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PROBABILISTIC b -SUPRAMETRIC SPACES AND NONLINEAR CONTRACTIONS, WITH AN APPLICATION TO (ζ_1, ζ_2) -DECOMPOSABLE MEASURES

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Abstract. This paper introduces the concept of probabilistic b -suprametric spaces (PbSMS). We conduct a systematic investigation of the fundamental topological properties inherent to these spaces. A key result is the proof of a fixed point theorem for nonlinear Banach-type contractions within the PbSMS framework, which is supported by illustrative examples. Finally, we demonstrate the practical utility of our main result by applying it to the theory of (ζ_1, ζ_2) -decomposable measures.

Keywords: ψ -contraction; fixed point; decomposable measure; menger probabilistic b -metric space; menger probabilistic suprametric space; b -suprametric space.

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1. INTRODUCTION

Stefan Banach laid the foundation for the metric fixed-point theory in 1922. His Banach Contraction Principle [3] plays a vital role in many mathematical applications and inspired a major research effort to find fixed points for contractive mappings. This has resulted in numerous new

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theorems, achieved by either weakening the original contraction conditions or by defining new types of generalized metric spaces.

Among these generalizations is the probabilistic metric space, a concept introduced by Karl Menger in [23]. Menger's innovation was to substitute nonnegative real numbers with distribution functions as metric values and to replace the standard triangle inequality with the more general notion of triangle norms. The principal advancement and refinement of this theory, however, is credited to Schweizer, Sklar, and their associates.(see [30],[27],[28],[29]).

In [4], Czerwik used a relaxed triangular inequality governed by a real coefficient $b \geq 1$, to introduce b -metric spaces equipped with a distance satisfying such an inequality. Moreover, He established a fixed point theorem within the framework of b -metric spaces. Given the significance of this distance function, it has since been generalized in multiple ways and applied across diverse fields, such as Multivalued fractals and fractional differential equations, (see [9], [2], [20]).

Mbarki and Oubrahim [22] introduced probabilistic b -metric spaces by modifying the triangle inequality in probabilistic metric spaces and subsequently developed a fixed point theory that incorporates both topological and geometric properties (see [1] ,[13], [15] ,[22] for more details). As a result, a significant portion of existing fixed point theorems originally formulated for classical metric and b -metric spaces can now be extended to probabilistic metric and probabilistic b -metric spaces, provided a sufficiently small triangular norm is used.

Recently Berzig proposed in [6] the idea of a modified triangle inequality which lead to new generalized spaces called suprametric spaces. Later on Panda et al [25] extended on this idea to prove the existence of solutions to stochastic integral equations of the Ito-Doob type. Moreover, Panda et al [26] proved the utility of the suprametric notions in studying the stability as well as the existence of equilibrium points in complex valued, fractional-order neural networks.

Unifying the two concepts of b -metric spaces and suprametric spaces lead to the introduction of the b -suprametric spaces in [7], merging the advantages of both the b -metric and the suprametric spaces. Berzig established a fixed point theorem for mappings satisfying nonlinear

contractions via Matkowski functions [7]. The Banach contraction principle has been successfully extended to both suprametric and b -suprametric spaces, with demonstrated applications in solving nonlinear integral equations and matrix equations (see [6], [7], [8]).

This paper is structured as follows:

In Section 2, we present the foundational results in metric probabilistic spaces, which will be utilized in the subsequent sections. In Section 3, we define probabilistic Menger b -suprametric spaces and explore the key topological principles governing them. In Section 4, we establish a fixed point theorem for nonlinear contractions in Menger probabilistic b -suprametric spaces. Section 5, We extend the foundational definitions of (ζ_1, ζ_2) -decomposable measure and (NSA_2) -type measures (see Definitions 12 and 13) to incorporate a pair of t -norms coherent with our generalized triangle inequality, we construct a bridge to our main theorem and subsequently prove the existence of fixed points for a class of random operators in this enriched measurable setting.

First, for the sake of the readers, we will state some basic definitions and expressions that will be used throughout the study.

2. PRELIMINARIES

Throughout this paper, we respectively denote by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ the sets of all real numbers, all nonnegative real numbers and all nonnegative natural numbers.

In 1993, Czerwik formally defined the b -metric space as follows:

Definition 1. *Let X be a non empty set and $s \in [1, +\infty)$. A function $d : X \times X \rightarrow \mathbb{R}_+$ is called a b -metric on X if for all $u, v, w \in X$, we have*

- (i) $d(u, v) = 0$ iff $u = v$,
- (ii) $d(u, v) = d(v, u)$,
- (iii) $d(u, v) \leq s(d(u, w) + d(w, v))$.

The pair (X, d) is called a b -metric space. If $s = 1$, the pair (X, d) is called a metric space.

We can find the notion of b -metric space introduced in [5] and [4]. Fixed point theorems in b -metric spaces can be found in [6] and [7]. Recall now the concept of b -suprametric spaces introduced in [6] by Berzig.

Definition 2. Let us consider that X is a nonempty set, $\mu \in \mathbb{R}_+$ and $d_\mu : X \times X \rightarrow [0, +\infty)$ is a functional, satisfying the following:

- (i) $d_\mu(u, v) = 0$, iff $u = v$,
- (ii) $d_\mu(u, v) = d_\mu(v, u)$, for all $u, v \in X$,
- (iii) $d_\mu(u, v) \leq s(d_\mu(u, w) + d_\mu(w, v)) + \mu d_\mu(u, w)d_\mu(w, v)$, for all $u, v, w \in X$, where $s \geq 1$.

Then, d_μ is a b -suprametric on X , and (X, d_μ) is a b -suprametric space.

If $s = 1$, d_μ is a suprametric on X , and (X, d_μ) is a suprametric space.

Example 1. [7] Let $\ell_p(\mathbb{R}) := \{(u_n)_{n \geq 1} \subset \mathbb{R} \text{ such that } \sum_{n=1}^{+\infty} \|u_n\|_p < \infty\}$, with $0 < p < 1$, equipped with a b -metric $\|\cdot\|_p : \ell_p(\mathbb{R}) \times \ell_p(\mathbb{R}) \rightarrow \mathbb{R}_+$ is given by

$$\|(u_n)_{n \geq 1} - (v_n)_{n \geq 1}\|_p = \left(\sum_{n=1}^{+\infty} |u_n - v_n|_p \right)^{\frac{1}{p}} \text{ for all } (u_n)_{n \geq 1}, (v_n)_{n \geq 1} \in \ell_p(\mathbb{R}),$$

with $s = 2^{\frac{1}{p}}$.

We consider the fonction $d_\mu : \ell_p(\mathbb{R}) \times \ell_p(\mathbb{R}) \rightarrow \mathbb{R}_+$ defined by

$$d_\mu(u, v) = \|(u_n)_{n \geq 1} - (v_n)_{n \geq 1}\|_p (\|(u_n)_{n \geq 1} - (v_n)_{n \geq 1}\|_p + 1), \text{ for all } u, v \in \ell_p(\mathbb{R}).$$

Then, $(\ell_p(\mathbb{R}), d_\mu)$ is a b -suprametric, with $s = 4^{\frac{1}{p}}$ and $\mu = 8^{\frac{1}{p}}$.

We begin by briefly recalling some definitions and notions from probabilistic metric spaces theory that we will use in the sequel. For more details, we refer the reader to [30].

A distance distribution function is a nondecreasing function h defined on $\mathbb{R}_+ \cup \{\infty\}$ such that

- (i) $h(0) = 0$ and $h(\infty) = 1$,
- (ii) h is left continuous on $(0, \infty)$.

The set of all distribution function will be denoted by Δ^+ , and the set of all $h \in \Delta^+$ such that $\lim_{t \rightarrow +\infty} h(t) = 1$ by $\mathcal{D}^+$.

A simple example of a distribution function is the Heaviside step function, also known as the unit step function, which is a commonly used example of a distribution function in $\mathcal{D}^+$. It is

defined as:

$$\mathcal{H}(t) = \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

The first field where triangular norms (t -norm briefly) played a major role was the theory of probabilistic metric spaces (see [19]).

Definition 3. (See [15].) A fonction $\tau : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a t -norm if for every $u, v, w \in [0, 1]$, the following requirement are verified:

- (i) $\tau(u, v) = \tau(v, u)$;
- (ii) $\tau(u, \tau(v, w)) = \tau(\tau(u, v), w)$;
- (iii) $(v \leq w \Rightarrow \tau(u, v) \leq \tau(u, w))$;
- (iv) $\tau(u, 1) = \tau(1, u) = u$.

A t -norm τ is said to be continuous if τ is continuous at each point $(u, v) \in [0, 1] \times [0, 1]$.

Typical continuous t -norms are

- (a) The minimum t -norm $\tau_M(u, v) = \min\{u, v\}$.
- (b) The product t -norm $\tau_P(u, v) = uv$.
- (c) The Lukasiewicz t -norm $\tau_L(u, v) = \max\{u + v - 1, 0\}$.

Definition 4. [15] The mapping $\zeta : [0, 1]^2 \rightarrow [0, 1]$ is a triangular conorm (briefly t -conorm) if, for all $u, v, w \in [0, 1]$, the following conditions are satisfied:

- (i) $\zeta(u, v) = \zeta(v, u)$;
- (ii) $\zeta(u, \zeta(v, w)) = \zeta(\zeta(u, v), w)$;
- (iii) $(v \leq w \Rightarrow \zeta(u, v) \leq \zeta(u, w))$;
- (iv) $\tau(u, 0) = \tau(0, u) = u$.

The connection between triangular norms and triangular conorms is given by the following result.

