3), (4) and (8), we find:*

$$T_{RJGS} = T_J T_{GS} \, and \, C_{RJGS} = T_J C_{GS} + C_J. \tag{9}$$

Theorem 2.2. *Let A be a strictly diagonally dominant (SDD) matrix, then the RJGS method is convergent for any initial guess* $x^{[0]}$.

Proof.

Let x^* be the exact solution of linear system (1). Because the matrix A is a strictly diagonally dominant, The Jacobi and GS methods are convergent [2, 4]. If $x^{[k+1]}$ be the $(k+1)^{th}$ approximation to the solution of linear system (1) by the RJGS method in (8), then we have

$$||x^{[k+1]} - x^*|| = ||x^{[vir]} + D^{-1}(b - Ax^{[vir]}) - x^*||$$
$$\leq ||x^{[vir]} - x^*|| + ||D^{-1}||||b - Ax^{[vir]}||$$

Where, $x^{[vir]}$ is assumed to be calculated by the GS method. Thus $||x^{[k+1]} - x^*||$ converges to the zero vector due to the convergence of the GS method. Hence, the RJGS method converges to the solution of linear system (1).

2.2. Gauss-Seidel Jacobi (RGSJ) Composite Refinement. The iterative formulation of the RGSJ method can be written in the form

$$x^{[k+1]} = T_{RGSJ}x^{[k]} + C_{RGSJ};$$
$$x^{[vir]} = D^{-1}(L+U)x^{[k]} + D^{-1}b,$$

where

$$T_{RGSJ} = (D-L)^{-1}UD^{-1}(L+U),$$
$$C_{RGSJ} = (D-L)^{-1}[I+UD^{-1}]b, \tag{10}$$

and this can be obtained by direct application of formulas (3) and (4).

Remark 2.3. *From (3), (4) and (10), we find:*

$$T_{RGSJ} = T_{GS}T_J \text{ and } C_{RGSJ} = T_{GS}C_J + C_{GS}. \tag{11}$$

Theorem 2.4. *Let A be a strictly diagonally dominant (SDD) matrix, then the RGSJ method is convergent for any initial guess $x^{[0]}$.*

Proof.

Let x^* be the exact solution of linear system (1). Due to the strict diagonal dominance of the matrix A, The Jacobi and GS methods are convergent [2, 4]. If $x^{[k+1]}$ be the $(k+1)^{th}$ approximation to the solution of linear system (1) by the RGSJ method in (10), then we have

$$\begin{aligned}\|x^{[k+1]} - x^*\| &= \|x^{[vir]} + (D-L)^{-1}(b - Ax^{[vir]}) - x^*\| \\ &\leq \|x^{[vir]} - x^*\| + \|(D-L)^{-1}\|\|b - Ax^{[vir]}\|\end{aligned}$$

Where, $x^{[vir]}$ is assumed to be calculated by the Jacobi method.

Thus $||x^{[k+1]} - x^*||$ converges to the zero vector because of the convergence of the Jacobi method. Hence, the RGSJ method converges to the solution of linear system (1).

Theorem 2.5. *For any two convergent methods M_1 and M_2*

$$\begin{aligned}[min(\rho(T_{M_1}), \rho(T_{M_2}))]^2 &\leq \rho(T_{RM_1M_2}) \\ &\leq min(\rho(T_{M_1}), \rho(T_{M_2})) \\ &\leq max(\rho(T_{M_1}), \rho(T_{M_2}))\end{aligned}$$

Proof.

Let T_{M_1} be the iteration matrix (with eigenvalues λ_i) of the method M_1, Let T_{M_2} be the iteration matrix (with eigenvalues μ_i) of the method M_2 and Let $T_{RM_1M_2}$ be the iteration matrix of the composite refinement technique of the two methods (with eigenvalues β_i). We have

$$T_{RM_1M_2} = T_{M_1}T_{M_2}$$

$$det(T_{RM_1M_2}) = det(T_{M_1})det(T_{M_2})$$

$$\prod_{i=1}^{n}\beta_i = \prod_{s=1}^{n}\lambda_s\prod_{r=1}^{n}\mu_r,$$

but we have $|\lambda_s| < 1$, $|\mu_r| < 1\ \forall r,s$

$$max|\beta_i| = (max|\lambda_s|)(max|\mu_r|) \leq [max(max|\lambda_i|, max|\mu_i|)]^2$$

Similarly,

$$max|\beta_i| = (max|\lambda_s|)(max|\mu_r|) \geq [min(max|\lambda_i|, max|\mu_i|)]^2$$

Accordingly, We can write

$$\begin{aligned}[min(\rho(T_{M_1}), \rho(T_{M_2}))]^2 &\leq \rho(T_{RM_1M_2}) \\ &\leq min(\rho(T_{M_1}), \rho(T_{M_2})) \\ &\leq max(\rho(T_{M_1}), \rho(T_{M_2}))\end{aligned}$$

Algorithm 2.6. *A general refinement algorithm of two iterative methods M_1 and M_2 for solving the linear system $Ax = b$*

Step 1: Input the coefficient matrix A and the constant matrix b.

