OPERATORS OF EXPONENTIAL TYPE AND THE ABSTRACT CAUCHY PROBLEM

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Abstract. In this paper, we introduce closed operators of exponential type, and use it to study the solution of the homogeneous abstract Cauchy problem of the first order, usual and fractional.

Keywords: abstract Cauchy problem; exponential type operators; fractional derivatives.

2010 AMS Subject Classification: 26A33.

1. INTRODUCTION

Let X be a Banach space and I = [0, ∞). Let C(I) be the Banach space of all bounded continuous real valued functions defined on I,

and let C(I,X) be the set of all bounded continuous function from I to X.

Now, the first order nonhomogeneous Abstract Cauchy Problem is

\[ \begin{cases} u'(t) = Au(t) + f(t) \\ u(0) = x_0 \end{cases} \] .................(1)

Here, u is a differentiable function from I to X, and A is a densely defined closed linear operators on X. Such equation appears in many applications in physics and applied sciences.

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Received December 8, 2019
The solution of such equation depends mainly on the operator $A$.

We will discuss in this paper the solution of (1), when $f = 0$. Further, and we will discuss equation (1), when the derivative is replaced by fractional derivative.

So let us recall some basics of the conformable derivative.

In [3], the authors gave a new definition of fractional derivative which is a natural extension to the usual first derivative as follows:

Given a function $f : [0, \infty) \rightarrow \mathbb{R}$. Then for all $t > 0$, $\alpha \in (0, 1)$, let

$$D_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

$D_\alpha f$ is called the conformable fractional derivative of $f$ of order $\alpha$.

Let $f^{(\alpha)}(t)$ stands for $D_\alpha(f)(t)$. Hence $f^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$.

If $f$ is $\alpha-$differentiable in some $(0, b)$, $b > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then let

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

The conformable derivative satisfies all the classical properties of the usual first derivative.

Further, according to this derivative, the following statements are true, see [3].

1. $D_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$,
2. $D_\alpha(\sin \frac{1}{\alpha} t^\alpha) = \cos \frac{1}{\alpha} t^\alpha$,
3. $D_\alpha(\cos \frac{1}{\alpha} t^\alpha) = -\sin \frac{1}{\alpha} t^\alpha$,
4. $D_\alpha(e^{\frac{1}{\alpha} t^\alpha}) = e^{\frac{1}{\alpha} t^\alpha}$.

The $\alpha-$fractional integral of a function $f$ starting from $a \geq 0$ is:

$$I_a^\alpha(f)(t) = I_1^\alpha(t^\alpha-1 f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

In this paper we will study the Abstract Cauchy Problem:

$$\begin{cases}
    u^{(\alpha)}(t) = Au(t) \\
    u(0) = x_0
\end{cases} \quad \text{..................(2)}$$

2. **Exponential Type Operators**

In this section we introduce a class of operators to be called of exponential type.

**Definition 2.1.** Let $A : Dom(A) \subseteq X \rightarrow X$, be a densely defined linear operator. The operator $A$ is called of exponential type if

(i) The operator $B(t) = e^{tA}$ exists and well defined for all $x \in Dom(A)$, in the sense:

$$B(t)x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x$$

converges absolutely for all $x \in Dom(A)$. That is

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A^n x\| < \infty.$$ 

(ii) $C(x,A) = \{x, Ax, A^2 x, \ldots\} \subseteq Dom(A)$.

**Examples.**

(1) Clearly every bounded linear operator on a Banach space $X$ is of exponential type.

(2) Consider the operator $T : \ell^2 \rightarrow \ell^2$ defined by $T(\delta_n) = n \delta_1$. Clearly $T$ is densely defined. Further:

$$T(a_1 \delta_1 + \ldots a_k \delta_k) = (\sum_{i=1}^k ia_i) \delta_1$$

Hence $T^2 x = T x$. In fact $T^n x = T x$, for any $x$ which can be written as a finite linear combination of the basis elements $\{\delta_1, \ldots, \delta_n, \ldots\}$

Hence

$$\|A^n x\| \leq \frac{k(k+1)}{2} \sup_{1 \leq j \leq k} |a_j|$$

Now

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A^n x\| \leq \sup_{1 \leq j \leq k} |a_j| \sum_{n=0}^{\infty} \frac{t^n k(k+1)}{2} < \infty$$

Thus, $T$ is of exponential type.