Proposition 1. [15] The function $\zeta : [0, 1]^2 \rightarrow [0, 1]$ is a triangular conorm if and only if there is a triangular norm τ such that for each $(u, v) \in [0, 1]^2$,

$$\zeta(u, v) = 1 - \tau(1 - u, 1 - v).$$

The opposite statement is also true.

Basic examples of triangular conorms are the following ones:

$$\zeta_m(u, v) = \max\{u, v\}, \quad \zeta_p(u, v) = u + v - uv, \quad \zeta_l(u, v) = \min\{u + v, 1\}.$$

Definition 5. [15]

- A t -conorm ζ (resp. t -norm) will be called Archimedean if it's continuous and $\zeta(a, a) > a$ (resp. $\tau(a, a) < a$), for all $a \in (0, 1)$.
- An Archimedean t -conorm ζ will be called strict if ζ is (strictly) increasing in $(0, 1) \times (0, 1)$.

Theorem 1. [15] A function $\tau : [0, 1]^2 \rightarrow [0, 1]$ is a continuous Archimedean triangular norm if and only if there exists a continuous, strictly decreasing function $t : [0, 1] \rightarrow [0, +\infty]$ with $t(1) = 0$ such that for all $x, y \in [0, 1]$,

$$\tau(x, y) = t^{-1}(\min\{t(x) + t(y), t(0)\}) = t^{(-1)}(t(x) + t(y)),$$

where $t^{(-1)}$ is the pseudo-inverse of t , defined by

$$t^{(-1)}(a) = \begin{cases} t^{-1}(a) & \text{if } a \in [0, t(0)], \\ 0 & \text{if } a \in [t(0), +\infty]. \end{cases}$$

The function t is then called an additive generator of τ ; it is uniquely determined by τ up to a positive multiplicative constant.

Triangular norms of H -type (see [15]) represent a very important class, especially in the theory of fixed point.

Definition 6. Let τ be a triangular norm and $\tau^n : [0, 1] \rightarrow [0, 1]$ a mapping defined in the following way:

$$\tau^1(u) = \tau(u, u), \quad \tau^{n+1}(u) = \tau(u, \tau^n(u)), \quad n \in \mathbb{N} \setminus \{0\}, \quad u \in [0, 1].$$

A triangular norm is H -type if the family $(\tau^n(u))_{n \geq 1}$ is equicontinuous at the point $u = 1$, that is, for all $\varepsilon \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that

$$u > 1 - \lambda \implies \tau^n(u) > 1 - \varepsilon, \quad \text{for every } n \geq 1.$$

Definition 7. [30]. A Menger space is a triple $(\Omega, \mathcal{F}, \tau)$, where Ω is a nonempty set, $\mathcal{F}$ is a function from $\Omega \times \Omega$ into $\mathcal{D}^+$, τ is a t -norm and the following conditions are satisfied for all $u, v, w \in \Omega$ and $x, y > 0$,

- (i) $\mathcal{F}_{u,v} = \mathcal{H}$ if and only if $u = v$,
- (ii) $\mathcal{F}_{u,v} = \mathcal{F}_{v,u}$,
- (iii) $\mathcal{F}_{u,v}(x+y) \geq \tau(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y))$.

In the sequel, we denote by Ψ the set of Matkowski functions [21], $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that satisfy

- (i) ψ is increasing on $\mathbb{R}_+$,
- (ii) $\lim_{n \rightarrow +\infty} \psi^n(x) = 0$, for all $x > 0$.

Theorem 2. [30]. Let $(X, \mathcal{F}, \tau)$ be a complete probabilistic metric space under a continuous t -norm τ of H -type such that $\text{Ran}\mathcal{F} \subset \mathcal{D}_+$. Let $f : X \rightarrow X$ be a ψ -probabilistic contraction, where $\psi \in \Psi$ i.e. for all $u, v \in X$,

$$\mathcal{F}_{f(u), f(v)}(\psi(x)) \geq \mathcal{F}_{u,v}(x).$$

Then, f has a unique fixed point u_0 and, for any $u \in X$, $\lim_{n \rightarrow +\infty} f_n(u) = u_0$.

Here we present an H_2 -type definition of two t -norms τ_1 and τ_2 (see [11]).

Let τ_1 and τ_2 be two t -norms. Let $u, v \in [0, 1]$ and $n \in \mathbb{N}$. We defined the fonction $h_u^n : [0, 1] \rightarrow [0, 1]$ by

$$h_u^n(v) = \begin{cases} \tau_1(u, \tau_1(v, \tau_2(u, v))) & \text{if } n = 1, \\ \tau_1(u, \tau_1(h_u^{n-1}(v), \tau_2(u, h_u^{n-1}(v)))) & \text{if } n \geq 2. \end{cases}$$

Definition 8. [11] Let τ_1 and τ_2 be two t -norms. We say that a pair (τ_1, τ_2) is of H_2 -type if the family $(h_u^n(u))_{n \geq 0}$ is equicontinuous at $u = 1$. It means that for all $\varepsilon \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that

$$u > 1 - \lambda \implies h_u^n(u) > 1 - \varepsilon, \text{ for all } n \geq 1.$$

The coupled t -norms (τ_M, τ_M) is a trivial example of H_2 -type.

3. MAIN RESULTS

3.1. Probabilistic b -suprametric space (briefly PbSMS).

The notion of the probabilistic b -suprametric space, which is larger than the class of probabilistic b -metric spaces, is introduced in this section.

Definition 9. Let Ω be a nonempty set, τ_1 and τ_2 be continuous t -norms, and $\mathcal{F}$ be a mapping from $\Omega \times \Omega$ into $\mathcal{D}^+$, with $\mathcal{D}^+$ being the set of Menger distance distribution functions, such that $\mathcal{F}_{u,v}$ is the value of $\mathcal{F}$ at (u, v) . Then the quadruple $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ is called a probabilistic b -suprametric space (briefly PbSMS) if for every $u, v, w \in \Omega$ and $x, y \in (0, +\infty)$,

- (i) $\mathcal{F}_{u,u}(x) = \mathcal{H}$;
- (ii) $\mathcal{F}_{u,v}(x) = \mathcal{H}$ implies $u = v$;
- (iii) $\mathcal{F}_{u,v}(x) = \mathcal{F}_{v,u}(x)$;
- (iv) $\mathcal{F}_{u,v}(s(x+y) + \mu xy) \geq \tau_1(\mathcal{F}_{u,w}(x), \tau_1(\mathcal{F}_{w,v}(y), \tau_2(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y))))$, with $s \geq 1$.

We call the condition (iv) of the definition 9, the generalized triangular inequality. If $s = 1$, then the quadruple $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ is called a probabilistic suprametric space (see [11]).

Example 2. Let $\Omega = \{0, 1, 2, n, \dots\} = \mathbb{N}$ and $\mu = 2$. Define the application $\mathcal{F} : \Omega \times \Omega \rightarrow \mathcal{D}_+$ as follows:

$$\mathcal{F}_{u,v}(x) = \begin{cases} \mathcal{H}(x) & \text{if } u = v, \\ \mathcal{H}(2x - 7) & \text{if } u, v \in \{0, 1\} \text{ and } u \neq v, \\ \mathcal{H}(x - (1 - e^{-|u-v|})) & \text{if } u, v \in \{0\} \cup \{2n : n \in \mathbb{N} \setminus \{0\}\} \text{ and } u \neq v, \\ \mathcal{H}(x - 2) & \text{otherwise.} \end{cases}$$

Actually, we only need to demonstrate the probabilistic triangle inequality since that (i), (ii) and (iii) of Definition 9 are trivially verified. Let $u, v, w \in \Omega$ and $x, y \in \mathbb{R}_+ \setminus \{0\}$.

Proving the triangle inequality comes down to proving the following:

$$\begin{aligned} \mathcal{F}_{u,v}(s(x+y) + \mu xy) &\geq \tau_L(\mathcal{F}_{u,w}(x), \tau_L(\mathcal{F}_{w,v}(y), \tau_P(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y)))) \\ &\geq \tau_L(\mathcal{F}_{u,w}(x), \tau_L(\mathcal{F}_{w,v}(y), \mathcal{F}_{u,w}(x) \times \mathcal{F}_{w,v}(y))) \\ &\geq \max \{ \mathcal{F}_{u,w}(x) + \tau_L(\mathcal{F}_{w,v}(y), \mathcal{F}_{u,w}(x) \times \mathcal{F}_{w,v}(y)) - 1, 0 \}. \end{aligned}$$

- If $(u, v) = (0, 2n)$ and $w = 0$, where $n \in \mathbb{N} \setminus \{0\}$, then

$$\mathcal{F}_{u,v}(2(x+y) + \mu xy) = \mathcal{H}(2(x+y) + \mu xy - (1 - e^{-2n})),$$

$$\mathcal{F}_{u,w}(x) = \mathcal{H}(x) = 1 \text{ and } \mathcal{F}_{w,v}(y) = \mathcal{H}(y - (1 - e^{-2n})).$$

Case 1: $y > 1 - e^{-2n}$, so $2(x+y) + \mu xy > 1 - e^{-2n}$, and then

$$\mathcal{H}(2(x+y) + \mu xy - (1 - e^{-2n})) = 1.$$

thus,

$$\mathcal{F}_{u,v}(s(x+y) + \mu xy) \geq \max\{\mathcal{F}_{u,w}(x) + \tau_L(\mathcal{F}_{w,v}(y), \mathcal{F}_{u,w}(x) \times \mathcal{F}_{w,v}(y)) - 1, 0\}.$$

