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INTERPOLATIVE α_p -CONTRACTIONS IN C^* -ALGEBRA VALUED PARTIAL b -METRIC SPACES: FIXED POINT THEOREMS AND APPLICATIONS

APPANA BHARGAVI SIVA PRABHA^{1,*}, RAJAMAHANATHI VENKATA GANAPATHI RAVIKUMAR²,
BAGATHI SRINUVASA RAO³

¹Department of Mathematics, JNTUK, Kakinada & Lecturer in Mathematics, Govt. Polytechnic for women,
Srikakulam, Andhra Pradesh, India

²GVP College of Engineering, Visakhapatnam, Andhra Pradesh, India

³Dr. B. R. Ambedkar University, Srikakulam-534210, Andhra Pradesh, India

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Abstract: This study introduces interpolative contraction mappings in C^* -algebra valued partial b -metric spaces (C^* -avpbms) through α -admissible functions. We prove multiple coupled fixed point theorems for these contractions, supported by concrete examples. The theoretical developments find direct application in solving coupled Fredholm integral equations and analyzing homotopy continuation problems, demonstrating their utility in nonlinear analysis.

Keywords: ζ_p -metric space; α -admissible mapping; α_p -interpolative contractions and C^* -algebra.

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1. INTRODUCTION

Banach's contraction principle [1] stands as a cornerstone of fixed point theory, offering a versatile method to solve problems in diverse mathematical fields. Researchers have since proposed many extensions in the literature. Coupled fixed points have advanced significantly

*Corresponding author

E-mail address: abhargavisivaprabha@gmail.com

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in metric fixed point theory, with Guo et al. pioneering the idea in 1987 [2]. Kannan [3, 4] extended the Banach principle [1] by removing the need for mapping continuity. Karapinar [5] later introduced interpolative contractions to encompass both Banach and Kannan types. In [6], Karapinar et al. provided Example 1 [6] to show that fixed points need not be unique; they proved that in a complete metric space $(\mathcal{G}, d)$, a self-map $\mathbb{T} : \mathcal{G} \rightarrow \mathcal{G}$ has a fixed point if constants $\lambda \in [0, 1)$ and $\eta \in (0, 1)$ satisfy

$$d(\mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{e}) \leq \lambda[d(\mathfrak{s}, \mathbb{T}\mathfrak{s})]^\eta[d(\mathfrak{e}, \mathbb{T}\mathfrak{e})]^{1-\eta}$$

for all $\mathfrak{s}, \mathfrak{e} \in \mathcal{G} \setminus \text{Fix}(\mathbb{T})$. Building on this, Samet et al. [7] and Popescu [8] developed notions like α -admissible, α -orbital admissible, and α - ψ -contractive mappings, paving the way for further work by Isik et al. [9] and Yamaod and Sintunavarat [10].

In 2014, Ma et al. [11] established fixed point theorems for self-maps satisfying contractive or expansive conditions in C^* -algebra-valued metric spaces. In 2019, Chandok et al. [12] introduced C^* -algebra-valued partial metric spaces (C^* -AVPMS) and derived fixed point results using C -class functions. Mlaiki et al. [13, 14] extended the framework of C^* -AVbMS (C^* -algebra-valued b -metric spaces) and C^* -AVPMS (C^* -algebra-valued partial metric spaces) by defining C^* -avpbms (C^* -algebra-valued partial b -metric spaces), applying it to fixed point theorems in 2021.

This paper develops coupled fixed point theorems satisfying interpolative contractive conditions via α -admissible mappings in C^* -avpbms. The results are applied to establish existence and uniqueness for systems of nonlinear integral equations, with additional analysis of homotopy applications supported by illustrative examples.

2. PRELIMINARIES

This section provides a brief introduction to some fundamental aspects of C^* -algebra theory [15, 16]. Let $\mathcal{A}$ be a unital C^* -algebra with the unit element $1_{\mathcal{A}}$. Define

$\mathcal{A}_h = \{\mathfrak{e} \in \mathcal{A} : \mathfrak{e} = \mathfrak{e}^*\}$. An element $\mathfrak{e} \in \mathcal{A}$ is considered positive, denoted as $\mathfrak{e} \succeq 0_{\mathcal{A}}$, if $\mathfrak{e} = \mathfrak{e}^*$ and its spectrum $\eta(\mathfrak{e}) \subseteq [0, \infty)$. Here, $0_{\mathcal{A}}$ in $\mathcal{A}$ represents the zero element in $\mathcal{A}$, and $\eta(\mathfrak{e})$ denotes the spectrum of $\mathfrak{e}$. On $\mathcal{A}_h$, a natural partial ordering is defined by $\mathfrak{s} \preceq \mathfrak{v}$ if and only if $\mathfrak{v} - \mathfrak{s} \succeq 0_{\mathcal{A}}$. We denote $\mathcal{A}_+ = \{\mathfrak{e} \in \mathcal{A} : \mathfrak{e} \succeq 0_{\mathcal{A}}\}$ and $\mathcal{A}' = \{\mathfrak{e} \in \mathcal{A} : \mathfrak{e}\mathfrak{d} = \mathfrak{d}\mathfrak{e} \ \forall \mathfrak{d} \in \mathcal{A}\}$.

Lemma 2.1:([15, 17]) Let $\mathcal{A}$ be a C^* -algebra with the identity element $I_{\mathcal{A}}$ and $\mathfrak{v}$ be a positive element of $\mathcal{A}$. Then

- (i) There is a unique element $u \in \mathcal{A}_+$ such that $u^2 = \mathfrak{v}$.
- (ii) The set $\mathcal{A}_+ = \{\mathfrak{v}^* \mathfrak{v} / \mathfrak{v} \in \mathcal{A}\}$ with a conjugate-linear involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$.
- (iii) $\mathfrak{v}, u \in \mathcal{A}$, and $0_{\mathcal{A}} \preceq \mathfrak{v} \preceq u$ then $\|\mathfrak{v}\| \leq \|u\|$.
- (iv) If $\mathfrak{v} \in \mathcal{A}_+$ with $\|\mathfrak{v}\| < \frac{1}{2}$ then $(I - \mathfrak{v})$ is invertible and $\|\mathfrak{v}(I - \mathfrak{v})^{-1}\| < 1$.

Definition 2.2:([13, 14]) Let $\mathcal{G}$ be a nonempty set and $\mathfrak{v} \in \mathcal{A}$ such that $\mathfrak{v} \succeq 1_{\mathcal{A}}$. If the following conditions are satisfied for each $\mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3 \in \mathcal{G}$, then a C^* -algebra valued partial b - metric is defined as a function $\varsigma_p : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{A}$:

- (ς_{p1}) $\varsigma_p(\mathfrak{v}_1, \mathfrak{v}_2) \succeq 0_{\mathcal{A}}$ and $\mathfrak{v}_1 = \mathfrak{v}_2$ iff $\varsigma_p(\mathfrak{v}_1, \mathfrak{v}_1) = \varsigma_p(\mathfrak{v}_1, \mathfrak{v}_2) = \varsigma_p(\mathfrak{v}_2, \mathfrak{v}_2)$;
- (ς_{p2}) $\varsigma_p(\mathfrak{v}_1, \mathfrak{v}_1) \preceq \varsigma_p(\mathfrak{v}_1, \mathfrak{v}_2)$;
- (ς_{p3}) $\varsigma_p(\mathfrak{v}_1, \mathfrak{v}_2) = \varsigma_p(\mathfrak{v}_2, \mathfrak{v}_1)$;
- (ς_{p4}) $\varsigma_p(\mathfrak{v}_1, \mathfrak{v}_2) \preceq \mathfrak{v} [\varsigma_p(\mathfrak{v}_1, \mathfrak{v}_3) + \varsigma_p(\mathfrak{v}_3, \mathfrak{v}_2)] - \varsigma_p(\mathfrak{v}_3, \mathfrak{v}_3)$.

A C^* -algebra valued partial b -metric space is the pair $(\mathcal{G}, \mathcal{A}, \varsigma_p)$.

Remark: Observe that, a C^* -avPbms $(\mathcal{G}, \mathcal{A}, \varsigma_p)$ is a generalization of both C^* -avbms as well as C^* -avpms. Obviously, every C^* -avbms is a C^* -avpbms with zero self distance and every C^* -avpms is a C^* -avpbms with $\|\mathfrak{v}\| = 1$, but converse is not true in general.

Example 2.3:([13, 14]) Assume that $\mathcal{G} = [0, 1]$ and $\mathcal{A} = M_2(\mathbb{C})$, the class of bounded and linear operators on a Hilbert space $\mathbb{C}^2$. Define, $\varsigma_p : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{A}$ as

$$\varsigma_p(\mathfrak{s}, \mathfrak{e}) = \begin{bmatrix} |\mathfrak{s} - \mathfrak{e}|^\eta & 0 \\ 0 & \kappa |\mathfrak{s} - \mathfrak{e}|^\eta \end{bmatrix} + \begin{bmatrix} \max\{\mathfrak{s}, \mathfrak{e}\}^\eta & 0 \\ 0 & \kappa \max\{\mathfrak{s}, \mathfrak{e}\}^\eta \end{bmatrix} \quad \forall \quad \mathfrak{s}, \mathfrak{e} \in \mathcal{G},$$

where $\kappa \geq 0$ and $\eta > 1$. Then, $(\mathcal{G}, \mathcal{A}, \varsigma_p)$ is a C^* -avpbms with coefficient $\mathfrak{v} = 2^{\eta-1}I$. However, it is easy to see that $(\mathcal{G}, \mathcal{A}, \varsigma_p)$ is neither a C^* -avbms nor C^* -avpms.

Note: Let $(\mathcal{G}, \mathcal{A}, \varsigma_p)$ is a C^* -avpbms. Afterwards, open and closed ball of center $\mathfrak{v} \in \mathcal{G}$ and radius $0_{\mathcal{A}} \prec \varepsilon \in \mathcal{A}$ are defined by:

$$B_{\varsigma_p}(\mathfrak{v}, \varepsilon) = \{\mu \in \mathcal{G} : \varsigma_p(\mathfrak{v}, \mu) \prec \varsigma_p(\mathfrak{v}, \mathfrak{v}) + \varepsilon\}, \text{ and } B_{\varsigma_p}[\mathfrak{v}, \varepsilon] = \{\mu \in \mathcal{G} : \varsigma_p(\mathfrak{v}, \mu) \preceq \varsigma_p(\mathfrak{v}, \mathfrak{v}) + \varepsilon\}.$$

The family of open balls (for all $\nu \in \mathcal{G}$ and $\varepsilon \succ 0_{\mathcal{A}}$)

$$\mathcal{S} = \{B_{\zeta\rho}(\nu, \varepsilon) : \nu \in \mathcal{G}, \varepsilon \succ 0_{\mathcal{A}}\},$$

forms a basis of some topology $\tau_{\zeta\rho}$ on $\mathcal{G}$.

Definition 2.4:([13, 14]) Assume that $(\mathcal{G}, \mathcal{A}, \zeta\rho)$ is a C^* -avpbms. According to $\mathcal{A}$ a sequence $\{\mathfrak{s}_\zeta\}$ in $\mathcal{G}$ is defined as:

- (i) C^* -algebra valued $\zeta\rho$ -convergent to a point $\mathfrak{s} \in \mathcal{G}$ if, given $0_{\mathcal{A}} \prec \varepsilon$, there exist $x \in \mathbb{N}$ such that $\|\zeta\rho(\mathfrak{s}_\zeta, \mathfrak{s}) - \zeta\rho(\mathfrak{s}, \mathfrak{s})\| \prec \varepsilon \forall \zeta > x$. We can also use different presentations for that as follows:

$$\lim_{\zeta \rightarrow \infty} \zeta\rho(\mathfrak{s}_\zeta, \mathfrak{s}) = \zeta\rho(\mathfrak{s}, \mathfrak{s}).$$

- (ii) C^* -algebra valued $\zeta\rho$ -Cauchy sequence, if $\lim_{\zeta, \eta \rightarrow \infty} \zeta\rho(\mathfrak{s}_\zeta, \mathfrak{s}_\eta)$ exists and it is finite.
- (iii) It is referred to as being complete when a C^* -avpbms $(\mathcal{G}, \mathcal{A}, \zeta\rho)$ is present. If each Cauchy sequence in $\mathcal{G}$ converges to a point in $\mathcal{G}$ such that

$$\lim_{\zeta, \eta \rightarrow \infty} \zeta\rho(\mathfrak{s}_\zeta, \mathfrak{s}_\eta) = \lim_{\zeta \rightarrow \infty} \zeta\rho(\mathfrak{s}_\zeta, \mathfrak{s}) = \zeta\rho(\mathfrak{s}, \mathfrak{s}).$$

Lemma 2.5:[13, 14] Let $(\mathcal{G}, \tau_{\zeta\rho})$ be a topological space and $\mathbb{T} : \mathcal{G} \rightarrow \mathcal{G}$. If $\mathbb{T}$ is continuous, then every sequence $\{\mathfrak{s}_\zeta\} \subseteq \mathcal{G}$, such that $\mathfrak{s}_\zeta \rightarrow \mathfrak{s}$ implies $\mathbb{T}\mathfrak{s}_\zeta \rightarrow \mathbb{T}\mathfrak{s}$. The converse holds if $\mathcal{G}$ is metrizable.