Step 2: Use the splitting form of the matrix A as $A = D - L - U$.

Step 3: Calculate the iteration matrix $T_{RM_1M_2} = T_{M_1}T_{M_2}$.

Step 4: Calculate the matrix $C_{RM_1M_2} = T_{M_1}C_{M_2} + C_{M_1}$.

Step 5: Input the initial guess $x^{[0]} = 0$.

Step 6: Loop $x^{[i]} = T_{RM_1M_2}x^{[i-1]} + C_{RM_1M_2}$ until the specified number of iterations.

3. NUMERICAL EXAMPLES

To illustrate the theoretical results, we consider three distinguished numerical examples. The first example is an important example given in [1] to illustrate the superiority of Gauss-Seidel over the Jacobi method in case of convergence, [10]. The second example appears in [4] to illustrate that the convergence of Jacobi method does not guarantee the convergence of Gauss-Seidel method. While the third example is to illustrate the theoretical results obtained in theorems 2.2 and 2.4 of the introduced composite refinement of two convergent iterative techniques.

Example 3.1. We consider the linear system of equations, [1]

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1, \\ 2x_1 + 2x_2 + 2x_3 &= 4, \\ -x_1 - x_2 + 2x_3 &= -5. \end{aligned} \tag{12}$$

with exact solution is $x_1 = 1, x_2 = 2, x_3 = -1$.

The spectral radius of the Jacobi iteration matrix is greater than one ($\rho(T_J) = 1.11803 > 1$) so Jacobi method (3) is divergent (table 1). The spectral radius of the Gauss-Seidel iteration matrix is ($\rho(T_{GS}) = 0.5 < 1$) so Gauss-Seidel method (4) is convergent (table 2). It is a surprising result that when solving this system by the RJGS and RGSJ methods the solution convergent (tables 3, 4). The spectral radius of the iteration matrix of RJGS method for the system (12) is ($\rho(T_{RJGS}) = 0.5 < 1$). Also, The spectral radius of the iteration matrix of RGSJ method for the system (12) is ($\rho(T_{RGSJ}) = 0.5 < 1$).

TABLE 1. The results of the Jacobi method to the linear system (12).

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$
0	0	0	0
1	-0.5	2	-2.5
2	1.75	5	-1.75
24	-7.73115	-32.9246	7.73115
25	-20.8279	2	-22.8279

TABLE 2. The solution of the linear system (12) by the GS method.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$
0	0	0	0
1	-0.5	2.5	-1.5
2	1.5	2	-0.75
24	0.999999	2	-1
25	1	2	-1

TABLE 3. The solution of the linear system (12) by the RJGS method.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$
0	0	0	0
1	1.5	4	-1.5
2	0.5	0.5	-0.75
3	1.375	3	-1.125
24	0.999999	2	-1
25	1	2	-1

TABLE 4. The solution of the linear system (12) by the RGSJ method.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$
0	0	0	0
1	1.75	2.75	-0.25
2	-0.125	2.375	-1.375
3	1.9375	1.4375	-0.8125
24	0.999996	2	-1
25	1	2	-1

Example 3.2. We consider the linear system of equations, [4]

$$\begin{aligned} x_1+2x_2+4x_3&=1,\\ 0.125x_1+x_2+x_3&=3, \qquad (13)\\ -x_1+4x_2+x_3&=7. \end{aligned}$$

Without rearrangements, the spectral radius of the iteration matrix of Gauss-Seidel method is $\rho(T_{GS}) = 2.53802$ so the method is divergent, (table 6) and the iteration matrix of Jacobi method is $\rho(T_J) = 0.5$, (table 5). Also, the solution of this system is divergent with the RJGS and RGSJ methods (tables 7, 8). The spectral radius of the iteration matrix of RJGS method is $\rho(T_{RJGS}) = 1$. Also, The spectral radius of the iteration matrix of RGSJ method is $\rho(T_{RGSJ}) = 1$.

TABLE 5. The solution of the linear system (13) by the Jacobi method.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$
0	0	0	0
1	1	3	7
2	-33	-4.125	-4
20	33.3331	13.8333	-15
21	33.3333	13.8333	-15

TABLE 6. The results of the GS method to the linear system (13).

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$
0	0	0	0
1	1	2.875	-3.5
2	9.25	5.34375	-5.125
20	2.03384×10^7	2.1239×10^6	1.18428×10^7
21	-5.1619×10^7	-5.390×10^6	-3.0057×10^7

TABLE 7. The results of the RJGS method to the linear system (13).

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$
0	0	0	0
1	9.25	6.375	-3.5
2	51.0625	18.3438	-15.625
20	-1.01904	4.40777	-8.30024
21	-14.7628	-0.0503403	-0.122835

TABLE 8. The results of the RGSJ method to the linear system (13).