Case 2: $y \leq 1 - e^{-2n}$, so $\mathcal{F}_{w,v}(y) = \mathcal{H}(y - (1 - e^{-2n})) = 0$.

then

$$\max\{\mathcal{F}_{u,w}(x) + \tau_L(\mathcal{F}_{w,v}(y), \mathcal{F}_{u,w}(x) \times \mathcal{F}_{w,v}(y)) - 1, 0\} = \max\{1 + \tau_L(0, 0) - 1, 0\} = 0.$$

Thus,

$$\mathcal{F}_{u,v}(2(x+y) + \mu xy) \geq \tau_L(\mathcal{F}_{u,w}(x), \tau_L(\mathcal{F}_{w,v}(y), \tau_P(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y))))).$$

- If $(u, v) = (0, 2n)$ and $w = 2m + 1$, $n, m \in \mathbb{N} \setminus \{0\}$, then

$$\mathcal{F}_{u,v}(2(x+y) + \mu xy) = \mathcal{H}(2(x+y) + \mu xy - (1 - e^{-2n})),$$

$$\mathcal{F}_{u,w}(x) = \mathcal{H}(x - 2) \text{ and } \mathcal{F}_{w,v}(y) = \mathcal{H}(y - 2).$$

Case 1: $x \leq 2$ or $y \leq 2$, in this case we either have $\mathcal{F}_{w,v}(y) = \mathcal{H}(y - 2) = 0$ or $\mathcal{F}_{u,w}(x) = \mathcal{H}(x - 2) = 0$.

Suppose that $\mathcal{F}_{u,w}(x) = \mathcal{H}(x - 2) = 0$, then we have

$$\begin{aligned} \max\{\mathcal{F}_{u,w}(x) + \tau_L(\mathcal{F}_{w,v}(y), \mathcal{F}_{u,w}(x) \times \mathcal{F}_{w,v}(y)) - 1, 0\} &= \max\{0 + \tau_L(\mathcal{F}_{w,v}(y), 0 \times \mathcal{F}_{w,v}(y)) - 1, 0\} \\ &= \max\{\tau_L(\mathcal{F}_{w,v}(y), 0) - 1, 0\} \\ &= \max\{\max\{\mathcal{F}_{w,v}(y) - 1, 0\} - 1, 0\} = 0. \end{aligned}$$

thus the generalized triangular inequality is verified; the case where $\mathcal{F}_{w,v}(y) = 0$ is treated similarly.

Case 2: $x > 2$ and $y > 2$. We have $2(x+y) + \mu xy > 8 > 1 - e^{-2n}$, we obtain the generalized triangular inequality. Since

$$\mathcal{F}_{u,v}(2(x+y) + \mu xy) = 1.$$

- If $(u, v) = (0, 2n)$ and $w = 1$, $n \in \mathbb{N} \setminus \{0\}$, then

$$\mathcal{F}_{u,v}(2(x+y) + \mu xy) = \mathcal{H}(2(x+y) + \mu xy - (1 - e^{-2n})),$$

$$\mathcal{F}_{u,w}(x) = \mathcal{H}(2x - 7) \text{ and } \mathcal{F}_{w,v}(y) = \mathcal{H}(y - 2).$$

Case 1: $x \leq \frac{7}{2}$ or $y \leq 2$, we obtain the generalized triangular inequality. Since weither $\mathcal{F}_{u,w}(x) = 0$ or $\mathcal{F}_{w,v}(y) = 0$ we have

$$\max \{ \mathcal{F}_{u,w}(x) + \tau_L(\mathcal{F}_{w,v}(y), \mathcal{F}_{u,w}(x) \times \mathcal{F}_{w,v}(y)) - 1, 0 \} = 0.$$

Case 2: $x > \frac{7}{2}$ and $y > 2$. We have $2(x+y) + \mu xy > 8 > 1 - e^{-2n}$, we obtain the generalized triangular inequality. Since

$$\mathcal{F}_{u,v}(2(x+y) + \mu xy) = 1.$$

- If $(u, v) = (0, 2n)$ and $w = 2m$, $n, m \in \mathbb{N} \setminus \{0\}$, then

$$\mathcal{F}_{u,v}(2(x+y) + \mu xy) = \mathcal{H}(2(x+y) + \mu xy - (1 - e^{-2n})),$$

$$\mathcal{F}_{u,w}(x) = \mathcal{H}(x - (1 - e^{-2n})) \text{ and } \mathcal{F}_{w,v}(y) = \mathcal{H}(y - (1 - e^{-2|n-m|})).$$

Case 1: $x \leq 1 - e^{-2n}$ or $y \leq 1 - e^{-2|n-m|}$, we obtain the generalized triangular inequality. Since weither $\mathcal{F}_{u,w}(x) = 0$ or $\mathcal{F}_{w,v}(y) = 0$, we have

$$\max \{ \mathcal{F}_{u,w}(x) + \tau_L(\mathcal{F}_{w,v}(y), \mathcal{F}_{u,w}(x) \times \mathcal{F}_{w,v}(y)) - 1, 0 \} = 0.$$

Case 2: $x > 1 - e^{-2n}$ and $y > 1 - e^{-2|n-m|}$. So

$$\begin{aligned} 2(x+y) + \mu xy &> 2(2 - e^{-2n} - e^{-2|n-m|}) + \mu (1 - e^{-2n}) (1 - e^{-2|n-m|}) \\ &= 2(2 - e^{-2n} - e^{-2|n-m|}) + \mu (1 - e^{-2|n-m|} - e^{-2n} + e^{-2|n-m|-2n}). \end{aligned}$$

Since $\mu = 2$, we have

$$\begin{aligned} 2(x+y) + \mu xy &> 2(2 - e^{-2n} - e^{-2|n-m|}) + 2(1 - e^{-2|n-m|} - e^{-2n} + e^{-2|n-m|-2n}) \\ &= 6 - 4e^{-2n} - 2e^{-2|n-m|} + 2(-e^{-2|n-m|} + e^{-2|n-m|-2n}). \end{aligned}$$

Suppose $n \leq m$. We have

$$\begin{aligned} 2(x+y) + \mu xy &> 6 - 4e^{-2n} - 2e^{-2(m-n)} + 2 \left(-e^{-2(m-n)} + e^{-2(m-n)-2n} \right) \\ &= 6 - 4e^{-2(m-n)} - 2(2e^{-2n} - e^{-2m}). \end{aligned}$$

Then

$$\begin{aligned} 2(x+y) + \mu xy - (1 - e^{-2n}) &> 6 - 4e^{-2(m-n)} - 2(2e^{-2n} - e^{-2m}) - 1 + e^{-2n} \\ &= 5 - 4e^{-2(m-n)} - 3e^{-2n} + 2e^{-2m} \\ &\geq 5 - 4e^{-2(m-n)} - 3e^{-2n} \\ &\geq 5 - 4 - e^{-2n} = 1 - e^{-2n} > 0. \end{aligned}$$

Hence, $\mathcal{F}_{u,v}(2(x+y) + \mu xy) = 1$.

And we obtain the generalized triangular inequality.

• If $(u, v) = (2n, 2m)$ and $w = 0$, $n \in \mathbb{N} \setminus \{0\}$, $m \in \mathbb{N} \setminus \{0\}$, and $n \neq m$ then

$$\begin{aligned} \mathcal{F}_{u,v}(2(x+y) + \mu xy) &= \mathcal{H} \left(2(x+y) + \mu xy - \left(1 - e^{-2|n-m|} \right) \right), \\ \mathcal{F}_{u,w}(x) &= \mathcal{H} \left(x - \left(1 - e^{-2n} \right) \right) \text{ and } \mathcal{F}_{w,v}(y) = \mathcal{H} \left(y - \left(1 - e^{-2m} \right) \right). \end{aligned}$$

Case 1: $x \leq 1 - e^{-2n}$ or $y \leq 1 - e^{-2m}$, we obtain the generalized triangular inequality. Since either $\mathcal{F}_{u,w}(x) = 0$ or $\mathcal{F}_{w,v}(y) = 0$ we have

$$\max \{ \mathcal{F}_{u,w}(x) + \tau_L(\mathcal{F}_{w,v}(y), \mathcal{F}_{u,w}(x) \times \mathcal{F}_{w,v}(y)) - 1, 0 \} = 0.$$

Case 2: $x > 1 - e^{-2n}$ and $y > 1 - e^{-2m}$. So

$$\begin{aligned} 2(x+y) + \mu xy - \left(1 - e^{-2|n-m|} \right) &> 2(2 - e^{-2n} - e^{-2m}) + \mu (1 - e^{-2n} - e^{-2m} + e^{-2n}e^{-2m}) \\ &\quad - \left(1 - e^{-2|n-m|} \right). \end{aligned}$$

Since $\mu = 2$, we have

$$\begin{aligned} 2(x+y) + \mu xy - \left(1 - e^{-2|n-m|} \right) &> 6 - 2e^{-2n} - 2e^{-2m} + 2(-e^{-2n} - e^{-2m} + e^{-2n}e^{-2m}) \\ &\quad - \left(1 - e^{-2|n-m|} \right). \end{aligned}$$

$$2(x+y) + \mu xy - \left(1 - e^{-2|n-m|} \right) > 5 - 4(e^{-2n} + e^{-2m}) + 2e^{-2n}e^{-2m} + e^{-2|n-m|}.$$

Suppose $n \leq m$. We have

$$2(x+y) + \mu xy - \left(1 - e^{-2|n-m|}\right) > 5 - 8e^{-2n} + 2e^{-4m} + e^{-2(m-n)} \geq 5 - 8e^{-2n} \geq 5 - 8e^{-2} > 3.$$

Hence, $\mathcal{F}_{u,v}(2(x+y) + \mu xy) = 1$.

we obtain the generalized triangular inequality.