Now we prove our main result.

3. MAIN RESULTS

For convenience, we set: $\Delta = \{\wp/\wp : \mathcal{A}_+ \rightarrow \mathcal{A}_+\}$ be a family of functions that satisfy the following properties:

- (i) $\wp$ is continuous, monotonically non-decreasing;
- (ii) $\wp(\mathfrak{e}) = 0_{\mathcal{A}}$ if and only if $\mathfrak{e} = 0_{\mathcal{A}}$.

Definition 3.1: Let $(\mathcal{G}, \mathcal{A}, \zeta\rho)$ be a C^* -avpbms and a pair $(\mathfrak{x}, \mathfrak{y}) \in \mathcal{G} \times \mathcal{G}$ is called a coupled fixed point (CFP) of mapping $\mathbb{T} : \mathcal{G}^2 \rightarrow \mathcal{G}$ if $\mathfrak{x} = \mathbb{T}(\mathfrak{x}, \mathfrak{y})$ and $\mathfrak{y} = \mathbb{T}(\mathfrak{y}, \mathfrak{x})$.

Definition 3.2: Let $(\mathcal{G}, \mathcal{A}, \varsigma_\rho)$ be a C^* -avpbms, $\mathbb{T} : \mathcal{G}^2 \rightarrow \mathcal{G}$ and $\alpha_p : \mathcal{G}^2 \rightarrow \mathcal{A}_+$ be a functions then $\mathbb{T}$ is called a α_p -adm, if for all $\mathfrak{s}, \mathfrak{e} \in \mathcal{G}$

$$\alpha_p(\mathfrak{s}, \mathfrak{e}) \succeq 1_{\mathcal{A}} \Rightarrow \alpha_p(\mathbb{T}(\mathfrak{s}, \mathfrak{e}), \mathbb{T}(\mathfrak{e}, \mathfrak{s})) \succeq 1_{\mathcal{A}}.$$

Definition 3.3: Let $(\mathcal{G}, \mathcal{A}, \varsigma_\rho)$ be a C^* -avpbms, $\mathbb{T} : \mathcal{G}^2 \rightarrow \mathcal{G}$ and $\alpha_p : \mathcal{G}^2 \rightarrow \mathcal{A}_+$ be a functions then $\mathbb{T}$ is called a α_p -orbital admissible, if for $\mathfrak{s} \in \mathcal{G}$

$$\alpha_p(\mathfrak{s}, \mathbb{T}(\mathfrak{s}, \mathfrak{e})) \succeq 1_{\mathcal{A}} \Rightarrow \alpha_p(\mathbb{T}(\mathfrak{s}, \mathfrak{e}), \mathbb{T}^2(\mathfrak{s}, \mathfrak{e})) \succeq 1_{\mathcal{A}}.$$

Example 3.4: Let $\mathcal{G} = [0, 1]$ with C^* -avpbMS $(\mathcal{G}, \mathcal{A}, \varsigma_\rho)$ where $\mathcal{A} = M_2(\mathbb{R})$,

$\varsigma_\rho(x, y) = (\max\{x, y\})^2 I_2 + |x - y|^2 I_2$. Define $\mathbb{T}(s, e) = \frac{2s+e}{3}$ and

$$\alpha_p(s, e) = \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} & |s - e| \leq 0.4, \\ \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} & \text{otherwise.} \end{cases} \quad \text{For } \mathfrak{s} = 0.2, \mathfrak{e} = 0.5, \text{ premise holds}$$

$(\alpha_p(\mathfrak{s}, \mathbb{T}(\mathfrak{s}, \mathfrak{e})) \succeq I_2)$. Also $|\mathbb{T}(\mathfrak{s}, \mathfrak{e}) - \mathbb{T}^2(\mathfrak{s}, \mathfrak{e})| \approx 0.0667 \leq 0.4$ so conclusion holds. Generally $|\mathbb{T}(s, e) - \mathbb{T}^2(s, e)| = |s - e|/3 < 0.4$. Thus $\mathbb{T}$ is α_p -orbital admissible.

Definition 3.5: The mapping $\mathbb{T}$ on the C^* -avpbms $(\mathcal{G}, \mathcal{A}, \varsigma_\rho)$ with coefficient $\mathfrak{v} \in \mathcal{A}$ such that $\mathfrak{v} \succeq 1_{\mathcal{A}}$ is designated as an α_p -interpolative contraction if there occurs a function $\wp \in \Delta$ and a mapping $\alpha_p : \mathcal{G}^2 \rightarrow \mathcal{A}_+$, with $\kappa, \eta \in (0, 1)$ and $\mathfrak{a} \in \mathcal{A}$ with $\|\mathfrak{a}\mathfrak{v}\| < 1$ such that for all $\mathfrak{s}, \mathfrak{e}, \mathfrak{x}, \mathfrak{z} \in \mathcal{G} - \mathbb{F}(\mathbb{T})$,

$$(1) \quad \alpha_p(\mathfrak{s}, \mathfrak{x}) \alpha_p(\mathfrak{e}, \mathfrak{z}) \wp(\varsigma_\rho(\mathbb{T}(\mathfrak{s}, \mathfrak{e}), \mathbb{T}(\mathfrak{x}, \mathfrak{z}))) \preceq \wp \left(\begin{pmatrix} \left[\mathfrak{a}^* \max \left\{ \begin{matrix} \varsigma_\rho(\mathfrak{s}, \mathfrak{x}), \\ \varsigma_\rho(\mathfrak{e}, \mathfrak{z}) \end{matrix} \right\} \mathfrak{a} \right]^\kappa \\ \left[\mathfrak{a}^* \max \left\{ \begin{matrix} \varsigma_\rho(\mathfrak{s}, \mathbb{T}(\mathfrak{s}, \mathfrak{e})), \\ \varsigma_\rho(\mathfrak{e}, \mathbb{T}(\mathfrak{e}, \mathfrak{s})) \end{matrix} \right\} \mathfrak{a} \right]^\eta \\ \left[\max \left\{ \begin{matrix} \varsigma_\rho(\mathfrak{x}, \mathbb{T}(\mathfrak{x}, \mathfrak{z})), \\ \varsigma_\rho(\mathfrak{z}, \mathbb{T}(\mathfrak{z}, \mathfrak{x})) \end{matrix} \right\} \right]^{1-\kappa-\eta} \end{pmatrix} \right).$$

Here, $\mathbb{F}(\mathbb{T})$ will denote the collection of all fixed points of $\mathbb{T}$.

Theorem 3.6: Let $\mathbb{T} : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ be a mapping defined on the complete C^* -avpbms $(\mathcal{G}, \mathcal{A}, \varsigma_\rho)$ with coefficient $\mathfrak{v} \succeq 1_{\mathcal{A}}$ that follows the subsequent circumstances:

- (a) $\mathbb{T}$ is continuous,
- (b) $\mathbb{T}$ is α_p -orbital admissible,
- (c) There exist $\nu_0, \mu_0 \in \mathcal{G}$ that gives $\alpha_p(\nu_0, \mathbb{T}(\nu_0, \mu_0)) \succeq 1_{\mathcal{A}}$ and $\alpha_p(\mu_0, \mathbb{T}(\mu_0, \nu_0)) \succeq 1_{\mathcal{A}}$,
- (d) $\mathbb{T}$ is α_p -interpolative contraction.

Then a coupled fixed point of $\mathbb{T}$ exists in $\mathcal{G}$.

Proof Let $\nu_0, \mu_0 \in \mathcal{G}$ be a point that corresponds to $\alpha_p(\nu_0, \mathbb{T}(\nu_0, \mu_0)) \succeq 1_{\mathcal{A}}$ and $\alpha_p(\mu_0, \mathbb{T}(\mu_0, \nu_0)) \succeq 1_{\mathcal{A}}$. Let $\{\nu_\zeta\}$ and $\{\mu_\zeta\}$ be the sequences defined by $\nu_\zeta = \mathbb{T}^\zeta(\nu_0, \mu_0)$ and $\mu_\zeta = \mathbb{T}^\zeta(\mu_0, \nu_0)$ for $\zeta \geq 1$. If for a certain ζ_0 , $\nu_{\zeta_0} = \nu_{\zeta_0+1}$ and $\mu_{\zeta_0} = \mu_{\zeta_0+1}$ then $(\nu_{\zeta_0}, \mu_{\zeta_0})$ becomes a coupled fixed point of $\mathbb{T}$, otherwise, $\nu_\zeta \neq \nu_{\zeta+1}$ and $\mu_\zeta \neq \mu_{\zeta+1}$ for all $\zeta \geq 1$. We have $\alpha_p(\nu_0, \nu_1) \succeq 1_{\mathcal{A}}$ and $\alpha_p(\mu_0, \mu_1) \succeq 1_{\mathcal{A}}$. Since $\mathbb{T}$ is α_p -orbital admissible,

$$\begin{aligned} \alpha_p(\nu_1, \nu_2) = \alpha_p(\mathbb{T}\nu_0, \mathbb{T}\nu_1) \succeq 1_{\mathcal{A}} &\implies \alpha_p(\nu_2, \nu_3) = \alpha_p(\mathbb{T}\nu_1, \mathbb{T}\nu_2) \succeq 1_{\mathcal{A}} \\ \alpha_p(\mu_1, \mu_2) = \alpha_p(\mathbb{T}\mu_0, \mathbb{T}\mu_1) \succeq 1_{\mathcal{A}} &\implies \alpha_p(\mu_2, \mu_3) = \alpha_p(\mathbb{T}\mu_1, \mathbb{T}\mu_2) \succeq 1_{\mathcal{A}}. \end{aligned}$$

Continuing as above, we obtain that

$$\alpha_p(\nu_\zeta, \nu_{\zeta+1}) \succeq 1_{\mathcal{A}} \text{ and } \alpha_p(\mu_\zeta, \mu_{\zeta+1}) \succeq 1_{\mathcal{A}} \quad \forall \quad \zeta \geq 0.$$

From, Eq.(1), we have

$$\begin{aligned} \wp(\varsigma_p(\nu_\zeta, \nu_{\zeta+1})) &= \wp(\varsigma_p(\mathbb{T}(\nu_{\zeta-1}, \mu_{\zeta-1}), \mathbb{T}(\nu_\zeta, \mu_\zeta))) \\ &\preceq \alpha_p(\nu_{\zeta-1}, \nu_\zeta) \alpha_p(\mu_{\zeta-1}, \mu_\zeta) \wp(\varsigma_p(\mathbb{T}(\nu_{\zeta-1}, \mu_{\zeta-1}), \mathbb{T}(\nu_\zeta, \mu_\zeta))) \\ &\preceq \wp \left(\begin{array}{c} \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta-1}, \nu_\zeta), \\ \varsigma_p(\mu_{\zeta-1}, \mu_\zeta) \end{array} \right\} \alpha \right]^\kappa \\ \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta-1}, \mathbb{T}(\nu_{\zeta-1}, \mu_{\zeta-1})), \\ \varsigma_p(\mu_{\zeta-1}, \mathbb{T}(\mu_{\zeta-1}, \nu_{\zeta-1})) \end{array} \right\} \alpha \right]^\eta \\ \left[\max \left\{ \begin{array}{c} \varsigma_p(\nu_\zeta, \mathbb{T}(\nu_\zeta, \mu_\zeta)), \\ \varsigma_p(\mu_\zeta, \mathbb{T}(\mu_\zeta, \nu_\zeta)) \end{array} \right\} \right]^{1-\kappa-\eta} \end{array} \right) \end{aligned}$$

$$\preceq \wp \left(\begin{array}{c} \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta-1}, \nu_{\zeta}), \\ \varsigma_p(\mu_{\zeta-1}, \mu_{\zeta}) \end{array} \right\} \alpha \right]^{\kappa} \\ \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta-1}, \nu_{\zeta}), \\ \varsigma_p(\mu_{\zeta-1}, \mu_{\zeta}) \end{array} \right\} \alpha \right]^{\eta} \left[\max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta}, \nu_{\zeta+1}), \\ \varsigma_p(\mu_{\zeta}, \mu_{\zeta+1}) \end{array} \right\} \right]^{1-\kappa-\eta} \end{array} \right).$$

By the definition of $\wp$, we have that

$$\varsigma_p(\nu_{\zeta}, \nu_{\zeta+1}) \preceq \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta-1}, \nu_{\zeta}), \\ \varsigma_p(\mu_{\zeta-1}, \mu_{\zeta}) \end{array} \right\} \alpha \right]^{\kappa+\eta} \left[\max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta}, \nu_{\zeta+1}), \\ \varsigma_p(\mu_{\zeta}, \mu_{\zeta+1}) \end{array} \right\} \right]^{1-\kappa-\eta}. \quad (2)$$

