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NUMERICAL METHOD FOR FRACTAL-FRACTIONAL FINANCE SYSTEM WITH HATTAF MIXED FRACTAL-FRACTIONAL DERIVATIVE

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Abstract. In this article, we investigate a new financial model with interest rate, investment demand and price index. The proposed model is developed by fractal-fractional differential equations (FFDEs) including Hattaf mixed fractal-fractional (HMFF) derivative. Firstly, we prove that our fractional model is mathematically and financially well-posed by means of fixed point theory. Additionally, a numerical scheme is implemented to generate system trajectories. Finally, numerical simulations are presented, providing a detailed discussion of how variations in the fractal and fractional parameters influence the system's behavior.

Keywords: finance; Hattaf fractal derivative; mathematical modeling; fractal-fractional calculus; fixed point theory; numerical method.

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1. INTRODUCTION

In recent years, fractional differential equations (FDEs) have become an indispensable tool for describing the dynamics of various complex phenomena incorporating memory effects in

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the fields of science and engineering. These FDEs allow for the accurate modeling of systems where the future state depends on past history, an aspect that integer-order models often fail to capture. To give these models more flexibility, several types of operators have been introduced. For instance, Hattaf [1] developed a new definition of a fractional derivative that mixes the definitions of fractional derivatives with singular and non-singular kernels, along with its application in biology. In 2026, Toufiq et al. [2] modeled the dynamics of financial system by formulating a nonlinear system of FDEs with the new generalized Hattaf fractional (GHF) derivative [3].

In parallel, the emergence of fractal-fractional calculus (FFC) has made it possible to associate the memory of fractional derivatives with the self-similarity and scale invariance of fractal geometry. These unified frameworks have found fruitful applications in modeling several real-world phenomena in many fields, such as ecology [4], chemistry [5] and epidemiology [6, 7].

Recently, Mamouni et al. [8] made a further advance by introducing the HMFF derivative. This innovative operator combines the advantages of singular and non-singular kernels within a structure that includes a weight function and a generalized fractal measure. This formulation has been successfully applied to model the complex dynamics of HIV/AIDS, and the authors have rigorously established the existence and uniqueness of solutions through the fixed-point theory. However, despite the theoretical robustness of this mixed operator, the complexity of the resulting equations often makes obtaining analytical solutions impossible. Although, numerical schemes have been developed for classical fractal-fractional derivatives such as, Atangana-Baleanu fractal-fractional derivative [9], fractal-fractional Caputo derivative [10], and for the new fractal-fractional derivative introduced by Hattaf in 2023 [11], there is currently no numerical method specifically dedicated to solving equations involving the HMFF derivative.

The original contribution of this article is to study the dynamics of a nonlinear financial system modeled by mixed fractal-fractional differential equations in the sense of Hattaf. Furthermore, we propose a new explicit numerical method, based on Lagrange polynomial interpolation, to solve nonlinear FFDEs with the HMFF derivative. To do this, the paper is organized as follows. Section 2 provides the formulation of the model and some interesting concepts

needed to the elaboration of this study. Section 3 presents the existence and uniqueness of solution by means of Banach contraction mapping principle as well as, it discusses the existence of equilibrium points. Section 4 focuses on the new numerical method. Finally, some numerical simulations are given in Section 5.

2. PRELIMINARY RESULTS AND MODEL FORMULATION

In this section, we first recall the definition of Hattaf fractal derivative. After, we present the HMFF derivative and integral operators.

Definition 2.1. [11] *Let I be an open interval of $\mathbb{R}$. The Hattaf fractal derivative of function $u(t)$ with respect to the fractal measure $g(\varepsilon, t)$ is defined as follows:*

$$(1) \quad \frac{d_g}{dt^\varepsilon} u(t) = \lim_{\tau \rightarrow t} \frac{u(t) - u(\tau)}{g(\varepsilon, t) - g(\varepsilon, \tau)}, \quad \varepsilon > 0.$$

When $\frac{d_g}{dt^\varepsilon} u(t)$ exists for all $t \in I$, we say that u is fractal differentiable in the interval I with order ε .

Note that when $g(\varepsilon, t) = t^\varepsilon$, Definition 2.1 reduced to the Hausdorff fractal derivative introduced by Chen [12]. Furthermore, if $g(\varepsilon, t) = h(t)$ with $h'(t) > 0$ and $u(t)$ is differentiable, then we find the general derivative proposed by Yang [13] and Definition 2.1 becomes

$$(2) \quad \frac{d_g}{dt^\varepsilon} u(t) = \frac{1}{h'(t)} \frac{du(t)}{dt}.$$

Next, we introduce the definition of the new Hattaf mixed fractal-fractional derivative in the sense of Caputo and Riemann-liouville.