It is easy to verify the condition (iv) for all other cases of (u, v) and w . Thus,

$$\mathcal{F}_{u,v}(s(x+y) + \mu xy) \geq \tau_L(\mathcal{F}_{u,w}(x), \tau_L(\mathcal{F}_{w,v}(y), \tau_P(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y)))) ,$$

for all $u, v, w \in \Omega$ and $x, y \in \mathbb{R}_+ \setminus \{0\}$, where $s = 2$.

Hence, $(\Omega, \mathcal{F}, (\tau_L, \tau_P), \mu)$ is a PbSMS.

However, we cannot use the results from [6] in this case as $(\Omega, \mathcal{F}, \tau_M)$ and $(\Omega, \mathcal{F}, \tau_P)$ are not standard probabilistic metric spaces since the triangle inequality is not satisfied:

If $(u, v) = (0, 1)$ and $w = 2$, and $(x, y) = (1, \frac{5}{2})$, then $\mathcal{F}_{u,w}(x) = \mathcal{H}(x - (1 - e^{-2})) = 1$ and $\mathcal{F}_{w,v}(y) = \mathcal{H}(y - 2) = 1$. We have

$$\mathcal{F}_{u,v}(x+y) = \mathcal{H}(2(x+y) - 7) = \mathcal{H}(0) = 0 < \tau_P(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y)),$$

$$\text{and } \mathcal{F}_{u,v}(x+y) < \tau_L(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y)).$$

Remak 1. (a) If $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ is a PbSMS, then for every $u, v, w \in \Omega$ and $x, y \in (0, +\infty)$,

$$\mathcal{F}_{u,v}(s(x+y) + \mu xy) \geq \tau_1 \left(\tau_1(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y)), \tau_2(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y)) \right).$$

(b) If also $\tau_1 = \tau_M$ and $\tau_2 = \tau_P$, then for every $u, v, w \in \Omega$ and $x, y \in (0, +\infty)$,

$$\mathcal{F}_{u,v}(s(x+y) + \mu xy) \geq \tau_P(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y)).$$

In particular, if $\mu = 0$, then $(\Omega, \mathcal{F}, (\tau_M, \tau_P), \theta)$ is standard probabilistic b-metric space.

Let $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ be a PbSMS, then the neighborhoods family

$$\mathcal{R} = \{R_u(\varepsilon, \lambda) : u \in \Omega, 0 < \varepsilon \text{ and } \lambda > 0\}, \text{ where}$$

$$R_u(\varepsilon, \lambda) = \{v \in \Omega : \mathcal{F}_{u,v}(\varepsilon) > 1 - \lambda\},$$

is a topology in $(\Omega, \mathcal{F})$, for all $u \in \Omega$.

Lemma 1. *Let $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ be a PbSMS. If the t -norms τ_1 and τ_2 are continuous, then the (ε, λ) -topology is a Hausdorff topology.*

Proof. Let $u, v \in \Omega$ such that $u \neq v$. Since $\mathcal{F} \in \mathcal{D}^+$, we put $\mathcal{F}_{u,v}(\varepsilon) = \lambda$ for some $\lambda \in (0, 1)$ and $\varepsilon > 0$. There exists $\varepsilon' > 0$ such that $\varepsilon = s\varepsilon' + \mu(\frac{\varepsilon'}{2})^2$. By the uniform continuity of τ_1 , we observe that for any $\lambda_0 \in (\lambda, 1)$, there exists $\lambda'_0 \in (0, 1)$ satisfying $\tau_1(\lambda'_0, \lambda'_0) \geq \lambda_0$. Given this λ'_0 , there exists $\lambda_1 \in (0, 1)$ such that $\tau_1(\lambda_1, \lambda_1) \geq \lambda'_0$. Furthermore, for λ_1 , there exists $\lambda_2 \in (0, 1)$ such that $\tau_2(\lambda_2, \lambda_2) \geq \lambda_1$.

Suppose that there exists $w \in R_u(\frac{\varepsilon'}{2}, 1 - \eta) \cap R_v(\frac{\varepsilon'}{2}, 1 - \eta)$, where $\eta = \max\{\lambda'_0, \lambda_1, \lambda_2\}$. Then, we have

$$\mathcal{F}_{u,w}(\frac{\varepsilon'}{2}) > 1 - (1 - \eta) = \eta \text{ and } \mathcal{F}_{w,v}(\frac{\varepsilon'}{2}) > 1 - (1 - \eta) = \eta.$$

Applying the generalized triangle inequality, it follows that

$$\begin{aligned} \lambda = \mathcal{F}_{u,v}(s\varepsilon' + \mu(\frac{\varepsilon'}{2})^2) &\geq \tau_1\left(\mathcal{F}_{u,w}\left(\frac{\varepsilon'}{2}\right), \tau_1\left(\mathcal{F}_{w,v}\left(\frac{\varepsilon'}{2}\right), \tau_2\left(\mathcal{F}_{u,w}\left(\frac{\varepsilon'}{2}\right), \mathcal{F}_{w,v}\left(\frac{\varepsilon'}{2}\right)\right)\right)\right) \\ &> \tau_1\left(\lambda'_0, \tau_1(\lambda_1, \tau_2(\lambda_2, \lambda_2))\right) \geq \tau_1\left(\lambda'_0, \tau_1(\lambda_1, \lambda_1)\right) \\ &\geq \tau_1\left(\lambda'_0, \lambda'_0\right) \geq \lambda_0 \\ &\geq \lambda. \end{aligned}$$

This yields a contradiction. Therefore, $R_u(\frac{\varepsilon'}{2}, 1 - \eta) \cap R_v(\frac{\varepsilon'}{2}, 1 - \eta) = \emptyset$. Consequently, the (ε, λ) -topology is Hausdorff. $\square$

The notions of a Cauchy sequence, completeness and convergence can all be applied to the case of a PbSMS as shown below.

Definition 10. *A sequence $(u_n)_n$ in a $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ is*

- (i) *Convergent to $u \in \Omega$ if for any given $\lambda > 0$ and $\varepsilon > 0$, there exist $n_0(\varepsilon, \lambda) \in \mathbb{N}$ that satisfies $\mathcal{F}_{u_n, u}(\varepsilon) > 1 - \lambda$, whenever $n \geq n_0(\varepsilon, \lambda)$;*
- (ii) *A Cauchy sequence if for any $\lambda > 0$ and $\varepsilon > 0$, there exist $n_0(\varepsilon, \lambda) \in \mathbb{N}$ that satisfies $\mathcal{F}_{u_n, u_m}(\varepsilon) > 1 - \lambda$, whenever $n, m \geq n_0(\varepsilon, \lambda)$;*
- (iii) *A PbSMS is complete if each Cauchy sequence in Ω is convergent in Ω .*

Lemma 2. *Let $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ be a PbSMS with the continuous t -norms τ_1 and τ_2 . Then, every convergent sequence is a Cauchy sequence.*

Proof. Let $x > 0$ and $\varepsilon \in (0, 1)$. There exists $x' > 0$ such that $x = sx' + \mu \left(\frac{x'}{2}\right)^2$. By the continuity of τ_1 and τ_2 , there exists $\varepsilon_1 \in (0, 1)$ that satisfies $\tau_2(1 - \varepsilon, 1 - \varepsilon) > 1 - \varepsilon_1$, and there exists $\varepsilon_2 \in (0, 1)$ that satisfies $\tau_1(1 - \varepsilon_1, 1 - \varepsilon_1) > 1 - \varepsilon_2$. For ε_2 , there exists $\varepsilon_3 \in (0, 1)$ that satisfies $\tau_1(1 - \varepsilon_2, 1 - \varepsilon_2) > 1 - \varepsilon_3$.

We put $\varepsilon = \min\{\varepsilon, \varepsilon_1, \varepsilon_2\}$. Take a sequence $(u_n)_n$ in Ω that converges to $u \in \Omega$. Then, there exists $n_0 \in \mathbb{N}$, which satisfies $\mathcal{F}_{u_n, u}\left(\frac{x'}{2}\right) > 1 - \varepsilon$, for each $n \geq n_0$.

For all $n, m \geq n_0$, we have

$$\begin{aligned}
 \mathcal{F}_{u_n, u_m}(x) &= \mathcal{F}_{u_n, u_m}\left(sx' + \mu \left(\frac{x'}{2}\right)^2\right) \\
 &\geq \tau_1\left(\mathcal{F}_{u_n, u}\left(\frac{x'}{2}\right), \tau_1\left(\mathcal{F}_{u, u_m}\left(\frac{x'}{2}\right), \tau_2\left(\mathcal{F}_{u_n, u}\left(\frac{x'}{2}\right), \mathcal{F}_{u, u_m}\left(\frac{x'}{2}\right)\right)\right)\right) \\
 &> \tau_1(1 - \varepsilon, \tau_1(1 - \varepsilon, \tau_2(1 - \varepsilon, 1 - \varepsilon))) \\
 &\geq \tau_1(1 - \varepsilon_2, \tau_1(1 - \varepsilon_1, \tau_2(1 - \varepsilon, 1 - \varepsilon))) \\
 &\geq \tau_1(1 - \varepsilon_2, \tau_1(1 - \varepsilon_1, 1 - \varepsilon_1)) \\
 &\geq \tau_1(1 - \varepsilon_2, 1 - \varepsilon_2) \\
 &\geq 1 - \varepsilon_3.
 \end{aligned}$$

Therefore, $\mathcal{F}_{u_n, u_m}(x) > 1 - \varepsilon_3$, which completes the proof. $\square$

Lemma 3. *Let (Ω, d_μ) be a b -suprametric space. We define $\mathcal{F} : \Omega \times \Omega \rightarrow \mathcal{D}^+$ by*

$$(1) \quad \mathcal{F}_{u, v}(x) = \mathcal{H}(x - d_\mu(u, v)) = \begin{cases} 0 & \text{if } x \leq d_\mu(u, v), \\ 1 & \text{if } d_\mu(u, v) < x. \end{cases}$$

Then, $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ is a PbSMS.