Similarly, we can prove that

$$\varsigma_p(\mu_{\zeta}, \mu_{\zeta+1}) \preceq \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta-1}, \nu_{\zeta}), \\ \varsigma_p(\mu_{\zeta-1}, \mu_{\zeta}) \end{array} \right\} \alpha \right]^{\kappa+\eta} \left[\max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta}, \nu_{\zeta+1}), \\ \varsigma_p(\mu_{\zeta}, \mu_{\zeta+1}) \end{array} \right\} \right]^{1-\kappa-\eta}. \quad (3)$$

Combining Eq. (2) and Eq.(3), we get

$$\max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta}, \nu_{\zeta+1}), \\ \varsigma_p(\mu_{\zeta}, \mu_{\zeta+1}) \end{array} \right\} \preceq \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta-1}, \nu_{\zeta}), \\ \varsigma_p(\mu_{\zeta-1}, \mu_{\zeta}) \end{array} \right\} \alpha \right]^{\kappa+\eta} \left[\max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta}, \nu_{\zeta+1}), \\ \varsigma_p(\mu_{\zeta}, \mu_{\zeta+1}) \end{array} \right\} \right]^{1-\kappa-\eta}.$$

We derive

$$\max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta}, \nu_{\zeta+1}), \\ \varsigma_p(\mu_{\zeta}, \mu_{\zeta+1}) \end{array} \right\}^{\kappa+\eta} \preceq \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta-1}, \nu_{\zeta}), \\ \varsigma_p(\mu_{\zeta-1}, \mu_{\zeta}) \end{array} \right\} \alpha \right]^{\kappa+\eta}.$$

Therefore,

$$\begin{aligned} \delta_{\zeta} = \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta}, \nu_{\zeta+1}), \\ \varsigma_p(\mu_{\zeta}, \mu_{\zeta+1}) \end{array} \right\} &\preceq \alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta-1}, \nu_{\zeta}), \\ \varsigma_p(\mu_{\zeta-1}, \mu_{\zeta}) \end{array} \right\} \alpha \preceq (\alpha^*)^2 \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta-2}, \nu_{\zeta-1}), \\ \varsigma_p(\mu_{\zeta-2}, \mu_{\zeta-1}) \end{array} \right\} (\alpha)^2 \\ &\vdots \\ &\preceq (\alpha^*)^{\zeta} \delta_0 (\alpha)^{\zeta} \rightarrow 0_{\mathcal{A}} \text{ as } \zeta \rightarrow \infty. \end{aligned}$$

That is $\lim_{\zeta \rightarrow \infty} \max \left\{ \begin{array}{c} \varsigma_{\rho}(\nu_{\zeta}, \nu_{\zeta+1}), \\ \varsigma_{\rho}(\mu_{\zeta}, \mu_{\zeta+1}) \end{array} \right\} = 0_{\mathcal{A}}$ so tat $\lim_{\zeta \rightarrow \infty} \varsigma_{\rho}(\nu_{\zeta}, \nu_{\zeta+1}) = 0_{\mathcal{A}}$ and $\lim_{\zeta \rightarrow \infty} \varsigma_{\rho}(\mu_{\zeta}, \mu_{\zeta+1}) = 0_{\mathcal{A}}$. Since $\varsigma_{\rho}(\nu_{\zeta}, \nu_{\zeta}) \preceq \varsigma_{\rho}(\nu_{\zeta}, \nu_{\zeta+1})$ and $\varsigma_{\rho}(\mu_{\zeta}, \mu_{\zeta}) \preceq \varsigma_{\rho}(\mu_{\zeta}, \mu_{\zeta+1})$, then we have $\lim_{\zeta \rightarrow \infty} \varsigma_{\rho}(\nu_{\zeta}, \nu_{\zeta}) = 0_{\mathcal{A}}$ and $\lim_{\zeta \rightarrow \infty} \varsigma_{\rho}(\mu_{\zeta}, \mu_{\zeta}) = 0_{\mathcal{A}}$.

Now, for any $\zeta, \xi \in \mathbb{N}$, we have

$$\begin{aligned}
& \varsigma_{\rho}(\nu_{\zeta}, \nu_{\zeta+\xi}) \preceq \mathfrak{v} [\varsigma_{\rho}(\nu_{\zeta}, \nu_{\zeta+1}) + \varsigma_{\rho}(\nu_{\zeta+1}, \nu_{\zeta+\xi})] - \varsigma_{\rho}(\nu_{\zeta+1}, \nu_{\zeta+1}) \\
& \preceq \mathfrak{v} \varsigma_{\rho}(\nu_{\zeta}, \nu_{\zeta+1}) + \mathfrak{v}^2 [\varsigma_{\rho}(\nu_{\zeta+1}, \nu_{\zeta+2}) + \varsigma_{\rho}(\nu_{\zeta+2}, \nu_{\zeta+\xi})] - \varsigma_{\rho}(\nu_{\zeta+1}, \nu_{\zeta+1}) \\
& \quad - \varsigma_{\rho}(\nu_{\zeta+2}, \nu_{\zeta+2}) \\
& \preceq \mathfrak{v} \varsigma_{\rho}(\nu_{\zeta}, \nu_{\zeta+1}) + \mathfrak{v}^2 \varsigma_{\rho}(\nu_{\zeta+1}, \nu_{\zeta+2}) \\
& \quad + \dots + \mathfrak{v}^{\zeta+\xi-1} [\varsigma_{\rho}(\nu_{\zeta+\xi-2}, \nu_{\zeta+\xi-1}) + \varsigma_{\rho}(\nu_{\zeta+\xi-1}, \nu_{\zeta+\xi})] \\
& \quad - \varsigma_{\rho}(\nu_{\zeta+1}, \nu_{\zeta+1}) - \varsigma_{\rho}(\nu_{\zeta+2}, \nu_{\zeta+2}) \dots - \varsigma_{\rho}(\nu_{\zeta+\xi-1}, \nu_{\zeta+\xi-1}) \\
& \preceq \mathfrak{v}(\mathfrak{a}^*)^{\zeta} \delta_0(\mathfrak{a})^{\zeta} + \mathfrak{v}^2(\mathfrak{a}^*)^{\zeta+1} \delta_0(\mathfrak{a})^{\zeta+1} + \dots + \mathfrak{v}^{\xi-1}(\mathfrak{a}^*)^{\zeta+\xi-2} \delta_0(\mathfrak{a})^{\zeta+\xi-2} \\
& \quad + \mathfrak{v}^{\xi-1}(\mathfrak{a}^*)^{\zeta+\xi-1} \delta_0(\mathfrak{a})^{\zeta+\xi-1} - (\mathfrak{a}^*)^{\zeta+1} \delta_0(\mathfrak{a})^{\zeta+1} - (\mathfrak{a}^*)^{\zeta+2} \delta_0(\mathfrak{a})^{\zeta+2} \\
& \quad - (\mathfrak{a}^*)^{\zeta+\xi-1} \delta_0(\mathfrak{a})^{\zeta+\xi-1} \\
& \preceq \sum_{\mathfrak{r}=1}^{\xi-1} \mathfrak{v}^{\mathfrak{r}}(\mathfrak{a}^*)^{\zeta+\mathfrak{r}-1} \delta_0(\mathfrak{a})^{\zeta+\mathfrak{r}-1} + \mathfrak{v}^{\xi-1}(\mathfrak{a}^*)^{\zeta+\xi-1} \delta_0(\mathfrak{a})^{\zeta+\xi-1} - \sum_{\mathfrak{r}=\zeta+1}^{\zeta+\xi-1} (\mathfrak{a}^*)^{\mathfrak{r}} \delta_0(\mathfrak{a})^{\mathfrak{r}} \\
& \preceq \sum_{\mathfrak{r}=1}^{\xi-1} \left((\mathfrak{a}^*)^{\zeta+\mathfrak{r}-1} \mathfrak{v}^{\frac{\mathfrak{r}}{2}} \delta_0^{\frac{1}{2}} \right) \left(\delta_0^{\frac{1}{2}} \mathfrak{v}^{\frac{\mathfrak{r}}{2}} (\mathfrak{a})^{\zeta+\mathfrak{r}-1} \right) - \sum_{\mathfrak{r}=\zeta+1}^{\zeta+\xi-1} \left((\mathfrak{a}^*)^{\mathfrak{r}} \delta_0^{\frac{1}{2}} \right) \left(\delta_0^{\frac{1}{2}} (\mathfrak{a})^{\mathfrak{r}} \right) \\
& \quad + \left((\mathfrak{a}^*)^{\zeta+\xi-1} \mathfrak{v}^{\frac{\xi-1}{2}} \delta_0^{\frac{1}{2}} \right) \left(\delta_0^{\frac{1}{2}} \mathfrak{v}^{\frac{\xi-1}{2}} (\mathfrak{a})^{\zeta+\xi-1} \right) \\
& \preceq \sum_{\mathfrak{r}=1}^{\xi-1} \left(\delta_0^{\frac{1}{2}} \mathfrak{v}^{\frac{\mathfrak{r}}{2}} \mathfrak{a}^{\zeta+\mathfrak{r}-1} \right)^* \left(\delta_0^{\frac{1}{2}} \mathfrak{v}^{\frac{\mathfrak{r}}{2}} \mathfrak{a}^{\zeta+\mathfrak{r}-1} \right) - \sum_{\mathfrak{r}=\zeta+1}^{\zeta+\xi-1} \left(\delta_0^{\frac{1}{2}} \mathfrak{a}^{\mathfrak{r}} \right)^* \left(\delta_0^{\frac{1}{2}} \mathfrak{a}^{\mathfrak{r}} \right) \\
& \quad + \left(\delta_0^{\frac{1}{2}} \mathfrak{v}^{\frac{\xi-1}{2}} \mathfrak{a}^{\zeta+\xi-1} \right)^* \left(\delta_0^{\frac{1}{2}} \mathfrak{v}^{\frac{\xi-1}{2}} \mathfrak{a}^{\zeta+\xi-1} \right) \\
& \preceq \sum_{\mathfrak{r}=1}^{\xi-1} |\delta_0^{\frac{1}{2}} \mathfrak{v}^{\frac{\mathfrak{r}}{2}} \mathfrak{a}^{\zeta+\mathfrak{r}-1}|^2 - \sum_{\mathfrak{r}=\zeta+1}^{\zeta+\xi-1} |\delta_0^{\frac{1}{2}} \mathfrak{a}^{\mathfrak{r}}|^2 + |\delta_0^{\frac{1}{2}} \mathfrak{v}^{\frac{\xi-1}{2}} \mathfrak{a}^{\zeta+\xi-1}|^2 \\
& \preceq \sum_{\mathfrak{r}=1}^{\xi-1} \|\delta_0^{\frac{1}{2}} \mathfrak{v}^{\frac{\mathfrak{r}}{2}} \mathfrak{a}^{\zeta+\mathfrak{r}-1}\|^2 1_{\mathcal{A}} - \sum_{\mathfrak{r}=\zeta+1}^{\zeta+\xi-1} \|\delta_0^{\frac{1}{2}} \mathfrak{a}^{\mathfrak{r}}\|^2 1_{\mathcal{A}} + \|\delta_0^{\frac{1}{2}} \mathfrak{v}^{\frac{\xi-1}{2}} \mathfrak{a}^{\zeta+\xi-1}\|^2 1_{\mathcal{A}}
\end{aligned}$$

$$\begin{aligned}
& \preceq \|\delta_0\| \left(\sum_{r=1}^{\xi-1} \|\mathfrak{v}^{\frac{\xi}{2}} \mathfrak{a}^{\zeta+r-1}\|^2 1_{\mathcal{A}} - \sum_{r=\zeta+1}^{\infty} \|\mathfrak{a}^r\|^2 1_{\mathcal{A}} + \|\mathfrak{v}^{\frac{\xi-1}{2}} \mathfrak{a}^{\zeta+\xi-1}\|^2 1_{\mathcal{A}} \right) \\
& \preceq \|\delta_0\| \left(\sum_{r=1}^{\xi-1} \|\mathfrak{v}^r\| \|\mathfrak{a}^{2(\zeta+r-1)}\| 1_{\mathcal{A}} - \sum_{r=\zeta+1}^{\infty} \|\mathfrak{a}^{2r}\| 1_{\mathcal{A}} + \|\mathfrak{v}^{\xi-1}\| \|\mathfrak{a}^{2(\zeta+\xi-1)}\| 1_{\mathcal{A}} \right) \\
& \preceq \|\delta_0\| \left(\|\mathfrak{v}\| \|\mathfrak{a}^{2\zeta}\| \frac{1 - (\|\mathfrak{v}\| \|\mathfrak{a}\|^2)^{\xi-1}}{1 - \|\mathfrak{v}\| \|\mathfrak{a}\|^2} - \frac{\|\mathfrak{a}^{2\zeta}\|^{\zeta+1}}{1 - \|\mathfrak{a}\|^2} + \|\mathfrak{v}^{\xi-1}\| \|\mathfrak{a}^{2\zeta+\xi-1}\| 1_{\mathcal{A}} \right) \\
(4) \quad & \rightarrow 0 \text{ as } \zeta, \xi \rightarrow \infty.
\end{aligned}$$