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$
0	0	0	0
1	-33	0.125	-26.5
2	39.75	24.5313	-51.375
20	-8.79131	-6.47066	24.0913
21	-55.3805	-14.1688	8.2946

Example 3.3. We consider the linear system of equations,

$$\begin{aligned} 4x_1 - x_2 - x_3 &= 2, \\ -x_1 + 4x_2 - x_4 &= 1, \\ -x_1 + 4x_3 - x_4 &= 1, \\ -x_2 - x_3 + 4x_4 &= 6. \end{aligned} \tag{14}$$

with exact solution $x_1 = x_2 = x_3 = 1$ and $x_4 = 2$.

The spectral radius of the iteration matrix of Gauss-Seidel method is $\rho(T_{GS}) = 0.25$, so the method is convergent, (table 10) and the iteration matrix of Jacobi method is $\rho(T_J) = 0.5$, (table 9). Accordingly theorems 2.2 and 2.4, the solution of this system is convergent by the RJGS and RGSJ methods (tables 11, 12). The spectral radius of the iteration matrix of RJGS method for the above system is $\rho(T_{RJGS}) = 0.177692$. Also, The spectral radius of the iteration matrix of RGSJ method for the above system is $\rho(T_{RGSJ}) = 0.177692$.

TABLE 9. The solution of the linear system (14) by the Jacobi method.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$	$x_4^{[k]}$
0	0	0	0	0
1	0.5	0.25	0.25	1.5
2	0.625	0.75	0.75	1.625
21	1	0.999999	0.999999	2
22	1	1	1	2

TABLE 10. The solution of the linear system (14) by the GS method.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$	$x_4^{[k]}$
0	0	0	0	0
1	0.5	0.375	0.375	1.6875
2	0.6875	0.84375	0.84375	1.92188
11	0.999999	0.999999	0.999999	2
12	1	1	1	2

TABLE 11. The solution of the linear system (14) by the RJGS method.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$	$x_4^{[k]}$
0	0	0	0	0
1	0.6875	0.796875	0.796875	1.6875
2	0.948242	0.96167	0.96167	1.94824
8	0.999998	0.999999	0.999999	2
9	1	1	1	2

4. DISSCUSION AND RESULTS

The problem of solving large linear system is one of fundamental problems in science in general,[11, 12, 13]. Because of its importance there are many books specialized in iterative solutions of large linear systems [1, 2, 4, 7]. It is generally accepted that iterative techniques are preferable to direct methods for solving large sparse systems. One of the main disadvantage of iterative methods is the slow rate of convergence. There are different techniques introduced to

TABLE 12. The solution of the linear system (14) by the RGSJ method.

k	$x_1^{[k]}$	$x_2^{[k]}$	$x_3^{[k]}$	$x_4^{[k]}$
0	0	0	0	0
1	0.625	0.78125	0.78125	1.89063
2	0.939453	0.95752	0.95752	1.97876
8	0.999998	0.999999	0.999999	2
9	1	1	1	2

increase the convergence rate of the stationary iterative technique. Relaxation techniques, which contains a relaxation parameter, the choice of the optimum value restricts the efficient use of the relaxation methods. Double sweep method is a technique used to increase the convergence rate of any iterative technique. In double sweep methods, the main computational costs occur in the first step (calculation of the iteration matrix). In classical refinement methods, a virtual iteration is assumed with the same method (iteration matrix).

In this work, the concept of composite refinement technique is introduced. In the introduced composite refinement techniques to different iterative methods are used successively. As proved in the theorems above the refinement approach increases the rate of convergence moreover the composite refinement can change the behavior of the method from divergence to convergence (example 3.1). Tables (3, 4) illustrate that RJGS and RGSJ methods can be convergent when the Gauss-Seidel method is convergent. Tables (7, 8) illustrate that RJGS and RGSJ methods may be divergent when the Gauss-Seidel method is divergent. It is well known that when both Jacobi and Gauss Seidel methods are convergent, Gauss-Seidel method is faster (Stein-Rosenberg theorem [1]). The RJGS and RGSJ methods are convergent even if the coefficient matrix of the linear system is not a SDD matrix. Moreover, the RJGS and RGSJ methods may be divergent even if the Jacobi method is convergent (example 3.2).

5. CONCLUSION

The main objective is to introduce a composite form of the refinement techniques. It is proved that the composite refinement technique of two iterative methods M_1 and M_2 converges with rate of convergence between the rates of convergence of the classical refinement of M_1 and the classical refinement of M_2 (theorem 2.5 and example 3.3) and this will be much interesting when one of the iterative techniques is divergent (examples 3.1 and 3.2). It is illustrated that the convergence of the Gauss-Seidel (GS) method dominates the divergence of the Jacobi method in the introduced composite refinement technique. Finally, we mention the following main concluding remarks:

1. The composite refinement treatment is introduced.
2. An inequality for the convergence rate in case of composite refinement is well established (theorem 2.5).

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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