Proof. It is obvious that $\mathcal{F}_{u, v}$ satisfies conditions (i), (ii) and (iii) of Definition 9. Let us now prove the following inequality for every $u, v, w \in \Omega$ and $x, y \in (0, +\infty)$:

$$(2) \quad \mathcal{F}_{u, v}(s(x + y) + \mu xy) \geq \tau_1(\mathcal{F}_{u, w}(x), \tau_1(\mathcal{F}_{w, v}(y), \tau_2(\mathcal{F}_{u, w}(x), \mathcal{F}_{w, v}(y)))).$$

Since $\mathcal{F}_{u,v}(s(x+y) + \mu xy) \in \{0, 1\}$, then (2) holds if

$$\tau_1(\mathcal{F}_{u,w}(x), \tau_1(\mathcal{F}_{w,v}(y), \tau_2(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y)))) = 0.$$

Suppose that $\tau_1(\mathcal{F}_{u,w}(x), \tau_1(\mathcal{F}_{w,v}(s), \tau_2(\mathcal{F}_{u,w}(x), \mathcal{F}_{w,v}(y)))) = 1$. Then, $d_\mu(u, w) < x$ and $d_\mu(w, v) < y$. Since (Ω, d_μ) is a b -suprametric space, one has

$$d_\mu(u, v) \leq s(d_\mu(u, w) + d_\mu(w, v)) + \mu d_\mu(u, w) d_\mu(w, v) < s(x+y) + \mu xy.$$

Hence, $\mathcal{F}_{u,v}(s(x+y) + \mu xy) = 1$, which proves that (2) also holds. Therefore, $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ is a $PbSMS$.

However, it is straightforward to confirm that $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ is complete if and only if (Ω, d_μ) is complete. $\square$

3.2. Fixed point theorem in probabilistic b -suprametric space.

For this section, we represent by Ψ the class of all function $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that ψ is nondecreasing, and $\lim_{n \rightarrow +\infty} \psi^n(x) = 0$, for all $x \in \mathbb{R}_+$.

Remak 2. If $\psi \in \Psi$, then $\psi(0) = 0$ and $\psi(x) < x$, for all $x \in \mathbb{R}_+ \setminus \{0\}$.

Definition 11. Let $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ be a $PbSMS$. A mapping $f : \Omega \rightarrow \Omega$ is called a probabilistic ψ -contraction if there exists a $\psi \in \Psi$ such that for every $u, v \in \Omega$ and $x \in \mathbb{R}_+$, we have

$$\mathcal{F}_{fu, fv}(\psi(x)) \geq \mathcal{F}_{u,v}(sx),$$

where $s \geq 1$.

Theorem 3. Let $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ be a complete $PbSMS$ with $\text{Ran } \mathcal{F} \subset \mathcal{D}^+$ and continuous t -norms (τ_1, τ_2) of H_2 -type. Let $f : \Omega \rightarrow \Omega$ be a probabilistic ψ -contraction, where $\psi \in \Psi$. Then, f admits a unique fixed point.

Proof. Let $u_0 \in \Omega$, and define a sequence $(u_n)_{n \geq 0}$ by

$$u_n = f(u_{n-1}) = f^n(u_0), \quad \text{for all } n \geq 1.$$

Step 1: For each $n \in \mathbb{N} \setminus \{0\}$ and $x > 0$, we have $x > \psi(x)$ and ψ is nondecreasing, then $x > \psi^n(x)$, by using the monotonicity of $\mathcal{F}$ and the contractivity condition, we obtain

$$\begin{aligned}
\mathcal{F}_{u_n, u_{n+1}}(x) &> \mathcal{F}_{u_n, u_{n+1}}(\psi^n(x)) \\
&\geq \mathcal{F}_{u_{n-1}, u_n}(s\psi^{n-1}(x)) \geq \mathcal{F}_{u_{n-1}, u_n}(\psi^{n-1}(x)) \\
&\geq \mathcal{F}_{u_{n-2}, u_{n-1}}(s\psi^{n-2}(x)) \geq \mathcal{F}_{u_{n-2}, u_{n-1}}(\psi^{n-2}(x)) \\
&\geq \dots \geq \mathcal{F}_{u_0, u_1}(sx) \geq \mathcal{F}_{u_0, u_1}(x).
\end{aligned}$$

Next, we shall prove that $\lim_{n \rightarrow +\infty} \mathcal{F}_{u_n, u_{n+1}}(x) = 1$, for all $x > 0$.

Since $\text{Ran}\mathcal{F} \subset \mathcal{D}^+$, then there exists a $x_0 \in (0, x)$ such that for any $\lambda \in (0, 1]$, we have $\mathcal{F}_{u_0, u_1}(x_0) > 1 - \lambda$. As $\psi \in \Psi$, then there is a $n_0 \in \mathbb{N}$ such that $\psi^n(x_0) < \lambda$, for each $n \geq n_0$.

Using the monotonicity of $\mathcal{F}$ we get

$$\mathcal{F}_{u_n, u_{n+1}}(x) \geq \mathcal{F}_{u_n, u_{n+1}}(\psi^n(x_0)) \geq \mathcal{F}_{u_0, u_1}(x_0) > 1 - \lambda, \quad \text{for all } n \geq n_0,$$

which implies that

$$(3) \quad \lim_{n \rightarrow +\infty} \mathcal{F}_{u_n, u_{n+1}}(x) = 1, \quad \text{for all } x > 0.$$

Step 2: For an element $x > 0$, we can find $x_1 > 0$ such that $x = s(x_1 + \psi(x)) + \mu x_1 \psi(x)$.

We demonstrate by induction that for any $m \geq 1$

$$(4) \quad \mathcal{F}_{u_n, u_{n+m}}(x) \geq h_{\mathcal{F}_{u_n, u_{n+1}}(x_1)}^{m-1}(\mathcal{F}_{u_n, u_{n+1}}(x_1)).$$

Inequality (4) is satisfied for $m = 1$. Case where $m = 2$.

$$\begin{aligned}
\mathcal{F}_{u_n, u_{n+2}}(x) &= \mathcal{F}_{u_n, u_{n+2}}(s(x_1 + \psi(x)) + \mu x_1 \psi(x)) \\
&\geq \tau_1(\mathcal{F}_{u_n, u_{n+1}}(x_1), \tau_1(\mathcal{F}_{u_{n+1}, u_{n+2}}(\psi(x)), \tau_2(\mathcal{F}_{u_n, u_{n+1}}(x_1), \mathcal{F}_{u_{n+1}, u_{n+2}}(\psi(x))))) \\
&\geq \tau_1(\mathcal{F}_{u_n, u_{n+1}}(x_1), \tau_1(\mathcal{F}_{u_n, u_{n+1}}(sx), \tau_2(\mathcal{F}_{u_n, u_{n+1}}(x_1), \mathcal{F}_{u_n, u_{n+1}}(sx)))) \\
&\geq \tau_1(\mathcal{F}_{u_n, u_{n+1}}(x_1), \tau_1(\mathcal{F}_{u_n, u_{n+1}}(x), \tau_2(\mathcal{F}_{u_n, u_{n+1}}(x_1), \mathcal{F}_{u_n, u_{n+1}}(x)))) \\
&\geq h_{\mathcal{F}_{u_n, u_{n+1}}(x_1)}^1(\mathcal{F}_{u_n, u_{n+1}}(x_1)).
\end{aligned}$$

Now suppose that (4) holds for $m \geq 1$

Using the contractivity condition and the monotonicity of τ_1 and τ_2 , we have

$$\begin{aligned}
\mathcal{F}_{u_n, u_{n+m+1}}(x) &= \mathcal{F}_{u_n, u_{n+m+1}}(s(x_1 + \psi(x)) + \mu x_1 \psi(x)) \\
&\geq \tau_1(\mathcal{F}_{u_n, u_{n+1}}(x_1), \tau_1(\mathcal{F}_{u_{n+1}, u_{n+m+1}}(\psi(x)), \tau_2(\mathcal{F}_{u_n, u_{n+1}}(x_1), \mathcal{F}_{u_{n+1}, u_{n+m+1}}(\psi(x)))))) \\
&\geq \tau_1(\mathcal{F}_{u_n, u_{n+1}}(x_1), \tau_1(\mathcal{F}_{u_n, u_{n+m}}(sx), \tau_2(\mathcal{F}_{u_n, u_{n+1}}(x_1), \mathcal{F}_{u_n, u_{n+m}}(sx)))) \\
&\geq \tau_1(\mathcal{F}_{u_n, u_{n+1}}(x_1), \tau_1(\mathcal{F}_{u_n, u_{n+m}}(x), \tau_2(\mathcal{F}_{u_n, u_{n+1}}(x_1), \mathcal{F}_{u_n, u_{n+m}}(x)))) \\
&\geq \tau_1\left(\mathcal{F}_{u_n, u_{n+1}}(x_1), \tau_1\left(h_{\mathcal{F}_{u_n, u_{n+1}}(x_1)}^{m-1}(\mathcal{F}_{u_n, u_{n+1}}(x_1))\right)\right), \\
&\quad \tau_2\left(\mathcal{F}_{u_n, u_{n+1}}(x_1), h_{\mathcal{F}_{u_n, u_{n+1}}(x_1)}^{m-1}(\mathcal{F}_{u_n, u_{n+1}}(x_1))\right)\right) \\
&= h_{\mathcal{F}_{u_n, u_{n+1}}(x_1)}^m(\mathcal{F}_{u_n, u_{n+1}}(x_1)).
\end{aligned}$$

Hence, (4) is proved for all $m \geq 1$. Thus,

$$(5) \quad \mathcal{F}_{u_n, u_{n+m}}(x) \geq h_{\mathcal{F}_{u_n, u_{n+1}}(x_1)}^m(\mathcal{F}_{u_n, u_{n+1}}(x_1)).$$

Step 3: We show that $(u_n)_{n \geq 0}$ is a Cauchy sequence.