Similarly, we can prove $\varsigma_p(\mu_\zeta, \mu_{\zeta+\xi}) \rightarrow 0_{\mathcal{A}}$ as $\zeta, \xi \rightarrow \infty$. Consequently, the series $\{\nu_\zeta\}$ and $\{\mu_\zeta\}$ are Cauchy sequences. Concerning the completeness of the C^* -avpbms $(\mathcal{G}, \mathcal{A}, \varsigma_p)$, we conclude that there exists an element $\nu, \mu \in \mathcal{G}$ such that

$$\lim_{\zeta, \xi \rightarrow \infty} \varsigma_p(\nu_\zeta, \nu_\xi) = \lim_{\zeta \rightarrow \infty} \varsigma_p(\nu_\zeta, \nu) = \varsigma_p(\nu, \nu) \quad \lim_{\zeta, \xi \rightarrow \infty} \varsigma_p(\mu_\zeta, \mu_\xi) = \lim_{\zeta \rightarrow \infty} \varsigma_p(\mu_\zeta, \mu) = \varsigma_p(\mu, \mu).$$

By employing Eq.(4), we have

$$(5) \quad \lim_{\zeta \rightarrow \infty} \varsigma_p(\nu_\zeta, \nu) = \varsigma_p(\nu, \nu) = 0_{\mathcal{A}} \quad \lim_{\zeta \rightarrow \infty} \varsigma_p(\mu_\zeta, \mu) = \varsigma_p(\mu, \mu) = 0_{\mathcal{A}}.$$

Given that $\mathbb{T}$ is continuous, we obtain

$$\nu = \lim_{\zeta \rightarrow \infty} \nu_{\zeta+1} = \lim_{\zeta \rightarrow \infty} \mathbb{T}(\nu_\zeta, \mu_\zeta) = \mathbb{T}\left(\lim_{\zeta \rightarrow \infty} \nu_\zeta, \lim_{\zeta \rightarrow \infty} \mu_\zeta\right) = \mathbb{T}(\nu, \mu)$$

and

$$\mu = \lim_{\zeta \rightarrow \infty} \mu_{\zeta+1} = \lim_{\zeta \rightarrow \infty} \mathbb{T}(\mu_\zeta, \nu_\zeta) = \mathbb{T}\left(\lim_{\zeta \rightarrow \infty} \mu_\zeta, \lim_{\zeta \rightarrow \infty} \nu_\zeta\right) = \mathbb{T}(\mu, \nu).$$

Hence, (ν, μ) is coupled fixed point of $\mathbb{T}$ in $\mathcal{G}$.

Theorem 3.7: Let $\mathbb{T} : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ be a mapping defined on the complete C^* -avpbms $(\mathcal{G}, \mathcal{A}, \varsigma_p)$ with coefficient $\mathfrak{v} \succeq 1_{\mathcal{A}}$ that follows the subsequent circumstances:

- (a) $\mathbb{T}$ is α_p -orbital admissible,
- (b) There exist $\nu_0, \mu_0 \in \mathcal{G}$ that gives $\alpha_p(\nu_0, \mathbb{T}(\nu_0, \mu_0)) \succeq 1_{\mathcal{A}}$ and $\alpha_p(\mu_0, \mathbb{T}(\mu_0, \nu_0)) \succeq 1_{\mathcal{A}}$,
- (c) If $\{\nu_\zeta\}, \{\mu_\zeta\} \subseteq \mathcal{G}$ such that $\alpha_p(\nu_\zeta, \nu_{\zeta+1}) \succeq 1_{\mathcal{A}}$ and $\alpha_p(\mu_\zeta, \mu_{\zeta+1}) \succeq 1_{\mathcal{A}}$ for each ζ and $\nu_\zeta \rightarrow \nu \in \mathcal{G}$ and $\mu_\zeta \rightarrow \mu \in \mathcal{G}$ as $\zeta \rightarrow \infty$, then there exists $\{\nu_{\zeta(z)}\}$ from $\{\nu_\zeta\}$ and $\{\mu_{\zeta(z)}\}$ from $\{\mu_\zeta\}$ such that $\alpha_p(\nu_{\zeta(k)}, \nu) \succeq 1_{\mathcal{A}}$ and $\alpha_p(\mu_{\zeta(k)}, \mu) \succeq 1_{\mathcal{A}}$ for each k ,
- (d) $\mathbb{T}$ is α_p -interpolative contraction.

Then a coupled fixed point of $\mathbb{T}$ exists in $\mathcal{G}$.

Proof According to the proof of Theorem 3.6, we deduce that the sequence $\{\nu_\zeta\}$ and $\{\mu_\zeta\}$ are Cauchy and that Eq.(5) is satisfied. Assume that condition (c) is satisfied. We employ proof by contradiction by presuming that $\nu \neq \mathbb{T}(\nu, \mu)$ and $\mu \neq \mathbb{T}(\mu, \nu)$. Note that $\nu_{\zeta(k)} \neq \mathbb{T}(\nu_{\zeta(k)}, \mu_{\zeta(k)})$ and $\mu_{\zeta(k)} \neq \mathbb{T}(\mu_{\zeta(k)}, \nu_{\zeta(k)})$ for every $k \geq 0$. As a result of (c), there exists a partial subsequence $\{\nu_{\zeta(k)}\}$ of $\{\nu_\zeta\}$ and $\{\mu_{\zeta(k)}\}$ of $\{\mu_\zeta\}$ such that $\alpha_p(\nu_{\zeta(k)}, \nu) \succeq 1_{\mathcal{A}}$ and $\alpha_p(\mu_{\zeta(k)}, \mu) \succeq 1_{\mathcal{A}}$ for all k . Since, $\varsigma_p(\nu_{\zeta(k)}, \nu) \rightarrow 0_{\mathcal{A}}$, $\varsigma_p(\mu_{\zeta(k)}, \mu) \rightarrow 0_{\mathcal{A}}$, $\varsigma_p(\nu_{\zeta(k)}, \mathbb{T}(\nu_{\zeta(k)}, \mu_{\zeta(k)})) \rightarrow 0_{\mathcal{A}}$, $\varsigma_p(\mu_{\zeta(k)}, \mathbb{T}(\mu_{\zeta(k)}, \nu_{\zeta(k)})) \rightarrow 0_{\mathcal{A}}$ and $\varsigma_p(\nu, \mathbb{T}(\nu, \mu)) \succ 0_{\mathcal{A}}$, $\varsigma_p(\mu, \mathbb{T}(\mu, \nu)) \succ 0_{\mathcal{A}}$.

There occurs $n \in \mathbb{N}$ such that, for every $k \geq n$,

$$\varsigma_p(\nu_{\zeta(k)}, \nu) \prec \varsigma_p(\nu, \mathbb{T}(\nu, \mu)), \quad \varsigma_p(\mu_{\zeta(k)}, \mu) \prec \varsigma_p(\mu, \mathbb{T}(\mu, \nu))$$

and

$$\varsigma_p(\nu_{\zeta(k)}, \mathbb{T}(\nu_{\zeta(k)}, \mu_{\zeta(k)})) \prec \varsigma_p(\nu, \mathbb{T}(\nu, \mu)), \quad \varsigma_p(\mu_{\zeta(k)}, \mathbb{T}(\mu_{\zeta(k)}, \nu_{\zeta(k)})) \prec \varsigma_p(\mu, \mathbb{T}(\mu, \nu)).$$

Then, from Eq.(1), we have

$$\begin{aligned} \wp(\varsigma_p(\nu_{\zeta(k)+1}, \mathbb{T}(\nu, \mu))) &= \wp(\varsigma_p(\mathbb{T}(\nu_{\zeta(k)}, \mu_{\zeta(k)}), \mathbb{T}(\nu, \mu))) \\ &\preceq \alpha_p(\nu_{\zeta(k)}, \nu) \alpha_p(\mu_{\zeta(k)}, \mu) \wp(\varsigma_p(\mathbb{T}(\nu_{\zeta(k)}, \mu_{\zeta(k)}), \mathbb{T}(\nu, \mu))) \\ (6) \quad &\preceq \wp \left(\begin{array}{c} \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta(k)}, \nu) \\ \varsigma_p(\mu_{\zeta(k)}, \mu) \end{array} \right\} \alpha \right]^\kappa \\ \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu_{\zeta(k)}, \mathbb{T}(\nu_{\zeta(k)}, \mu_{\zeta(k)})) \\ \varsigma_p(\mu_{\zeta(k)}, \mathbb{T}(\mu_{\zeta(k)}, \nu_{\zeta(k)})) \end{array} \right\} \alpha \right]^\eta \\ \left[\max \left\{ \begin{array}{c} \varsigma_p(\nu, \mathbb{T}(\nu, \mu)) \\ \varsigma_p(\mu, \mathbb{T}(\mu, \nu)) \end{array} \right\} \right]^{1-\kappa-\eta} \end{array} \right). \end{aligned}$$

Since $\wp$ is nondecreasing, it emerges from Eq.(6) that

$$\varsigma_p(\nu_{\zeta(k)+1}, \mathbb{T}(\nu, \mu)) \preceq \begin{array}{c} \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu, \mathbb{T}(\nu, \mu)) \\ \varsigma_p(\mu, \mathbb{T}(\mu, \nu)) \end{array} \right\} \alpha \right]^\kappa \left[\alpha^* \max \left\{ \begin{array}{c} \varsigma_p(\nu, \mathbb{T}(\nu, \mu)) \\ \varsigma_p(\mu, \mathbb{T}(\mu, \nu)) \end{array} \right\} \alpha \right]^\eta \\ \left[\max \left\{ \begin{array}{c} \varsigma_p(\nu, \mathbb{T}(\nu, \mu)) \\ \varsigma_p(\mu, \mathbb{T}(\mu, \nu)) \end{array} \right\} \right]^{1-\kappa-\eta} \end{array}.$$

Letting $k \rightarrow \infty$, we find that

$$\zeta_p(\nu, \mathbb{T}(\nu, \mu)) \preceq \left[\alpha^* \max \left\{ \zeta_p(\nu, \mathbb{T}(\nu, \mu)), \zeta_p(\mu, \mathbb{T}(\mu, \nu)) \right\} \alpha \right]^{\kappa+\eta} \left[\max \left\{ \zeta_p(\nu, \mathbb{T}(\nu, \mu)), \zeta_p(\mu, \mathbb{T}(\mu, \nu)) \right\} \right]^{1-\kappa-\eta}. \quad (7)$$

Similarly, we get that

$$\zeta_p(\mu, \mathbb{T}(\mu, \nu)) \preceq \left[\alpha^* \max \left\{ \zeta_p(\nu, \mathbb{T}(\nu, \mu)), \zeta_p(\mu, \mathbb{T}(\mu, \nu)) \right\} \alpha \right]^{\kappa+\eta} \left[\max \left\{ \zeta_p(\nu, \mathbb{T}(\nu, \mu)), \zeta_p(\mu, \mathbb{T}(\mu, \nu)) \right\} \right]^{1-\kappa-\eta}. \quad (8)$$

Combining Eq.(7) and Eq.(8), we have

$$\left[\max \left\{ \zeta_p(\nu, \mathbb{T}(\nu, \mu)), \zeta_p(\mu, \mathbb{T}(\mu, \nu)) \right\} \right]^{\kappa+\eta} \preceq \left[\alpha^* \max \left\{ \zeta_p(\nu, \mathbb{T}(\nu, \mu)), \zeta_p(\mu, \mathbb{T}(\mu, \nu)) \right\} \alpha \right]^{\kappa+\eta}.$$

Therefore,

$$\begin{aligned} 0 \leq \max \left\{ \begin{array}{l} \|\zeta_p(\nu, \mathbb{T}(\nu, \mu))\|, \\ \|\zeta_p(\mu, \mathbb{T}(\mu, \nu))\| \end{array} \right\} &\leq \|\alpha\|^2 \max \left\{ \begin{array}{l} \|\zeta_p(\nu, \mathbb{T}(\nu, \mu))\|, \\ \|\zeta_p(\mu, \mathbb{T}(\mu, \nu))\| \end{array} \right\} \\ &< \max \left\{ \begin{array}{l} \|\zeta_p(\nu, \mathbb{T}(\nu, \mu))\|, \\ \|\zeta_p(\mu, \mathbb{T}(\mu, \nu))\| \end{array} \right\} \end{aligned}$$

which is a contradiction. Thus $\mathbb{T}(\nu, \mu) = \nu$ and $\mathbb{T}(\mu, \nu) = \mu$.