Definition 2.2. [8] *Let $(\alpha, \beta) \in [0, 1]^2$, $r, m > 0$, and $u(t)$ be a differentiable in the interval (t_0, T) and fractal differentiable on (t_0, T) with order $0 < \varepsilon \leq 1$. The Hattaf mixed fractal-fractional derivative of the function $u(t)$ of order α in Caputo sense with respect to the weight function $\omega(t)$ is defined as follows:*

$$(3) \quad {}^{\text{FFC}}\mathbf{D}_{t_0, t, \omega, \delta}^{\alpha, \beta, r, m, \varepsilon} \mathbf{u}(t) = \frac{N(\alpha + \beta - 1)}{2 - \alpha - \beta} \frac{1}{\omega(t)} \int_{t_0}^t (t - \tau)^{\beta-1} \mathbf{E}_{r, \beta}[-\delta \mu_{\alpha, \beta}(t - \tau)^m] \frac{d_g}{dt^\varepsilon}(\omega u)(\tau) d\tau,$$

where $\delta \in \mathbb{R}^*$, $\omega \in \mathcal{C}^1(t_0, T)$, with $\omega > 0$ on $[t_0, T]$; $N(\cdot)$ a normalization function such that $N(0) = N(1) = 1$, $\mu_{\alpha, \beta} = \frac{\alpha + \beta - 1}{2 - \alpha - \beta}$; and $E_{r, \beta}(t) = \sum_{k=0}^{+\infty} \frac{t^k}{\Gamma(rk + \beta)}$ is the Mittag-Leffler function with two parameters r and β .

Definition 2.3. [8] Let $(\alpha, \beta) \in [0, 1]^2$, $r, m > 0$, and $u(t)$ be a continuous in the interval (t_0, T) and fractal differentiable on (t_0, T) with order $0 < \varepsilon \leq 1$. The Hattaf mixed fractal-fractional derivative of the function $u(t)$ of order α in Riemann-Liouville sense with respect to the weight function $\omega(t)$ is defined as follows:

$$(4) \quad {}^{FFR}D_{t_0, t, \omega, \delta}^{\alpha, \beta, r, m, \varepsilon} u(t) = \frac{N(\alpha + \beta - 1)}{2 - \alpha - \beta} \frac{1}{\omega(t)} \frac{d_g}{dt^\varepsilon} \int_{t_0}^t (t - \tau)^{\beta-1} E_{r, \beta}[-\delta \mu_{\alpha, \beta}(t - \tau)^m](\omega u)(\tau) d\tau.$$

The Hattaf mixed fractal-fractional derivative introduced in definition 2.3 provides a unified framework which contains several existing fractal-fractional derivatives involving both singular and non-singular kernels. For instance,

1. When $g(\varepsilon, t) = t^\varepsilon$, $\omega(t) = 1$ and $\beta = 1 - \alpha$, we get the fractal-fractional derivative with power-law type kernel [9], which is given by:

$${}^{FFP}D_{0, t}^{\alpha, \varepsilon} u(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d_g}{dt^\varepsilon} \int_0^t (t - \tau)^{-\alpha} u(\tau) d\tau.$$

2. When $g(\varepsilon, t) = t^\varepsilon$, $\omega(t) = 1$ and $\beta = r = m = \delta = 1$, we obtain the fractal-fractional derivative with exponential decay kernel [9], which is given by:

$${}^{FFE}D_{0, t}^{\alpha, \varepsilon} u(t) = \frac{N(\alpha)}{1 - \alpha} \frac{d_g}{dt^\varepsilon} \int_0^t \exp\left(-\frac{\alpha}{1 - \alpha}(t - s)\right) u(\tau) d\tau.$$

3. When $g(\varepsilon, t) = \frac{t^{2-\varepsilon}}{2-\varepsilon}$, $\omega(t) = \beta = \delta = 1$ and $r = m = \alpha$, we get the fractal-fractional derivative with generalized Mittag-Leffler kernel [10], which is given by:

$${}^{FFM}D_{0, t}^{\alpha, \varepsilon} u(t) = \frac{N(\alpha)}{1 - \alpha} \frac{d_g}{dt^\varepsilon} \int_0^t E_\alpha\left(-\frac{\alpha}{1 - \alpha}(t - s)^\alpha\right) u(\tau) d\tau,$$

where $N(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

Definition 2.4. [8] If $r = m$, then the fractal-fractional integral associated with the new Hattaf mixed fractal-fractional derivative is defined by:

$$(5) \quad {}^{FF}I_{t_0, w, \delta}^{\alpha, \beta, r, \varepsilon} u(t) = \begin{cases} \frac{2 - \alpha - \beta}{N(\alpha + \beta - 1)} \left[{}^{RL}\mathcal{J}_{t_0, \omega}^{1-\beta, \varepsilon} u(t) + \delta \mu_{\alpha, \beta} {}^{RL}\mathcal{J}_{t_0, \omega}^{1+r-\beta, \varepsilon} u(t) \right], & \text{if } \beta \neq 1, \\ \frac{1 - \alpha}{N(\alpha)} \left(\frac{\partial g(\varepsilon, t)}{\partial t} \right) u(t) + \frac{\delta \alpha}{N(\alpha)} {}^{RL}\mathcal{J}_{t_0, \omega}^{r, \varepsilon} u(t), & \text{if } \beta = 1. \end{cases}$$

where ${}^{RL}\mathcal{J}_{t_0,\omega}^{\alpha,\varepsilon}$ is the weighted Riemann-Liouville fractal-fractional integral of order α defined by

$$(6) \quad {}^{RL}\mathcal{J}_{t_0,\omega}^{\alpha,\varepsilon} u(t) = \frac{1}{\Gamma(\alpha)} \frac{1}{\omega(t)} \int_{t_0}^t (t-\tau)^{\alpha-1} \left(\frac{\partial g(\varepsilon, t)}{\partial t} \right) \omega(\tau) u(\tau) d\tau.$$

The existence and uniqueness of the solution for our fractal-fractional finance model, we require the following lemma.

Lemma 2.5. (*Krasnoselskii's fixed point theorem* [14, 15]) *Let M be a nonempty closed convex subset of a Banach space $(C, \|\cdot\|)$. Suppose that A and B map M into C such that*

- (i): $A\psi_1 + B\psi_2 \in M$, for all $\psi_1, \psi_2 \in M$;
- (ii): A is a contraction mapping;
- (iii): B is continuous and $B(M)$ is contained in a compact subset of C .

Then $A + B$ has a fixed point $\psi \in M$.

Now, we propose the following financial model involving the new Hattaf mixed fractal-fractional derivative:

$$(7) \quad \begin{cases} {}^{FFR}D_{0,t,\omega,\delta}^{\alpha,\beta,r,r,\varepsilon} x(t) = z(t) + (y(t) - a)x(t), \\ {}^{FFR}D_{0,t,\omega,\delta}^{\alpha,\beta,r,r,\varepsilon} y(t) = 1 - by(t) - x^2(t), \\ {}^{FFR}D_{0,t,\omega,\delta}^{\alpha,\beta,r,r,\varepsilon} z(t) = -x(t) - cz(t), \end{cases}$$

where $x(t)$, $y(t)$ and $z(t)$ denote the interest rate, the investment demand and the price index at time t , respectively. Further, the positive parameters a , b and c are, respectively, the saving amount, the cost per investment and elasticity of the demand of the commercial markets.

It is important that our financial model presented by system (7) improves and generalizes numerous financial models existing in the literature. For example,

- When $\omega(t) = 1$ and $\beta = r = \delta = 1$, we get the fractal-fractional model presented in [16] with fractal-fractional derivative involving a generalized Mittag-Leffler kernel type.
- When $\omega(t) = 1$ and $\beta = 1 - \alpha$, we obtain the model proposed by Agathiyan et al. [16] using fractal-fractional derivative with power law kernel type.

For simplicity, we denote ${}^{FFR}D_{0,t,\omega,\delta}^{\alpha,\beta,r,r,\varepsilon}$ by $\mathcal{D}_{0,\omega,\delta}^{\alpha,\beta,r,\varepsilon}$ and ${}^{FF}I_{0,t,\omega,\delta}^{\alpha,\beta,r,\varepsilon}$ by $\mathcal{J}_{0,\omega,\delta}^{\alpha,\beta,r,\varepsilon}$.

3. THE EXISTENCE OF SOLUTION AND EQUILIBRIUM POINTS

In this section, we first investigate the existence and uniqueness of solutions of system (7) by means of fixed point theory.

Let $\mathcal{C} = C([0, T], \mathbb{R}^3)$ be the Banach space of continuous functions f from $[0, T]$ into $\mathbb{R}^3$ equipped with the sup-norm

$$\|f\| = \sup_{t \in [0, T]} |f(t)|.$$

Let $I = [0, T]$. System (7) can be rewriting as

$$(8) \quad \begin{cases} \mathcal{D}_{0, \omega, \delta}^{\alpha, \beta, r, \varepsilon} h(t) = G(t, h(t)), & \forall t \in I, \\ h(0) = h_0, \end{cases}$$

where $h(t) = (x(t), y(t), z(t))^T$, $h(0) = h_0 = (x(0), y(0), z(0))^T$ and the vector function G is given

$$\text{by } G = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \begin{pmatrix} z + (y - a)x \\ 1 - by + x^2 \\ -x - cz \end{pmatrix}.$$