Let $\varepsilon \in (0, 1)$, since (τ_1, τ_2) is of H_2 -type, there exists $\lambda \in (0, 1)$ such that

$$u > 1 - \lambda \implies h_u^m(u) > 1 - \varepsilon, \quad \text{for all } m \geq 1,$$

then by $\lim_{n \rightarrow +\infty} \mathcal{F}_{u_n, u_{n+1}}(x_1) = 1$, there is a $N \in \mathbb{N}$ such that $\mathcal{F}_{u_n, u_{n+1}}(x_1) > 1 - \lambda$, for all $n \geq N$.

Hence,

$$\begin{aligned}
\mathcal{F}_{u_n, u_{n+1}}(x) &\geq h_{\mathcal{F}_{u_n, u_{n+1}}(x_1)}^m(\mathcal{F}_{u_n, u_{n+1}}(x_1)) \\
&\geq 1 - \varepsilon, \quad \text{for all } n \geq N \text{ and } m \geq 1.
\end{aligned}$$

Thus, $(u_n)_{n \geq 0}$ is a Cauchy sequence. From the completeness of Ω , the sequence $(u_n)_{n \geq 0}$ converges to an element $u \in \Omega$.

Step 4: We shall now prove that u is a unique fixed point of f .

Let $a > 0$ and $a_1 = \frac{a - \psi(a)}{1 + \mu\psi(a)}$. The generalized triangular inequality and (2) implies that

$$\begin{aligned}
\mathcal{F}_{u, fu}(a) &\geq \tau_1(\mathcal{F}_{u, u_n}(a_1), \tau_1(\mathcal{F}_{u_n, fu}(\psi(a)), \tau_2(\mathcal{F}_{u, u_n}(a_1), \mathcal{F}_{u_n, fu}(\psi(a))))) \\
&\geq \tau_1(\mathcal{F}_{u, u_n}(a_1), \tau_1(\mathcal{F}_{fu_{n-1}, fu}(\psi(a)), \tau_2(\mathcal{F}_{u, u_n}(a_1), \mathcal{F}_{fu_{n-1}, fu}(\psi(a))))) \\
&\geq \tau_1(\mathcal{F}_{u, u_n}(a_1), \tau_1(\mathcal{F}_{u_{n-1}, u}(sa), \tau_2(\mathcal{F}_{u, u_n}(a_1), \mathcal{F}_{u_{n-1}, u}(sa))))
\end{aligned}$$

$$\geq \tau_1 \left(\mathcal{F}_{u,u_n}(a_1), \tau_1 \left(\mathcal{F}_{u_{n-1},u}(a), \tau_2 \left(\mathcal{F}_{u,u_n}(a_1), \mathcal{F}_{u_{n-1},u}(a) \right) \right) \right).$$

Letting $n \rightarrow +\infty$, we get

$$\mathcal{F}_{u,fu}(a) \geq 1, \quad \text{for all } a > 0,$$

this implies that $fu = u$ and u is a fixed point of f .

We now establish the uniqueness of u . Assume that there exists another element $v \in \Omega$, with $v \neq u$, satisfying $fv = v$. Let $x > 0$, we have

$$\mathcal{F}_{u,v}(\psi(x)) = \mathcal{F}_{fu,fv}(\psi(x)) \geq \mathcal{F}_{u,v}(sx) \geq \mathcal{F}_{u,v}(x),$$

and since $\psi(x) < x$, then $\mathcal{F}_{u,v}(\psi(x)) \leq \mathcal{F}_{u,v}(x)$. Therefore $\mathcal{F}_{u,v}(\psi(x)) = \mathcal{F}_{u,v}(x)$.

Using (6), we can easily show, by induction, that

$$(6) \quad \mathcal{F}_{u,v}(\psi^n(x)) = \mathcal{F}_{u,v}(x), \quad \text{for all } n \geq 1 \text{ and } x > 0.$$

Since $\lim_{n \rightarrow +\infty} \psi^n(x) = 0$, we conclude that $u = v$, which is absurd. Thus, u is the unique fixed point of f . $\square$

Example 3. Let $p \in (0, 1)$ and $\Omega := L_p([0, 1], \mathbb{R})$, where

$$L_p([0, 1], \mathbb{R}) := \left\{ f : X \rightarrow \mathbb{R} : \int_0^1 |f(t)|^p < +\infty \right\}.$$

Define $d_p, d_\mu : \Omega \times \Omega \rightarrow \mathbb{R}_+$ as

$$d_p(f, g) = \left(\int_0^1 |f(t) - g(t)|^p \right)^{\frac{1}{p}} \text{ and } d_\mu(f, g) = d_p(f, g)(d_p(f, g) + 1),$$

where $f, g \in \Omega$. Then (Ω, d_p) is a b -suprametric space [7-9], with coefficient $2^{1/p}$, and (Ω, d_μ) is b -suprametric space, with $s = 4^{\frac{1}{p}}$ and $\mu = 8^{\frac{1}{p}}$ (see [7], Example 2.9). Let $\mathcal{F}$ be defined by $\mathcal{F}_{f,g}(x) = \mathcal{H}(x - d_\mu(f, g))$, for all $f, g \in \Omega$. By Lemma 3, the space $(\Omega, \mathcal{F}, (\tau_p, \tau_L), \mu)$ is a complete PbSMS.

Let $\mathcal{T} : \Omega \rightarrow \Omega$ be the function defined by $\mathcal{T}(f)(t) := \sin \left(\frac{f(t)}{4^{\frac{1}{p}}} \right)$, for all $t \in [0, 1]$. For each $(f, g) \in \Omega^2$, we have

$$\begin{aligned} d_\mu(\mathcal{T}(f), \mathcal{T}(g)) &= d_p((\mathcal{T}(f), (\mathcal{T}(g)))(d_p((\mathcal{T}(f), (\mathcal{T}(g))) + 1) \\ &\leq \frac{1}{4^{\frac{2}{p}}} d_p(f, g) \left(\frac{1}{4^{\frac{2}{p}}} d_p(f, g) + 1 \right) \end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{4^{\frac{2}{p}}} d_p(f, g)(d_p(f, g) + 1) \\ &= \frac{1}{4^{\frac{2}{p}}} d_\mu(f, g) = \psi\left(\frac{1}{4^{\frac{1}{p}}} d_\mu(f, g)\right), \end{aligned}$$

where $\psi(x) = \frac{1}{4^{\frac{1}{p}}}x$, for all $x \in \mathbb{R}_+$. We have $\psi \in \Psi$. Let us show that

$$\mathcal{F}_{\mathcal{T}(f), \mathcal{T}(g)}(\psi(x)) \geq \mathcal{F}_{f, g}(sx), \text{ for all } x \in \mathbb{R}_+.$$

If $sx \leq d_\mu(f, g)$, then the inequality is true.

If $sx > d_\mu(f, g)$ and since ψ is nondecreasing on $\mathbb{R}_+$, then

$$\psi(x) > \psi\left(\frac{1}{s} d_\mu(f, g)\right) \geq d_\mu(\mathcal{T}(f), \mathcal{T}(g)).$$

Thus, $\mathcal{F}_{\mathcal{T}(f), \mathcal{T}(g)}(\psi(x)) \geq \mathcal{F}_{f, g}(sx)$, for all $x \in \mathbb{R}_+$.

Hence, $\mathcal{T}$ is probabilistic Ψ -contraction, by Theorem 3, $\mathcal{T}$ has unique fixed point that is 0.

As a corollary we give the result of Chaira and El Harmouchi [11] concerning Probabilistic suprametric spaces (PSMS):

Corollary 1. Let $(\Omega, \mathcal{F}, (\tau_1, \tau_2), \mu)$ be a complete PSMS with $\text{Ran } \mathcal{F} \subset \mathcal{D}^+$ and continuous t -norms (τ_1, τ_2) of H_2 -type. Suppose f is a self-mapping of Ω into itself satisfying

$$\mathcal{F}_{fu, fv}(kx) \geq \mathcal{F}_{u, v}(sx),$$

with $k \in [0, 1[$ and $s \geq 1$. Then, f admits a unique fixed point.

3.3. Application to (ζ_1, ζ_2) -Decomposable measures.

In this section $(\Omega, \mathcal{A})$ always denotes some measurable space. We give a generalization of the ζ -decomposable measure which was defined by Weber (see article [31]):

Definition 12. Let ζ_1, ζ_2 be two t -conorms. A (ζ_1, ζ_2) -decomposable measure m is a set function $m : \mathcal{A} \rightarrow [0, 1]$ such that $m(\emptyset) = 0$, $m(\Omega) = 1$ and

$$m(A \cup B) = \zeta_1\left(\zeta_1(m(A), m(B)), \zeta_2(m(A), m(B))\right),$$

whenever $A, B \in \mathcal{A}$, with $A \cap B = \emptyset$.