Corollary 3.8: Let $\mathbb{T} : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ be a mapping defined on the complete C^* -avpbms $(\mathcal{G}, \mathcal{A}, \zeta_p)$ with coefficient $\mathfrak{v} \succeq 1_{\mathcal{A}}$ such that

$$\zeta_p(\mathbb{T}(\mathfrak{s}, \mathfrak{e}), \mathbb{T}(\mathfrak{x}, \mathfrak{z})) \preceq \left[\alpha^* \max \left\{ \zeta_p(\mathfrak{s}, \mathfrak{x}), \zeta_p(\mathfrak{e}, \mathfrak{z}) \right\} \alpha \right]^{\kappa} \left[\alpha^* \max \left\{ \zeta_p(\mathfrak{s}, \mathbb{T}(\mathfrak{s}, \mathfrak{e})), \zeta_p(\mathfrak{e}, \mathbb{T}(\mathfrak{e}, \mathfrak{s})) \right\} \alpha \right]^{\eta} \left[\max \left\{ \zeta_p(\mathfrak{x}, \mathbb{T}(\mathfrak{x}, \mathfrak{z})), \zeta_p(\mathfrak{z}, \mathbb{T}(\mathfrak{z}, \mathfrak{x})) \right\} \right]^{1-\kappa-\eta}.$$

$\forall \mathfrak{s}, \mathfrak{e}, \mathfrak{x}, \mathfrak{z} \in \mathcal{G} - \mathbb{F}(\mathbb{T})$ where $\kappa, \eta \in (0, 1)$ and $\alpha \in \mathcal{A}$ with $\|\alpha \mathfrak{v}\| < 1$. Then $\mathbb{T}$ possesses a coupled fixed point in $\mathcal{G}$.

Proof Invoke Theorem 3.6 with the assignments $\alpha_p(\mathfrak{s}, \mathfrak{x}) = 1_{\mathcal{A}}$, $\alpha_p(\mathfrak{e}, \mathfrak{z}) = 1_{\mathcal{A}}$, and $\mathcal{P}(\mathfrak{t}) = \mathfrak{t}$.

The desired result follows immediately.

Corollary 3.9: Let $\mathbb{T} : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ be a mapping defined on the complete C^* -avpbms $(\mathcal{G}, \mathcal{A}, \varsigma_\rho)$ with coefficient $\mathfrak{v} \succeq 1_{\mathcal{A}}$ such that

$$\varsigma_\rho(\mathbb{T}(\mathfrak{s}, \mathfrak{e}), \mathbb{T}(\mathfrak{x}, \mathfrak{z})) \preceq \mathfrak{a}^* \max \left\{ \varsigma_\rho(\mathfrak{s}, \mathfrak{x}), \varsigma_\rho(\mathfrak{e}, \mathfrak{z}) \right\} \mathfrak{a}.$$

$\forall \mathfrak{s}, \mathfrak{e}, \mathfrak{x}, \mathfrak{z} \in \mathcal{G} - \mathbb{F}(\mathbb{T})$ where $\mathfrak{a} \in \mathcal{A}$ with $\|\mathfrak{a}\mathfrak{v}\| < 1$. Then $\mathbb{T}$ possesses a coupled fixed point in $\mathcal{G}$.

Proof Setting $\kappa = 1$ and $\eta = 0$ in Corollary 3.8 yields the result directly.

Example 3.10: Consider the C^* -algebra $\mathcal{A} = \mathcal{M}_2(\mathbb{R})$ equipped with the operator norm and partial order $\mathfrak{a} \preceq \mathfrak{b}$ whenever $\mathfrak{b} - \mathfrak{a}$ is positive semi-definite. Consider $\mathcal{G} = [0, \infty)$ equipped with the C^* -algebra valued partial b -metric

$$\varsigma_\rho(x, y) = \begin{pmatrix} |x-y|^2 & 0 \\ 0 & 2|x-y|^2 \end{pmatrix}, \quad \mathfrak{v} = \begin{pmatrix} 1.2 & 0 \\ 0 & 1.2 \end{pmatrix} \succeq I_2.$$

The space $(\mathcal{G}, \mathcal{A}, \varsigma_\rho)$ is complete C^* -avpbms.

Define the continuous mapping $\mathbb{T} : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ as $\mathbb{T}(s, e) = 0.25s + 0.35e$, with $\wp(t) = t \in \Delta$, $\alpha_p(s, x) = I_2$, $\kappa = 0.3$, $\eta = 0.2$, $\mathfrak{a} = 0.6I_2$.

Then clearly, $\mathfrak{a}\mathfrak{v} = 0.6I_2 \cdot 1.2I_2 = 0.72I_2$, $\|\mathfrak{a}\mathfrak{v}\| = 0.72 < 1$.

For(a): $\mathbb{T}$ is affine (linear combination), hence continuous on $\mathcal{G}$.

For(b): Since $\alpha_p(s, x) = I_2 \succeq I_2$ for all s, x , the implication holds trivially.

For(c): Take $\mathfrak{v}_0 = \mu_0 = 0.5$. Then $\mathbb{T}(0.5, 0.5) = 0.25(0.5) + 0.35(0.5) = 0.15$. Then

$\alpha_p(0.5, 0.15) = I_2 \succeq I_2$.

For(d): $\varsigma_\rho(\mathbb{T}(s, e), \mathbb{T}(x, z)) = \begin{pmatrix} |0.25(s-x) + 0.35(e-z)|^2 & 0 \\ 0 & 2|0.25(s-x) + 0.35(e-z)|^2 \end{pmatrix}.$

By triangle inequality:

$$|0.25(s-x) + 0.35(e-z)| \leq 0.25|s-x| + 0.35|e-z| \leq 0.6 \max\{|s-x|, |e-z|\}.$$

Thus LHS $\preceq (0.6)^2 = 0.36$ times reference metrics.

RHS: $[\mathfrak{a}^* M \mathfrak{a}]^{0.3} = (0.72)^{0.3} \approx 0.90 > 0.36$, so inequality holds.

Hence, all conditions of Theorem 3.6 satisfied. The system $\mathbb{T}(s, e) = s$, $\mathbb{T}(e, s) = e$ gives $s = e = 0$, confirming coupled fixed point $(0, 0)$.

Example 3.11: Let $\mathcal{A} = \mathcal{M}_2(\mathbb{R})$ with operator norm and positive semi-definite partial order. Consider $\mathcal{G} = [0, \infty)$ equipped with the C^* -algebra valued partial b -metric

$$\varsigma_\rho(x, y) = \begin{pmatrix} |x-y|^2 & 0 \\ 0 & 2|x-y|^2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.5 \end{pmatrix} \succeq I_2.$$

The space $(\mathcal{G}, \mathcal{A}, \varsigma_\rho)$ is complete C^* -avpbms.

Define the piecewise mapping $\mathbb{T} : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ as $\mathbb{T}(s, e) = \begin{cases} 0.3(s+e) & \text{if } s+e \leq 1, \\ 0 & \text{if } s+e > 1, \end{cases}$

with $\wp(t) = t \in \Delta$, $\alpha_p(s, x) = I_2$, $\kappa = 0.4$, $\eta = 0.3$, $\mathbf{a} = 0.4I_2$.

Then clearly,

$$\mathbf{av} = 0.4I_2 \cdot 1.5I_2 = 0.6I_2, \quad \|\mathbf{av}\| = 0.6 < 1.$$

For(a): Since $\alpha_p(s, x) = I_2 \succeq I_2$ for all $s, x \in \mathcal{G}$, the implication holds trivially.

For(b): Take $\nu_0 = \mu_0 = 0.3$. Then $s+e = 0.6 \leq 1$, so $\mathbb{T}(0.3, 0.3) = 0.3(0.6) = 0.18$. Then $\alpha_p(0.3, 0.18) = I_2 \succeq I_2$, $\alpha_p(0.3, 0.18) = I_2 \succeq I_2$.

For(c): Let $\{\nu_\zeta\}, \{\mu_\zeta\}$ converge to ν, μ with $\alpha_p(\nu_\zeta, \nu_{\zeta+1}) \succeq I_2$. Since $\alpha_p \equiv I_2$, every subsequence satisfies $\alpha_p(\nu_{\zeta(k)}, \nu) = I_2 \succeq I_2$.

For(d): *Case 1:* $s+e \leq 1, x+z \leq 1$. Then $\mathbb{T}(s, e) = 0.3(s+e)$, $\mathbb{T}(x, z) = 0.3(x+z)$. Compute:

$$\varsigma_\rho(\mathbb{T}(s, e), \mathbb{T}(x, z)) = 0.09 \begin{pmatrix} |s+e-x-z|^2 & 0 \\ 0 & 2|s+e-x-z|^2 \end{pmatrix}.$$

By triangle inequality $|s+e-x-z| \leq |s-x| + |e-z|$, LHS $\preceq 0.09\varsigma_\rho(s, x)$. RHS starts with $(0.6)^{0.4} \approx 0.802 > 0.09$, so inequality holds.

Case 2: $s+e > 1, x+z > 1$. Then $\mathbb{T}(s, e) = \mathbb{T}(x, z) = 0$, so LHS = $0_{\mathcal{A}} \preceq$ RHS.

Case 3: $s+e > 1, x+z \leq 1$. Then $\mathbb{T}(s, e) = 0$, $\mathbb{T}(x, z) = 0.3(x+z)$. LHS:

$$\varsigma_\rho(0, 0.3(x+z)) = \begin{pmatrix} [0.3(x+z)]^2 & 0 \\ 0 & 2[0.3(x+z)]^2 \end{pmatrix} = 0.09\varsigma_\rho(0, x+z).$$

RHS contains $\wp(\varsigma_\rho(s, 0)) = \varsigma_\rho(s, 0) \succeq s^2 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Since $s > 1-e \geq 0$, this dominates LHS.

Case 4: Symmetric to Case 3.

Hence, all conditions (a)-(d) satisfied. By Theorem 3.7, a coupled fixed point exists. Indeed, $(0, 0)$ satisfies $\mathbb{T}(0, 0) = 0.3(0 + 0) = 0$.

4. APPLICATIONS

4.1. Application to the existence of solutions of integral equations.

In this part, we demonstrate how to prove the existence and uniqueness of a solution to a system of non-linear Fredholm integral equations using the coupled fixed point results obtained in our Corollary 3.9.

Let $\mathcal{G} = L^\infty(\mathcal{E})$ be the space of essentially bounded measurable functions on $\mathcal{E}$ and $\mathcal{H} = L^2(\mathcal{E})$ where $\mathcal{E}$ is a Lebesgue space. The set of bounded linear operators on Hilbert space $\mathcal{H}$, denoted by $L(\mathcal{H})$, is a C^* -algebra with operator norm $\|A\| = \sup_{\|a\|_{\mathcal{H}}=1} \|Aa\|_{\mathcal{H}}$. We equip $\mathcal{G}$ with $\zeta_\rho : \mathcal{G} \times \mathcal{G} \rightarrow L(\mathcal{H})$, defined by $\zeta_\rho(\mathfrak{p}, \mathfrak{q}) = \mathbb{M}_{|\mathfrak{p}-\mathfrak{q}|^t + I}$ for all $\mathfrak{p}, \mathfrak{q} \in \mathcal{G}$, $I \in \mathcal{A}$, $t \geq 1$, where $\mathbb{M}_\phi : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ is defined by $\mathbb{M}_\phi(\alpha) = \phi \circ \alpha$ (operator composition) for $\alpha \in \mathcal{H}$ and $\phi \in L(\mathcal{H})$. Therefore, $(\mathcal{G}, L(\mathcal{H}), \zeta_\rho)$ is a complete C^* -avpbms.

The following system of non-linear Fredholm integral equations is now examined:

$$(9) \quad \begin{cases} \mathfrak{x}(\mathfrak{v}) &= \mathfrak{f}(\mathfrak{v}) + \int_{\mathcal{E}} \mathcal{K}(\mathfrak{v}, u, \mathfrak{x}(u), \mathfrak{y}(u)) du, \mathfrak{v}, u \in \mathcal{E} \\ \mathfrak{y}(\mathfrak{v}) &= \mathfrak{f}(\mathfrak{v}) + \int_{\mathcal{E}} \mathcal{K}(\mathfrak{v}, u, \mathfrak{y}(u), \mathfrak{x}(u)) du, \mathfrak{v}, u \in \mathcal{E} \end{cases}$$

where $\mathcal{K} : \mathcal{E}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and $\mathfrak{f} : \mathcal{E} \rightarrow \mathbb{R}$ are continuous.

Define the operator $\mathbb{T} : \mathcal{G}^2 \rightarrow \mathcal{G}$ by $\mathbb{T}(\mathfrak{x}, \mathfrak{y})(\mathfrak{v}) = \mathfrak{f}(\mathfrak{v}) + \int_{\mathcal{E}} \mathcal{K}(\mathfrak{v}, u, \mathfrak{x}(u), \mathfrak{y}(u)) du$.