**Theorem 2.1.** Let $A : Dom(A) \subseteq X \rightarrow X$ be of exponential type. Then $\frac{d}{dt} B(t)x = \frac{d}{dt} e^{tA} x = Ae^{tA} x = AB(t)x$ for $x \in Dom(A)$

**Proof.** $B(t)x = B(t)x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x$. Then using classical tools we get

$$\frac{d}{dt} B(t)x = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} A^n x$$

$$= \sum_{n=1}^{\infty} A \frac{t^{n-1}}{(n-1)!} A^{n-1} x$$
This ends the proof.

**Theorem 2.2.** Let $A : Dom(A) \subseteq X \to X$ be of exponential type, and $u : [0, \infty) \to X$ be differentiable.

Then \[
\begin{cases}
  u'(t) = Au(t) \\
  u(0) = x_0
\end{cases}
\]
has a unique solution.

**Proof.** It follows from Theorem 2.1 that $u_1(t) = e^{tA}x_0$ is a solution.

Assume if possible that $u_2$ is another solution. Then $u_2'(t) = Au_2(t)$. But then

\[
u_2'(t) - u_1'(t) = A(u_2(t) - u_1(t))
\]

But this implies that $u_2(t) - u_1(t) = e^{tA}y$. Since $u_2(0) - u_1(0) = x_0 - x_0 = 0$, it follows that $y = 0$, and hence $u_2(t) = u_1(t)$.

3. **Fractional Abstract Cauchy Problem**

Let us write $D_\alpha(f)(t)$ for $f'(\alpha)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon}$. In this section we are interested in discussing

\[
\begin{cases}
  u'(\alpha)(t) = Au(t) \\
  u(0) = x_0
\end{cases}
\]

Let us recall that, [2]:

$T : [0, \infty) \to L(X)$, the space of bounded linear operators on $X$, is called an $\alpha$-fractional semigroup of operators if $T(0) = I$ and $T(s+t)^{\frac{1}{\alpha}} = T(s)^{\frac{1}{\alpha}} T(t)^{\frac{1}{\alpha}}$.

The generator of the semigroup $T(t)$ is just the $\alpha-$conformable derivative of $T(t)$ at $t = 0$.

We refer to [2] for more results on fractional semigroups of operators.

Now we have:

**Theorem 3.1.** Let $A : Dom(A) \subseteq X \to X$ be of exponential type. Then $T(t)x = e^{tA}x$ is an $\alpha-$fractional semigroup with $A$ as the generator.

**Proof.** That $T(t)x = e^{tA}x$ is an $\alpha-$fractional semigroup is straight forward computations.
Consider\( D_\alpha T(t)x = D_\alpha \sum_{n=0}^{\infty} \frac{(\alpha t)^n}{n!} A^n x.\)

Using the same ideas in Theorem 2.1, we get
\[
D_\alpha T(t)x = AT(t)x = e^{\frac{\alpha t}{\alpha} A} x. \quad \text{(2)}
\]

Note that we used \( D_\alpha (e^{\frac{1}{\alpha} t}) = e^{\frac{1}{\alpha} t}. \)

Taking the limit as \( t \to 0, \) we get \( D_\alpha T(0)x = A. \) That ends the proof.

Now we discuss the fractional Abstract Cauchy Problem

\[
\begin{cases}
  u^{(\alpha)}(t) = Au(t) \\
  u(0) = x_0
\end{cases} \quad \text{(3)}
\]

**Theorem 3.2.** If \( A \) is of exponential type, then (3) has a unique solution.

**Proof.** By (2) in Theorem 3.1, we get \( u_1(t) = e^{\frac{\alpha t}{\alpha}} x_0 \) as a solution of (3).

Using the same idea as in Theorem 2.2, we get our result.

**ACKNOWLEDGMENT**

This work is part of the Ph.D. thesis written by Rabahi Lahcene written under the supervision of the professors Al Horani, M. and Khalil, R.

Lahcene would like to thank his supervisors for the help and guidance throughout the work of the thesis.

**CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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