Example 4. Let $\Omega = \mathbb{N}$, $\mathcal{A} = 2^{\mathbb{N}}$ and $m(K) = \min\{\frac{\text{card}(K)}{m}, 1\}$, for a fixed non-zero natural number m , where $\text{card}(K)$ is the cardinal number of K . First $m(\Omega) = 1$ and $m(\emptyset) = 0$. Let A and B be two elements of $\mathcal{A}$ such that $A \cap B = \emptyset$. We have $m(A \cup B) = \min\{\frac{\text{card}(A) + \text{card}(B)}{m}, 1\}$.

If $\frac{\text{card}(A) + \text{card}(B)}{m} \leq 1$, then

$$\begin{aligned} \min\left\{\frac{\text{card}(A) + \text{card}(B)}{m}, 1\right\} &= \frac{\text{card}(A)}{m} + \frac{\text{card}(B)}{m} \\ &= \min\left\{\min\left\{\frac{\text{card}(A)}{m}, 1\right\} + \min\left\{\frac{\text{card}(B)}{m}, 1\right\}, 1\right\}. \end{aligned}$$

If $\frac{\text{card}(A) + \text{card}(B)}{m} > 1$, then

$$\min\left\{\frac{\text{card}(A) + \text{card}(B)}{m}, 1\right\} = 1 = \min\left\{\min\left\{\frac{\text{card}(A)}{m}, 1\right\} + \min\left\{\frac{\text{card}(B)}{m}, 1\right\}, 1\right\}.$$

We consider the two t -conorms: $\zeta_M(x, y) = \max\{x, y\}$ and $\zeta_L(x, y) = \min\{x + y, 1\}$, where $x, y \in [0, 1]$. We obtain

$$\begin{aligned} \zeta_M(\zeta_M(m(A), m(B)), \zeta_L(m(A), m(B))) &= \max\{\zeta_M(m(A), m(B)), \zeta_L(m(A), m(B))\} \\ &= \max\{\max\{m(A), m(B)\}, \min\{m(A) + m(B), 1\}\} \\ &= \min\{m(A) + m(B), 1\} \\ &= \min\left\{\min\left\{\frac{\text{card}(A)}{N}, 1\right\} + \min\left\{\frac{\text{card}(B)}{N}, 1\right\}, 1\right\} \\ &= m(A \cup B). \end{aligned}$$

Thus, m is (ζ_1, ζ_2) -decomposable measure.

Definition 13. A measure m is of $(NSA)_2$ -type if there exist two triangular conorms ζ_1 and ζ_2 continuous, not strict, Archimedean such that $z_1 \circ m$ or $z_2 \circ m$ is a finite additive measure, where z_1 and z_2 are two additive generators for ζ_1 and ζ_2 resp., and in relation to which m is (ζ_1, ζ_2) -decomposable and $((z_1(1), z_2(1)) = (1, 1))$.

Property 1. Let m be of $(NSA)_2$ -type with z_1 and z_2 are two additive generators for the triangular conorms ζ_1 and ζ_2 respectively. If m is (ζ_1, ζ_2) -decomposable measure, then m is monotone i.e. for all $A, B \in \mathcal{A}$,

$$A \subseteq B \Rightarrow m(A) \leq m(B).$$

Indeed, Let $A, B \in \mathcal{A}$, such that $A \subseteq B$. We have

$$\begin{aligned} m(B) &= m(A \cup (B \setminus A)) = \zeta_1 \left(\zeta_1(m(A), m(B \setminus A)), \zeta_2(m(A), m(B \setminus A)) \right) \\ &\geq \zeta_1 \left(\zeta_1(m(A), 0), \zeta_2(0, 0) \right) = \zeta_1(m(A), 0) = m(A). \end{aligned}$$

Lemma 4. *Let m be of $(NSA)_2$ -type. Suppose $\mathbf{z}_1$ and $\mathbf{z}_2$ strictly increases. Then for every $A, B \in \mathcal{A}$, $m(A \cap B) \geq \tau_1 \left(\tau_1(m(A), m(B)), \tau_2(m(A), m(B)) \right)$, where τ_1 and τ_2 are two t -norms with two additive generators $\mathbf{t}_1 = 1 - \mathbf{z}_1$ and $\mathbf{t}_2 = 1 - \mathbf{z}_2$.*

Proof. Since m is of $(NSA)_2$ -type, $\mathbf{z}_1 \circ m$ or $\mathbf{z}_2 \circ m$ are two finite additive measures, where $\mathbf{z}_1$ and $\mathbf{z}_2$ are mentioned in the definition 13. We have $\mathbf{t}_1 = 1 - \mathbf{z}_1$ and $\mathbf{t}_2 = 1 - \mathbf{z}_2$ are a generators of t -norm τ_1 and τ_2 resp. Suppose $\mathbf{z}_1 \circ m$ is finite additive measure, then

$$\begin{aligned} m(A \cap B) &= \tau_1(m(A \cap B), 1) \geq \tau_1 \left(m(A \cap B), \tau_2(m(A), m(B)) \right) \\ &= \tau_1 \left(\mathbf{z}_1^{-1}[\mathbf{z}_1(m(A \cap B))], \tau_2(m(A), m(B)) \right) \\ &\geq \tau_1 \left(\mathbf{z}_1^{-1}(\max\{\mathbf{z}_1 \circ m(A) + \mathbf{z}_1 \circ m(B) - 1, 0\}), \tau_2(m(A), m(B)) \right), \text{ (because } \mathbf{z}_1 \\ &\quad \text{is monotone increasing)} \\ &= \tau_1 \left(\mathbf{t}_1^{-1}(\min\{\mathbf{t}_1 \circ m(A) + \mathbf{t}_1 \circ m(B), 1\}), \tau_2(m(A), m(B)) \right) \\ &= \tau_1 \left(\tau_1(m(A), m(B)), \tau_2(m(A), m(B)) \right). \end{aligned}$$

If $\mathbf{z}_2 \circ m$ is finite additive measure, we proceed in the same way using

$$m(A \cap B) = \tau_1 \left(1, \tau_2(1, m(A \cap B)) \right) \geq \tau_1 \left(\tau_1(m(A), m(B)), m(A \cap B) \right).$$

Hence,

$$m(A \cap B) \geq \tau_1 \left(\tau_1(m(A), m(B)), \tau_2(m(A), m(B)) \right). \quad \square$$

Definition 14. *A set function $m : \mathcal{A} \rightarrow [0, 1]$ is continuous if m verifies the following properties:*

- (a) *if $A_n \in \mathcal{A}$ and $A_n \subseteq A_{n+1}$, for all $n \in \mathbb{N} \setminus \{0\}$, then $m\left(\bigcup_{n \geq 1} A_n\right) = \lim_{n \rightarrow +\infty} m(A_n)$;*

(b) if $A_n \in \mathcal{A}$ and $A_{n+1} \subseteq A_n$, for all $n \in \mathbb{N} \setminus \{0\}$, with $m(A_1) < \infty$, then

$$m\left(\bigcap_{n \geq 1} A_n\right) = \lim_{n \rightarrow +\infty} m(A_n).$$

We recall the density and separability which were mentioned in article [12].

Definition 15. Let $(\mathcal{M}, d_\mu)$ be a b -suprametric space and A is a nonempty subset of $\mathcal{M}$.

- 1) A is dense in $\mathcal{M}$ if $\overline{A}^{d_\mu} = \mathcal{M}$.
- 2) $\mathcal{M}$ is separable if it has a countable dense subset.

Example 5. We continue with the previous example 1. We consider the b -suprametric space $\mathcal{M} = (\ell_p(\mathbb{R}), d_\mu)$, with

$$d_\mu(u, v) = \|u - v\|_p (\|u - v\|_p + 1), \text{ for all } u, v \in \ell_p(\mathbb{R}),$$

and $s = 4^{\frac{1}{p}}$ and $\mu = 8^{\frac{1}{p}}$. Since $(\ell_p(\mathbb{R}), \|\cdot\|_p)$ is a separable space, so is $(\ell_p(\mathbb{R}), d_\mu)$. Indeed, there exists a countable subset $\mathcal{K}$ dense in $(\ell_p(\mathbb{R}), \|\cdot\|_p)$. If $u = (u_n)_{n \geq 1} \in \ell_p(\mathbb{R})$, there exists a sequence $(U_k)_{k \geq 1}$ of $\mathcal{K}$ such that $\lim_{k \rightarrow +\infty} \|U_k - u\|_p = 0$, and consequently

$$\lim_{k \rightarrow +\infty} d_\mu(U_k, u) = \lim_{k \rightarrow +\infty} \|U_k - u\|_p (\|U_k - u\|_p + 1) = 0.$$

Thus, $\ell_p(\mathbb{R}) = \overline{\mathcal{K}}^{d_\mu}$.

We follow the same steps as in the proof of Proposition 4 of article [12], except that in the next proposition we will use b -suprametric.

Proposition 2. Let $(\Omega, \mathcal{A})$ be a measurable space and $(\mathcal{M}, d_\mu)$ be a separable b -suprametric space. If for all $\omega \in \Omega$, the map $f(\omega, \cdot) : x \mapsto f(\omega, x)$ is continuous on $\mathcal{M}$. Then f is a random operator.