Note that Eq. (9) has a coupled solution if and only if $\mathbb{T}$ has a coupled fixed point (CFP).

Theorem 4.1.1: Suppose that, for all $\mathfrak{z}, \mathfrak{w}, \mathfrak{x}, \mathfrak{y} \in \mathcal{G}$,

(a) there exist a continuous function $\phi : \mathcal{E}^2 \rightarrow [0, \infty)$ and $\xi \in (0, 1)$ such that

$$|\mathcal{K}(\mathfrak{v}, u, \mathfrak{z}(u), \mathfrak{w}(u)) - \mathcal{K}(\mathfrak{v}, u, \mathfrak{x}(u), \mathfrak{y}(u))| \leq \xi |\phi(\mathfrak{v}, u)| \left(\max \left\{ \begin{array}{l} |\mathfrak{z}(u) - \mathfrak{x}(u)|, \\ |\mathfrak{w}(u) - \mathfrak{y}(u)| \end{array} \right\} + I - \xi^{-1} I \right)$$

for all $\mathfrak{v}, u \in \mathcal{E}$,

(b) $\sup_{\mathfrak{v} \in \mathcal{E}} \int_{\mathcal{E}} |\phi(\mathfrak{v}, u)| du \leq 1$.

Then the system of integral equations (9) has a unique solution.

Proof This proof will be carried out using Corollary 3.9. Because the function $\mathcal{K}$ is continuous, so is the operator $\mathbb{T} : \mathcal{G}^2 \rightarrow \mathcal{G}$ defined above.

Let $\kappa = \xi I$, then $\kappa \in \mathcal{A}$. For any $\mathfrak{k} \in \mathcal{H}$ and $t \geq 1$, we have

$$\begin{aligned}
& \|\zeta_p(\mathbb{T}(\mathfrak{z}, \mathfrak{w}), \mathbb{T}(\mathfrak{x}, \mathfrak{y}))\| = \sup_{\|\mathfrak{k}\|=1} (\mathbb{M}_{|\mathbb{T}(\mathfrak{z}, \mathfrak{w}) - \mathbb{T}(\mathfrak{x}, \mathfrak{y})|^{t+I}} \mathfrak{k}, \mathfrak{k}) \\
&= \sup_{\|\mathfrak{k}\|=1} \int_{\mathcal{E}} \left[\left| \int_{\mathcal{E}} (\mathcal{K}(\mathfrak{v}, \mathfrak{u}, \mathfrak{z}(\mathfrak{u}), \mathfrak{w}(\mathfrak{u})) - \mathcal{K}(\mathfrak{v}, \mathfrak{u}, \mathfrak{x}(\mathfrak{u}), \mathfrak{y}(\mathfrak{u}))) d\mathfrak{u} \right|^t \right] \mathfrak{k}(\mathfrak{v}) \overline{\mathfrak{k}(\mathfrak{v})} d\mathfrak{v} + \sup_{\|\mathfrak{k}\|=1} \int_{\mathcal{E}} \mathfrak{k}(\mathfrak{v}) \overline{\mathfrak{k}(\mathfrak{v})} d\mathfrak{v} \cdot I \\
&\leq \sup_{\|\mathfrak{k}\|=1} \int_{\mathcal{E}} \left[\left| \int_{\mathcal{E}} |\mathcal{K}(\mathfrak{v}, \mathfrak{u}, \mathfrak{z}(\mathfrak{u}), \mathfrak{w}(\mathfrak{u})) - \mathcal{K}(\mathfrak{v}, \mathfrak{u}, \mathfrak{x}(\mathfrak{u}), \mathfrak{y}(\mathfrak{u}))| d\mathfrak{u} \right|^t \right] |\mathfrak{k}(\mathfrak{v})|^2 d\mathfrak{v} + \sup_{\|\mathfrak{k}\|=1} \int_{\mathcal{E}} |\mathfrak{k}(\mathfrak{v})|^2 d\mathfrak{v} \cdot I \\
&\leq \sup_{\|\mathfrak{k}\|=1} \int_{\mathcal{E}} \left[\int_{\mathcal{E}} \xi |\phi(\mathfrak{v}, \mathfrak{u})| \left(\max \left\{ \begin{array}{l} |\mathfrak{z}(\mathfrak{u}) - \mathfrak{x}(\mathfrak{u})|, \\ |\mathfrak{w}(\mathfrak{u}) - \mathfrak{y}(\mathfrak{u})| \end{array} \right\} + I - \xi^{-1} I \right) d\mathfrak{u} \right]^t |\mathfrak{k}(\mathfrak{v})|^2 d\mathfrak{v} + I \\
&\leq \xi^t \sup_{\|\mathfrak{k}\|=1} \int_{\mathcal{E}} \left[\int_{\mathcal{E}} |\phi(\mathfrak{v}, \mathfrak{u})| d\mathfrak{u} \right]^t |\mathfrak{k}(\mathfrak{v})|^2 d\mathfrak{v} \max \left\{ \begin{array}{l} \|\mathfrak{z}(\mathfrak{u}) - \mathfrak{x}(\mathfrak{u})\|_{\infty}^t, \\ \|\mathfrak{w}(\mathfrak{u}) - \mathfrak{y}(\mathfrak{u})\|_{\infty}^t \end{array} \right\} + I \\
&\leq \xi \sup_{\mathfrak{v} \in \mathcal{E}} \int_{\mathcal{E}} |\phi(\mathfrak{v}, \mathfrak{u})| d\mathfrak{u} \sup_{\|\mathfrak{k}\|=1} \int_{\mathcal{E}} |\mathfrak{k}(\mathfrak{v})|^2 d\mathfrak{v} \cdot \max \left\{ \begin{array}{l} \|\mathfrak{z}(\mathfrak{u}) - \mathfrak{x}(\mathfrak{u})\|_{\infty}^t, \\ \|\mathfrak{w}(\mathfrak{u}) - \mathfrak{y}(\mathfrak{u})\|_{\infty}^t \end{array} \right\} + I \\
&\leq \xi \max \left\{ \begin{array}{l} \|\mathfrak{z}(\mathfrak{u}) - \mathfrak{x}(\mathfrak{u})\|_{\infty}^t, \\ \|\mathfrak{w}(\mathfrak{u}) - \mathfrak{y}(\mathfrak{u})\|_{\infty}^t \end{array} \right\} + I \\
&= \|\kappa\| \max \left\{ \|\zeta_p(\mathfrak{z}, \mathfrak{x})\|, \|\zeta_p(\mathfrak{w}, \mathfrak{y})\| \right\}
\end{aligned}$$

This shows that $\mathbb{T}$ satisfies the contraction with $\|\kappa\| < 1$. We thus infer that $\mathbb{T}$ has a unique coupled solution in $\mathcal{G}$ to the integral equation (9) since all the requirements of the Corollary 3.9 are met.

Example 4.1.2: Consider the symmetric coupled system of nonlinear Fredholm integral equations on $\mathcal{E} = [0, 1]$:

$$(10) \quad x(\mathfrak{v}) = \int_0^1 K(\mathfrak{v}, \mathfrak{u}, x(\mathfrak{u}), y(\mathfrak{u})) d\mathfrak{u},$$

$$(11) \quad y(\mathfrak{v}) = \int_0^1 K(\mathfrak{v}, \mathfrak{u}, y(\mathfrak{u}), x(\mathfrak{u})) d\mathfrak{u},$$

where the kernel is defined by

$$K(v, u, r, s) = \frac{vu}{5} \left(e^{r/2} + e^{s/2} \right), \quad v, u \in [0, 1], \quad r, s \in \mathbb{R}.$$

Define the space $\mathcal{G} = L^\infty([0, 1])$ equipped with the C^* -algebra valued partial b -metric

$\zeta_\rho(\mathfrak{p}, \mathfrak{q}) = \mathbb{M}_{|\mathfrak{p}-\mathfrak{q}|^2+I}$, $\mathfrak{p}, \mathfrak{q} \in \mathcal{G}$, where $\mathcal{A} = L(\mathcal{H})$, $\mathcal{H} = L^2([0, 1])$, I is the identity operator, and $\mathbb{M}_\phi(\alpha) = \phi \diamond \alpha$ denotes right composition.

Then the System (10)–(11) possesses a unique coupled solution $(x, y) \in \mathcal{G}^2$.

Let us, consider the operator $\mathbb{T} : \mathcal{G}^2 \rightarrow \mathcal{G}^2$ by $\mathbb{T}(x, y)(v) = \int_0^1 K(v, u, x(u), y(u)) du$.

Verify the contraction condition of Corollary 3.9 with $\xi = \frac{1}{5}$. For $(x, y), (z, w) \in \mathcal{G}^2$,

$$\begin{aligned} |K(v, u, z(u), w(u)) - K(v, u, x(u), y(u))| &= \frac{vu}{5} \left| e^{z(u)/2} + e^{w(u)/2} - e^{x(u)/2} - e^{y(u)/2} \right| \\ &\leq \frac{vu}{5} \max \left\{ |e^{z(u)/2} - e^{x(u)/2}|, |e^{w(u)/2} - e^{y(u)/2}| \right\} \\ &\leq \frac{vu}{5} \cdot \frac{1}{2} \max \{ |z(u) - x(u)|, |w(u) - y(u)| \}, \end{aligned}$$

using the mean value theorem ($|e^a - e^b| \leq \frac{1}{2}|a - b|$ for bounded domain). Thus,

$$|K(v, u, z(u), w(u)) - K(v, u, x(u), y(u))| \leq \frac{1}{5} \cdot vu \cdot \max \{ |z(u) - x(u)|, |w(u) - y(u)| \}.$$

Set $\phi(v, u) = vu$. Then condition (a) holds and

$$\sup_{v \in [0, 1]} \int_0^1 |\phi(v, u)| du = \sup_v \int_0^1 v u du = \sup_v \frac{v}{2} = \frac{1}{2} \leq 1,$$

satisfying (b). With $\kappa = \xi I = \frac{1}{5}I$, we have $\|\kappa\| = \frac{1}{5} < 1$.

Since $(\mathcal{G}, L(\mathcal{H}), \zeta_\rho)$ is complete C^* -avpbms and $\mathbb{T}$ is continuous, Corollary 3.9 guarantees a unique coupled fixed point $(x, y) \in \mathcal{G}^2$, which solves (10)–(11).

Remark: The trivial solution $x \equiv y \equiv 0$ satisfies the system. Uniqueness follows from the strict contraction $\frac{1}{5} < 1$, excluding other solutions regardless of initial approximation.

Example 4.1.3: Consider $\mathcal{E} = [0, 1]$ with Lebesgue measure and define

$$\mathcal{H}(v, u, x(u), y(u)) = e^{-v-u} \sin(x(u)) \cos(y(u)),$$

$$\mathfrak{f}(v) = v^2,$$

with $\phi(v, u) = e^{-u}$ and $\xi = \frac{1}{2}$. Then

$$\begin{aligned} |\mathcal{K}(v, u, z(u), w(u)) - \mathcal{K}(v, u, x(u), y(u))| &= e^{-v-u} |\sin(z(u)) \cos(w(u)) - \sin(x(u)) \cos(y(u))| \\ &\leq e^{-v-u} \cdot \frac{1}{2} (|z(u) - x(u)| + |w(u) - y(u)|) \\ &= \xi |\phi(v, u)| \max \{ |z(u) - x(u)|, |w(u) - y(u)| \} \end{aligned}$$

and

$$\sup_{v \in [0, 1]} \int_0^1 |\phi(v, u)| du = 1 - e^{-1} < 1.$$

The unique coupled solution is

$$x(v) = y(v) = v^2 + \frac{1 - e^{-1}}{2} \sin(v^2) \cos(v^2), \quad v \in [0, 1].$$

Direct substitution confirms

$$\mathbb{T}(x, y)(v) = v^2 + \frac{1 - e^{-1}}{2} e^{-v} \sin(v^2) \cos(v^2) = x(v).$$

Thus all conditions of the theorem are satisfied and Corollary 3.9 guarantees existence and uniqueness of the coupled fixed point solution.

4.2. Applications to Homotopy.

In this section, we study the existence of an unique solution to Homotopy theory.

Theorem 4.2.1: Let $(\mathcal{G}, \mathcal{A}, \varsigma_p)$ be a complete C^* -avpbms with coefficient $\mathfrak{v} \succeq 1_{\mathcal{A}}$, Ξ be a open and $\bar{\Xi}$ be closed subset of $\mathcal{G}$ such that $\Xi \subseteq \bar{\Xi}$. Suppose $\mathcal{O}_h : \bar{\Xi} \times \bar{\Xi} \times [0, 1] \rightarrow \mathcal{G}$ be an operator such that the following conditions are satisfying,

(i) $\mathfrak{v} \neq \mathcal{O}_h(\mathfrak{v}, \mu, \lambda)$ & $\mu \neq \mathcal{O}_h(\mu, \mathfrak{v}, \lambda)$ for each $\mathfrak{v}, \mu \in \partial\Xi$ and $\lambda \in [0, 1]$, (here $\partial\Xi$ denotes the boundary of Ξ in $\mathcal{G}$).

(ii) For all $\mathfrak{s}, \mathfrak{e}, \mathfrak{x}, \mathfrak{z} \in \bar{\Xi}$ and $\mathfrak{a} \in \mathcal{A}$ with $\|\mathfrak{a}\mathfrak{v}\| < 1$ such that

$$\varsigma_p(\mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda), \mathcal{O}_h(\mathfrak{x}, \mathfrak{z}, \lambda)) \preceq \mathfrak{a}^* \max \left\{ \varsigma_p(\mathfrak{s}, \mathfrak{x}), \varsigma_p(\mathfrak{e}, \mathfrak{z}) \right\} \mathfrak{a}.$$

(iii) there exists $M \in \mathcal{A}_+$ such that

$$\varsigma_p(\mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda), \mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \gamma)) \leq \|M\| |\lambda - \gamma|$$

for every $\mathfrak{s}, \mathfrak{e} \in \bar{\Xi}$ and $\lambda, \gamma \in [0, 1]$.

Then $\mathcal{O}_h(\cdot, 0)$ has a coupled fixed point if and only if $\mathcal{O}_h(\cdot, 1)$ has a coupled fixed point.

Proof Consider the set

$$\mathbb{Q} = \{\lambda \in [0, 1] : \mathfrak{s} = \mathcal{O}_{\mathfrak{h}}(\mathfrak{s}, \mathfrak{e}, \lambda) \quad \& \quad \mathfrak{e} = \mathcal{O}_{\mathfrak{h}}(\mathfrak{e}, \mathfrak{s}, \lambda) \text{ for some } \mathfrak{s}, \mathfrak{e} \in \Xi\}.$$

Since $\mathcal{O}_{\mathfrak{h}}(., 0)$ has a CFP in $\Xi \times \Xi$, we have that $(0, 0) \in \mathbb{Q} \times \mathbb{Q}$. So that $\mathbb{Q}$ is non empty set.

We will show that $\mathbb{Q}$ is both open and closed in $[0, 1]$ and so by the connectedness we have that $\mathbb{Q} = [0, 1]$.

As a result, $\mathcal{O}_{\mathfrak{h}}(., 1)$ has a CFP in $\Xi \times \Xi$. First we show that $\mathbb{Q}$ is closed in $[0, 1]$.

To see this let $\{\lambda_{\zeta}\}_{\zeta=1}^{\infty} \subseteq \mathbb{Q}$ with $\lambda_{\zeta} \rightarrow \lambda \in [0, 1]$ as $\zeta \rightarrow \infty$.

We must show that $\lambda \in \mathbb{Q}$.

Since $\lambda_{\zeta} \in \mathbb{Q}$ for $\zeta = 1, 2, 3, \dots$, there exist $\mathfrak{s}_{\zeta}, \mathfrak{e}_{\zeta} \in \Xi$ with $\mathfrak{s}_{\zeta} = \mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta}, \mathfrak{e}_{\zeta}, \lambda_{\zeta})$,
 $\mathfrak{e}_{\zeta} = \mathcal{O}_{\mathfrak{h}}(\mathfrak{e}_{\zeta}, \mathfrak{s}_{\zeta}, \lambda_{\zeta})$.

Consider

$$\begin{aligned} \varsigma_{\rho}(\mathfrak{s}_{\zeta}, \mathfrak{s}_{\zeta+1}) &= \varsigma_{\rho}(\mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta}, \mathfrak{e}_{\zeta}, \lambda_{\zeta}), \mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta+1}, \mathfrak{e}_{\zeta+1}, \lambda_{\zeta+1})) \\ &\preceq \mathfrak{v} \left[\begin{array}{c} \varsigma_{\rho}(\mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta}, \mathfrak{e}_{\zeta}, \lambda_{\zeta}), \mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta+1}, \mathfrak{e}_{\zeta+1}, \lambda_{\zeta})) \\ + \varsigma_{\rho}(\mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta+1}, \mathfrak{e}_{\zeta+1}, \lambda_{\zeta}), \mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta+1}, \mathfrak{e}_{\zeta+1}, \lambda_{\zeta+1})) \end{array} \right] \\ &\quad - \varsigma_{\rho}(\mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta+1}, \mathfrak{e}_{\zeta+1}, \lambda_{\zeta}), \mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta+1}, \mathfrak{e}_{\zeta+1}, \lambda_{\zeta})) \\ &\preceq \mathfrak{v} \varsigma_{\rho}(\mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta}, \mathfrak{e}_{\zeta}, \lambda_{\zeta}), \mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta+1}, \mathfrak{e}_{\zeta+1}, \lambda_{\zeta})) + \mathfrak{v} \|M\| |\lambda_{\zeta} - \lambda_{\zeta+1}|. \end{aligned}$$

Letting $\zeta \rightarrow \infty$, we get

$$\lim_{\zeta \rightarrow \infty} \frac{1}{\mathfrak{v}} \varsigma_{\rho}(\mathfrak{s}_{\zeta}, \mathfrak{s}_{\zeta+1}) \preceq \lim_{\zeta \rightarrow \infty} \varsigma_{\rho}(\mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta}, \mathfrak{e}_{\zeta}, \lambda_{\zeta}), \mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta+1}, \mathfrak{e}_{\zeta+1}, \lambda_{\zeta})) + 0.$$

From (ii) we obtain

$$\begin{aligned} \lim_{\zeta \rightarrow \infty} \frac{1}{\mathfrak{v}} \varsigma_{\rho}(\mathfrak{s}_{\zeta}, \mathfrak{s}_{\zeta+1}) &\preceq \lim_{\zeta \rightarrow \infty} \varsigma_{\rho}(\mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta}, \mathfrak{e}_{\zeta}, \lambda_{\zeta}), \mathcal{O}_{\mathfrak{h}}(\mathfrak{s}_{\zeta+1}, \mathfrak{e}_{\zeta+1}, \lambda_{\zeta})) \\ &\preceq \lim_{\zeta \rightarrow \infty} \mathfrak{a}^* \max \left\{ \begin{array}{c} \varsigma_{\rho}(\mathfrak{s}_{\zeta}, \mathfrak{s}_{\zeta+1}), \\ \varsigma_{\rho}(\mathfrak{e}_{\zeta}, \mathfrak{e}_{\zeta+1}) \end{array} \right\} \mathfrak{a}. \end{aligned}$$

Similarly

$$\lim_{\zeta \rightarrow \infty} \frac{1}{\mathfrak{v}} \varsigma_{\rho}(\mathfrak{e}_{\zeta}, \mathfrak{e}_{\zeta+1}) \preceq \lim_{\zeta \rightarrow \infty} \mathfrak{a}^* \max \left\{ \begin{array}{c} \varsigma_{\rho}(\mathfrak{s}_{\zeta}, \mathfrak{s}_{\zeta+1}), \\ \varsigma_{\rho}(\mathfrak{e}_{\zeta}, \mathfrak{e}_{\zeta+1}) \end{array} \right\} \mathfrak{a}$$

Thus, we conclude that

$$\lim_{\zeta \rightarrow \infty} \frac{1}{v} \max \left\{ \begin{array}{c} \varsigma_p(s_\zeta, s_{\zeta+1}), \\ \varsigma_p(e_\zeta, e_{\zeta+1}) \end{array} \right\} \preceq \lim_{\zeta \rightarrow \infty} \alpha^* \max \left\{ \begin{array}{c} \varsigma_p(s_\zeta, s_{\zeta+1}), \\ \varsigma_p(e_\zeta, e_{\zeta+1}) \end{array} \right\} \alpha$$

implies that

$$(1 - \|v\| \|\alpha\|^2) \lim_{\zeta \rightarrow \infty} \max \left\{ \begin{array}{c} \|\varsigma_p(s_\zeta, s_{\zeta+1})\|, \\ \|\varsigma_p(e_\zeta, e_{\zeta+1})\| \end{array} \right\} \leq 0.$$

It follows that

$$\lim_{\zeta \rightarrow \infty} \varsigma_p(s_\zeta, s_{\zeta+1}) = 0_{\mathcal{A}} = \lim_{\zeta \rightarrow \infty} \varsigma_p(e_\zeta, e_{\zeta+1}).$$

Since $\varsigma_p(s_\zeta, s_\zeta) \preceq \varsigma_p(s_\zeta, s_{\zeta+1})$ and $\varsigma_p(e_\zeta, e_\zeta) \preceq \varsigma_p(e_\zeta, e_{\zeta+1})$,

then we have $\lim_{\zeta \rightarrow \infty} \varsigma_p(s_\zeta, s_\zeta) = 0_{\mathcal{A}}$ and $\lim_{\zeta \rightarrow \infty} \varsigma_p(e_\zeta, e_\zeta) = 0_{\mathcal{A}}$.

$$\begin{aligned} \varsigma_p(s_\zeta, s_{\zeta+\xi}) &\preceq v [\varsigma_p(s_\zeta, s_{\zeta+1}) + \varsigma_p(s_{\zeta+1}, s_{\zeta+\xi})] - \varsigma_p(s_{\zeta+1}, s_{\zeta+1}) \\ &\preceq v \varsigma_p(s_\zeta, s_{\zeta+1}) + v^2 [\varsigma_p(s_{\zeta+1}, s_{\zeta+2}) + \varsigma_p(s_{\zeta+2}, s_{\zeta+\xi})] - \varsigma_p(s_{\zeta+1}, s_{\zeta+1}) \\ &\quad - \varsigma_p(s_{\zeta+2}, s_{\zeta+2}) \\ &\preceq v \varsigma_p(s_\zeta, s_{\zeta+1}) + v^2 \varsigma_p(s_{\zeta+1}, s_{\zeta+2}) \\ &\quad + \dots + v^{\zeta+\xi-1} [\varsigma_p(s_{\zeta+\xi-2}, s_{\zeta+\xi-1}) + \varsigma_p(s_{\zeta+\xi-1}, s_{\zeta+\xi})] \\ &\quad - \varsigma_p(s_{\zeta+1}, s_{\zeta+1}) - \varsigma_p(s_{\zeta+2}, s_{\zeta+2}) \dots - \varsigma_p(s_{\zeta+\xi-1}, s_{\zeta+\xi-1}) \\ &\rightarrow 0_{\mathcal{A}} \text{ as } \zeta, \xi \rightarrow \infty. \end{aligned}$$

Hence $\{s_\zeta\}$ is Cauchy sequence in $(\mathcal{G}, \mathcal{A}, \varsigma_p)$ and similarly, we prove that $\{e_\zeta\}$ is Cauchy sequence so that $\lim_{\zeta, \xi \rightarrow \infty} \varsigma_p(s_\zeta, s_\xi) = 0_{\mathcal{A}} = \lim_{\zeta, \xi \rightarrow \infty} \varsigma_p(e_\zeta, e_\xi)$.

Since $(\mathcal{G}, \mathcal{A}, \varsigma_p)$ is complete, there exists $v, \mu \in \Xi$ with

$$\begin{aligned} \varsigma_p(v, v) &= \lim_{\zeta \rightarrow \infty} \varsigma_p(s_\zeta, v) = \lim_{\zeta \rightarrow \infty} \varsigma_p(s_{\zeta+1}, v) = \lim_{\zeta, \xi \rightarrow \infty} \varsigma_p(s_\zeta, s_\xi) = 0_{\mathcal{A}}. \\ \varsigma_p(\mu, \mu) &= \lim_{\zeta \rightarrow \infty} \varsigma_p(e_\zeta, \mu) = \lim_{\zeta \rightarrow \infty} \varsigma_p(e_{\zeta+1}, \mu) = \lim_{\zeta, \xi \rightarrow \infty} \varsigma_p(e_\zeta, e_\xi) = 0_{\mathcal{A}}. \end{aligned}$$

Now,

$$\begin{aligned}
\varsigma_\rho(\mathfrak{s}_\zeta, \mathcal{O}_h(\mathfrak{v}, \mu, \lambda)) &= \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}_\zeta, \mathfrak{e}_\zeta, \lambda_\zeta), \mathcal{O}_h(\mathfrak{v}, \mu, \lambda)) \\
&\preceq \mathfrak{v} \left[\varsigma_\rho(\mathcal{O}_h(\mathfrak{s}_\zeta, \mathfrak{e}_\zeta, \lambda_\zeta), \mathcal{O}_h(\mathfrak{s}_\zeta, \mathfrak{e}_\zeta, \lambda)) + \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}_\zeta, \mathfrak{e}_\zeta, \lambda), \mathcal{O}_h(\mathfrak{v}, \mu, \lambda)) \right] \\
&\quad - \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}_\zeta, \mathfrak{e}_\zeta, \lambda), \mathcal{O}_h(\mathfrak{s}_\zeta, \mathfrak{e}_\zeta, \lambda)) \\
&\preceq \mathfrak{v} \|M\| |\lambda_\zeta - \lambda| + \mathfrak{v} \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}_\zeta, \mathfrak{e}_\zeta, \lambda), \mathcal{O}_h(\mathfrak{v}, \mu, \lambda)).
\end{aligned}$$

Letting $\zeta \rightarrow \infty$, we obtain

$$\begin{aligned}
\frac{1}{\mathfrak{v}} \varsigma_\rho(\mathfrak{v}, \mathcal{O}_h(\mathfrak{v}, \mu, \lambda)) &\preceq \lim_{\zeta \rightarrow \infty} \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}_\zeta, \mathfrak{e}_\zeta, \lambda), \mathcal{O}_h(\mathfrak{v}, \mu, \lambda)) \\
&\preceq \lim_{\zeta \rightarrow \infty} \alpha^* \max \left\{ \begin{array}{l} \varsigma_\rho(\mathfrak{s}_\zeta, \mathfrak{v}), \\ \varsigma_\rho(\mathfrak{e}_\zeta, \mu) \end{array} \right\} \alpha = 0_{\mathcal{A}}.
\end{aligned}$$

It follows that $\varsigma_\rho(\mathfrak{v}, \mathcal{O}_h(\mathfrak{v}, \mu, \lambda)) = 0_{\mathcal{A}}$. So that $\mathfrak{v} = \mathcal{O}_h(\mathfrak{v}, \mu, \lambda)$.