If $(\Omega, \mathcal{A})$ is a measurable space and $(\mathcal{M}, d_\mu)$ is a separable b -suprametric space, by $\mathfrak{S}$ we shall denote the set of all the equivalence classes of measurable mappings $X : \Omega \rightarrow \mathcal{M}$. An element from $\mathfrak{S}$ will be denoted by $\tilde{X}$ if $\{X(\omega)\} \in \tilde{X}$.

Proposition 3. Let $(\Omega, \mathcal{A})$ be a measurable space and m be a (ζ_1, ζ_2) -decomposable continuous measure and of $(NSA)_2$ -type, where ζ_1 and ζ_2 are two t -conorms with monotone increasing

generators $\mathbf{z}_1$ and $\mathbf{z}_2$ respectively. Then $(\mathfrak{S}, \mathcal{F}, (\tau_1, \tau_2), \mu)$ is a PbSMS, where $\mathcal{F}$ and the t -norms τ_1 and τ_2 are defined as follows:

$$\mathcal{F}(\tilde{X}, \tilde{Y}) := F_{\tilde{X}, \tilde{Y}}(u) = m\left(\left\{\omega \in \Omega : d_\mu(X(\omega), Y(\omega)) < u\right\}\right) = m(\{d_\mu(X, Y) < u\}),$$

$$\text{for every } \tilde{X}, \tilde{Y} \in \mathfrak{S}, u \in \mathbb{R}, \text{ and } \begin{cases} \tau_1(x, y) = \mathbf{z}_1^{-1}(\max\{0, \mathbf{z}_1(x) + \mathbf{z}_1(y) - 1\}), \\ \tau_2(x, y) = \mathbf{z}_2^{-1}(\max\{0, \mathbf{z}_2(x) + \mathbf{z}_2(y) - 1\}), \end{cases} \text{ for every } x, y \in [0, 1].$$

Proof. We have to prove that the mapping F maps $\mathfrak{S} \times \mathfrak{S}$ into $\mathcal{D}^+$, such that the following conditions are satisfied:

- (i) $(F_{\tilde{X}, \tilde{X}}(u) = 1, \text{ for every } u > 0;$
- (ii) $(F_{\tilde{X}, \tilde{Y}}(u) = 1, \text{ for every } u > 0) \Rightarrow \tilde{X} = \tilde{Y};$
- (iii) $F_{\tilde{X}, \tilde{Y}} = F_{\tilde{Y}, \tilde{X}}, \text{ for every } (\tilde{X}, \tilde{Y}) \in \mathfrak{S} \times \mathfrak{S};$
- (iv) $F_{\tilde{X}, \tilde{Z}}(s(u+v) + \mu uv) \geq \tau_1(F_{\tilde{X}, \tilde{Y}}(u), F_{\tilde{Y}, \tilde{Z}}(v)), \tau_2(F_{\tilde{X}, \tilde{Y}}(u), F_{\tilde{Y}, \tilde{Z}}(v)), \text{ for every } \tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{S} \text{ and every } u, v > 0.$

First $F_{\tilde{X}, \tilde{Y}} : \mathbb{R} \rightarrow [0, 1]$ is monotone nondecreasing, because m is monotone. Since $F_{\tilde{X}, \tilde{Y}}(0) = m(\emptyset) = 0$, in order to prove that $F_{\tilde{X}, \tilde{Y}} \in \mathcal{D}^+$ it remains to prove that $\lim_{u \rightarrow +\infty} F_{\tilde{X}, \tilde{Y}}(u) = 1$.

Let $A_n = \{d_\mu(X, Y) < n\}$, $n \in \mathbb{N} \setminus \{0\}$. We have $\{A_n\}_{n \geq 1}$ is an increasing sequence in the sense of inclusion, and from the continuity of m and $\bigcup_{n \geq 1} A_n = \Omega$, it follows that

$$1 = m(\Omega) = m\left(\bigcup_{n \geq 1} A_n\right) = \lim_{n \rightarrow +\infty} F_{\tilde{X}, \tilde{Y}}(n) = \lim_{u \rightarrow +\infty} F_{\tilde{X}, \tilde{Y}}(u).$$

We directly obtain (i) and (iii). We shall prove that (ii) holds. Suppose $F_{\tilde{X}, \tilde{Y}}(u) = 1$, for every $u > 0$, so $F_{\tilde{X}, \tilde{Y}}(\frac{1}{n}) = 1$, for every $n \in \mathbb{N} \setminus \{0\}$. We have that $m(A_n) = 1$, where $A_n = \{d(X, Y) < \frac{1}{n}\}$, for all $n \in \mathbb{N} \setminus \{0\}$. From $\{d(X, Y) = 0\} = \bigcap_{n \geq 1} A_n$ and the continuity of m it follows that $m(\{d(X, Y) = 0\}) = \lim_{n \rightarrow +\infty} m(A_n) = 1$, and therefore $\tilde{X} = \tilde{Y}$.

We shall prove that (iv) holds. Let $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{S}$ and $u, v > 0$. We consider the three sets

$$A = \{d(X, Y) < u\}, B = \{d(Y, Z) < v\} \text{ and } C = \{d(X, Z) < s(u+v) + \mu uv\}.$$

To prove (iv) means we need to show that $m(C) \geq \tau_1(\tau_1(m(A), m(B)), \tau_2(m(A), m(B)))$.

We have $A \cap B \subseteq C$ and since m is monotone, $m(A \cap B) \leq m(C)$. Lemma 4 implies that

$$m(C) \geq m(A \cap B) \geq \tau_1\left(\tau_1(m(A), m(B)), \tau_2(m(A), m(B))\right),$$

where τ_1 and τ_2 are given by

$$\tau_1(x, y) = z_1^{-1}(\max(0, z_1(x) + z_1(y) - 1)) \text{ and } \tau_2(x, y) = z_2^{-1}(\max(0, z_2(x) + z_2(y) - 1)),$$

for all $x, y \in [0, 1]$, which concludes the proof. $\square$

Lemma 5. *Let $(\Omega, \mathcal{A}, m)$ be as in Proposition 3 and $(\mathcal{M}, d_\mu)$ be a complete separable b -suprametric space. Then $(\mathfrak{S}, \mathcal{F}, (\tau_1, \tau_2), \mu)$ from Proposition 3 is complete.*

Proof. We follow the same steps as in the proof of Theorem 3 of the article by Pap and Hadžić [15], using z_1 associated with τ_1 and z_2 associated with τ_2 . $\square$

Theorem 4. *Let $(\Omega, \mathcal{A})$ be a measurable space and m be a (ζ_1, ζ_2) -decomposable continuous measure and of $(NSA)_2$ -type, where ζ_1 and ζ_2 are two t -conorms with monotone increasing generators z_1 and z_2 respectively, and τ_1, τ_2 be their dual t -norms continuous. Suppose (τ_1, τ_2) is of H_2 -type. Let $(\mathcal{M}, d_\mu)$ be a complete, separable b -suprametric space and $\xi : \Omega \times \mathcal{M} \rightarrow \mathcal{M}$ be a continuous random operator such that*

(7)

$$m(\{\omega \in \Omega : d_\mu(\xi(\omega, X(\omega)), \xi(\omega, Y(\omega))) < \psi(u)\}) \geq m(\{\omega \in \Omega : d_\mu(X(\omega), Y(\omega)) < su\}),$$

for all measurable mappings $X, Y : \Omega \mapsto \mathcal{M}$, where $\psi \in \Psi$. then there is a random fixed point for the operator ξ .

Proof. Since m be a (ζ_1, ζ_2) -decomposable continuous, measure of $(NSA)_2$ -type and $(\mathcal{M}, d_\mu)$ is complete, then according to the Proposition 3 and Lemma 5, $(\mathfrak{S}, \mathcal{F}, (\tau_1, \tau_2), \mu)$ is a complete PbSMS, where $\mathcal{F}$ and the t -norms τ_1 and τ_2 are defined as follows:

$$\mathcal{F}(\tilde{X}, \tilde{Y}) := F_{\tilde{X}, \tilde{Y}}(u) = m\left(\{\omega \in \Omega : d_\mu(X(\omega), Y(\omega)) < u\}\right) = m(\{d_\mu(X, Y) < u\}),$$

for every $\tilde{X}, \tilde{Y} \in \mathfrak{S}$, $u \in \mathbb{R}$, and
$$\begin{cases} \tau_1(x, y) = \mathbf{z}_1^{-1}(\max\{0, \mathbf{z}_1(x) + \mathbf{z}_1(y) - 1\}), \\ \tau_2(x, y) = \mathbf{z}_2^{-1}(\max\{0, \mathbf{z}_2(x) + \mathbf{z}_2(y) - 1\}), \end{cases} \quad \text{for every } x, y \in [0, 1].$$
 According to equation (7), the application $\hat{\xi} : \mathfrak{S} \rightarrow \mathfrak{S}$ defined by $(\hat{\xi}X)(\omega) = \xi(\omega, X(\omega))$, for every $\omega \in \Omega$ and $\tilde{X} \in \mathfrak{S}$, is probabilistic ψ -contraction. Hence, by Theorem 3 there is a random fixed point for the operator ξ i.e. there exists a measurable mapping $\tilde{X}_* \in \mathfrak{S}$ such that $\xi(\omega, X_*(\omega)) = X_*(\omega)$, for all $\omega \in \Omega$. $\square$

3.4. Conclusion. The probability suprametric space mentioned in the article [11] is a special case of the probability b -suprametric space. We also find the theorem of the existence and uniqueness of the fixed point for a probabilistic contraction of a probabilistic b -suprametric space. This provides us with an application to (ζ_1, ζ_2) -decomposable measures, and highlights the usefulness of our approach beyond the abstract setting.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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