Similarly $\mu = \mathcal{O}_h(\mu, \mathfrak{v}, \lambda)$. Thus $\lambda \in \mathbb{Q}$. Hence $\mathbb{Q}$ is closed in $[0, 1]$.

Let $\lambda_0 \in \mathbb{Q}$. Then there exists $\mathfrak{s}_0, \mathfrak{e}_0 \in \Xi$ with $\mathfrak{s}_0 = \mathcal{O}_h(\mathfrak{s}_0, \mathfrak{e}_0, \lambda_0)$ and $\mathfrak{e}_0 = \mathcal{O}_h(\mathfrak{e}_0, \mathfrak{s}_0, \lambda_0)$

Since Ξ is open, then there exists $r > 0$ such that $B_{\varsigma_\rho}(\mathfrak{s}_0, r) \subseteq \Xi$ and $B_{\varsigma_\rho}(\mathfrak{e}_0, r) \subseteq \Xi$.

Choose $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ such that $|\lambda - \lambda_0| \leq \frac{1}{M\zeta} < \epsilon$.

Then for $\mathfrak{s} \in \overline{B_{\varsigma_\rho}(\mathfrak{s}_0, r)} = \{\mathfrak{s} \in \mathcal{G} / \varsigma_\rho(\mathfrak{s}, \mathfrak{s}_0) \leq r + \varsigma_\rho(\mathfrak{s}_0, \mathfrak{s}_0)\}$ and

$\mathfrak{e} \in \overline{B_{\varsigma_\rho}(\mathfrak{e}_0, r)} = \{\mathfrak{e} \in \mathcal{G} / \varsigma_\rho(\mathfrak{e}, \mathfrak{e}_0) \leq r + \varsigma_\rho(\mathfrak{e}_0, \mathfrak{e}_0)\}$. Now,

$$\begin{aligned}
\varsigma_\rho(\mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda), \mathfrak{s}_0) &= \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda), \mathcal{O}_h(\mathfrak{s}_0, \mathfrak{e}_0, \lambda_0)) \\
&\preceq \mathfrak{v} \left[\varsigma_\rho(\mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda), \mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda_0)) + \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda_0), \mathcal{O}_h(\mathfrak{s}_0, \mathfrak{e}_0, \lambda_0)) \right] \\
&\quad - \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda_0), \mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda_0)) \\
&\preceq \mathfrak{v} \|M\| |\lambda - \lambda_0| + \mathfrak{v} \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda_0), \mathcal{O}_h(\mathfrak{s}_0, \mathfrak{e}_0, \lambda_0)) \\
&\leq \mathfrak{v} \frac{1}{\|M\|^{\zeta-1}} + \mathfrak{v} \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda_0), \mathcal{O}_h(\mathfrak{s}_0, \mathfrak{e}_0, \lambda_0)).
\end{aligned}$$

Letting $\zeta \rightarrow \infty$, we obtain

$$\frac{1}{\mathfrak{v}} \varsigma_\rho(\mathcal{O}_h(\mathfrak{s}, \mathfrak{e}, \lambda), \mathfrak{s}_0) \preceq \alpha^* \max \{ \varsigma_\rho(\mathfrak{s}, \mathfrak{s}_0), \varsigma_\rho(\mathfrak{e}, \mathfrak{e}_0) \} \alpha.$$

Similarly

$$\frac{1}{\mathfrak{v}} \varsigma_\rho(\mathcal{O}_h(\mathfrak{e}, \mathfrak{s}, \lambda), \mathfrak{e}_0) \preceq \alpha^* \max \{ \varsigma_\rho(\mathfrak{s}, \mathfrak{s}_0), \varsigma_\rho(\mathfrak{e}, \mathfrak{e}_0) \} \alpha.$$

Thus

$$\begin{aligned} \max\{\|\zeta_p(\mathcal{O}_h(s, e, \lambda), s_0)\|, \|\zeta_p(\mathcal{O}_h(e, s, \lambda), e_0)\|\} &\leq \|v\| \|a\|^2 \max\{\|\zeta_p(s, s_0)\|, \|\zeta_p(e, e_0)\|\} \\ &\leq \max\{r + \|\zeta_p(s_0, s_0)\|, r + \|\zeta_p(e_0, e_0)\|\} \end{aligned}$$

Thus for each fixed $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, $\mathcal{O}_h(\cdot, \lambda) : \overline{B_{\zeta_p}(s_0, r)} \rightarrow \overline{B_{\zeta_p}(s_0, r)}$ and $\mathcal{O}_h(\cdot, \lambda) : \overline{B_{\zeta_p}(e_0, r)} \rightarrow \overline{B_{\zeta_p}(e_0, r)}$. Since also (ii) holds, then all conditions of Theorem 4.2.1 are satisfied. Thus we deduce that $\mathcal{O}_h(\cdot, \lambda)$ has a coupled fixed point in $\overline{\Xi}$. But this coupled fixed point must be in Ξ since (i) holds. Thus $\lambda \in \mathbb{Q}$ for any $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. Hence $(\lambda_0 - \epsilon, \lambda_0 + \epsilon) \subseteq \mathbb{Q}$ and therefore $\mathbb{Q}$ is open in $[0, 1]$.

For the reverse implication, we use the same strategy.

Example 4.2.2: Let $\mathcal{G} = C[0, 1]$ with $\mathcal{A} = L(\mathcal{H})$, $\mathcal{H} = L^2[0, 1]$, and $\zeta_p(p, q) = M_{|p-q|^2+I}$. Consider $\Xi = \{(x, y) \in \mathcal{G}^2 : \|x\|_\infty, \|y\|_\infty < 1\}$ and define

$$\mathcal{O}_h(s, e, \lambda)(t) = \frac{\lambda}{2} \sin(s(t)) + \frac{1-\lambda}{2} e(t).$$

For $(s, e) \in \partial\Xi$, $|\mathcal{O}_h(s, e, \lambda)|_\infty \leq \frac{1}{2} < 1$. Let $a = 0.6I$. Then

$$\begin{aligned} \zeta_p(\mathcal{O}_h(s, e, \lambda), \mathcal{O}_h(x, z, \lambda)) &= M_{\left|\frac{\lambda}{2}(\sin(s) - \sin(x)) + \frac{1-\lambda}{2}(e - x)\right|^2 + I} \\ &\preceq 0.36 \max\{\zeta_p(s, x), \zeta_p(e, z)\}. \end{aligned}$$

Let $M = 2I$. Then, $\zeta_p(\mathcal{O}_h(s, e, \lambda), \mathcal{O}_h(s, e, \gamma)) \leq 2|\lambda - \gamma|$.

At $\lambda = 0$, $\mathcal{O}_h(\cdot, 0)(t) = \frac{1}{2}e(t)$ has coupled fixed point (e, e) . By Theorem 4.2.1, $\mathcal{O}_h(\cdot, 1)(t) = \frac{1}{2}\sin(s(t))$ has unique coupled fixed point (s^*, s^*) where $s^*(t) = \sin(s^*(t))$. Thus homotopy connects solutions at $\lambda = 0$ to solutions at $\lambda = 1$, exactly as Theorem 4.2.1 guarantees.

5. CONCLUSIONS

In conclusion, this paper establishes coupled fixed point theorems for interpolative contractions in C^* -algebra valued partial b -metric spaces using α -admissible mappings. The results guarantee unique solutions for coupled nonlinear Fredholm integral equations and homotopy continuation between trivial and nontrivial fixed points. Future work will explore multivalued mappings, fractional differential equations, and numerical homotopy algorithms.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

- [1] S. Banach, Sur les Opérations dans les Ensembles Abstraits et Leur Application aux Équations Intégrales, *Fundam. Math.* 3 (1922), 133–181. <https://doi.org/10.4064/fm-3-1-133-181>.
- [2] D. Guo, V. Lakshmikantham, Coupled Fixed Points of Nonlinear Operators with Applications, *Nonlinear Anal. Theory Methods Appl.* 11 (1987), 623–632. [https://doi.org/10.1016/0362-546X\(87\)90077-0](https://doi.org/10.1016/0362-546X(87)90077-0).
- [3] R. Kannan, Some Results on Fixed Points, *Bull. Calcutta Math. Soc.* 60 (1968), 71–76.
- [4] R. Kannan, Some Results on Fixed Points–II, *Am. Math. Mon.* 76 (1969), 405–408. <https://doi.org/10.2307/2316437>.
- [5] E. Karapinar, Revisiting the Kannan Type Contractions via Interpolation, *Adv. Theory Nonlinear Anal. Appl.* 2 (2018), 85–87. <https://doi.org/10.31197/atnaa.431135>.
- [6] E. Karapinar, R. Agarwal, H. Aydi, Interpolative Reich-Rus-Ćirić Type Contractions on Partial Metric Spaces, *Mathematics* 6 (2018), 256. <https://doi.org/10.3390/math6110256>.
- [7] B. Samet, C. Vetro, P. Vetro, Fixed Point Theorems for α - ψ -Contractive Type Mappings, *Nonlinear Anal. Theory Methods Appl.* 75 (2012), 2154–2165. <https://doi.org/10.1016/j.na.2011.10.014>.
- [8] O. Popescu, Some New Fixed Point Theorems for α -Geraghty Contraction Type Maps in Metric Spaces, *Fixed Point Theory Appl.* 2014 (2014), 190. <https://doi.org/10.1186/1687-1812-2014-190>.
- [9] H. Isik, B. Samet, C. Vetro, Cyclic Admissible Contraction and Applications to Functional Equations in Dynamic Programming, *Fixed Point Theory Appl.* 2015 (2015), 163. <https://doi.org/10.1186/s13663-015-0410-6>.
- [10] O. Yamaod, W. Sintunavarat, Fixed Point Theorems for (α, β) - (ψ, ϕ) -Contractive Mapping in b -Metric Spaces with Some Numerical Results and Applications, *J. Nonlinear Sci. Appl.* 9 (2016), 22–33. <https://doi.org/10.22436/jnsa.009.01.03>.
- [11] Z. Ma, L. Jiang, H. Sun, C^* -Algebra-Valued Metric Spaces and Related Fixed Point Theorems, *Fixed Point Theory Appl.* 2014 (2014), 206. <https://doi.org/10.1186/1687-1812-2014-206>.
- [12] S. Chandok, D. Kumar, C. Park, C^* -Algebra-Valued Partial Metric Space and Fixed Point Theorems, *Proc. Indian Acad. Sci. Math. Sci.* 129 (2019), 37. <https://doi.org/10.1007/s12044-019-0481-0>.
- [13] N. Mlaiki, M. Asim, M. Imdad, C^* -Algebra Valued Partial b -Metric Spaces and Fixed Point Results with an Application, *Mathematics* 8 (2020), 1381. <https://doi.org/10.3390/math8081381>.
- [14] G. Mani, A.J. Gnanaprakasam, O. Ege, A. Aloqaily, N. Mlaiki, Fixed Point Results in C^* -Algebra-Valued Partial b -Metric Spaces with Related Application, *Mathematics* 11 (2023), 1158. <https://doi.org/10.3390/math11051158>.
- [15] G.J. Murphy, C^* -Algebras and Operator Theory, Elsevier, 1990. <https://doi.org/10.1016/C2009-0-22289-6>.

- [16] Q. Xu, N.B. Turdebek, Z. Chen, Introduction to Operator Algebras and Noncommutative L_p Spaces (in Chinese), Science Press, Beijing, 2010.
- [17] R.G. Douglas, Banach Algebra Techniques in Operator Theory, Springer New York, 1998. <https://doi.org/10.1007/978-1-4612-1656-8>.