By applying the Hattaf mixed fractal-fractional integral to both sides of (8), it follows that:

$$(9) \quad \begin{aligned} h(t) &= \frac{\omega(0)h(0)}{\omega(t)} + \mathcal{J}_{0, \omega, \delta}^{\alpha, \beta, r, \varepsilon} G(t, h(t)) \\ &= \frac{\omega(0)h_0}{\omega(t)} + \frac{1 - \alpha}{N(\alpha)} \frac{\partial g(\varepsilon, t)}{\partial t} G(t, h(t)) + \frac{\alpha \delta}{N(\alpha) \Gamma(r) \omega(t)} \int_0^t (t - \tau)^{r-1} \omega(\tau) \frac{\partial g(\varepsilon, \tau)}{\partial \tau} G(\tau, h(\tau)) d\tau. \end{aligned}$$

To proceed, we consider the following hypotheses:

$(\mathcal{H}_1)$: There exist positive constants ϕ_1 , ϕ_2 and $q \in [0, 1)$ such that

$$G(t, h(t)) \leq \phi_1 \|h\|^q + \phi_2.$$

$(\mathcal{H}_2)$: There exists a positive constant $M_1 > 0$ for all $h, \tilde{h}$, such that

$$|G(t, h(t)) - G(t, \tilde{h}(t))| \leq M_1 \|h - \tilde{h}\|.$$

$(\mathcal{H}_3)$: There exists a positive constant $M_2 > 0$ such that for all $\varepsilon > 0$ and $t \in [0, T]$, we have

$$\left| \frac{\partial g(\varepsilon, t)}{\partial t} \right| \leq M_2.$$

Furthermore, we define the following operator $\mathcal{K} : C([0, T], \mathbb{R}^3) \longrightarrow C([0, T], \mathbb{R}^3)$ such that

$$\mathcal{K}h(t) = \mathcal{K}_1h(t) + \mathcal{K}_2h(t),$$

where

$$(10) \quad \begin{cases} \mathcal{K}_1h(t) = \frac{\omega(0)h_0}{\omega(t)} + \frac{1-\alpha}{N(\alpha)} \frac{\partial g(\varepsilon, t)}{\partial t} G(t, h(t)), \\ \mathcal{K}_2h(t) = \frac{\delta\alpha}{N(\alpha)\Gamma(r)} \frac{1}{\omega(t)} \int_0^t (t-\tau)^{r-1} \frac{\partial g(\varepsilon, \tau)}{\partial \tau} \omega(\tau) G(\tau, h(\tau)) d\tau. \end{cases}$$

Consequently, (9) can be written as

$$(11) \quad \begin{aligned} \mathcal{K}h(t) &= \frac{\omega(0)h_0}{\omega(t)} + \frac{1-\alpha}{N(\alpha)} \frac{\partial g(\varepsilon, t)}{\partial t} G(t, h(t)) \\ &\quad + \frac{\delta\alpha}{N(\alpha)\Gamma(r)} \frac{1}{\omega(t)} \int_0^t (t-\tau)^{r-1} \frac{\partial g(\varepsilon, \tau)}{\partial \tau} \omega(\tau) G(\tau, h(\tau)) d\tau. \end{aligned}$$

Theorem 3.1. Suppose that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ hold such that $\frac{M_1M_2(1-\alpha)}{N(\alpha)} < 1$. Then model (7) has at least one solution.

Proof. Firstly, we prove that $\mathcal{K}_1$ is a contraction. Consider

$$M = \{h \in \mathcal{C} : \|h\| \leq \mathcal{L}, \mathcal{L} > 0\},$$

which is closed and convex set. For all $h, \tilde{h} \in M$, we have

$$|\mathcal{K}_1h(t) - \mathcal{K}_1\tilde{h}(t)| \leq \frac{M_1M_2(1-\alpha)}{N(\alpha)} \|h - \tilde{h}\|.$$

Then

$$\|\mathcal{K}_1h - \mathcal{K}_1\tilde{h}\| \leq \frac{M_1M_2(1-\alpha)}{N(\alpha)} \|h - \tilde{h}\|.$$

Since $\frac{M_1M_2(1-\alpha)}{N(\alpha)} < 1$, we deduce that $\mathcal{K}_1$ is a contraction.

Secondly, we prove that $\mathcal{K}_2$ is compact. We have

$$\begin{aligned} \|\mathcal{K}_2h\| &\leq \max_{t \in [0, T]} \left| \frac{\delta\alpha}{N(\alpha)\Gamma(r)} \frac{1}{\omega(t)} \int_0^t (t-\tau)^{r-1} \frac{\partial g(\varepsilon, \tau)}{\partial \tau} \omega(\tau) G(\tau, h(\tau)) d\tau \right| \\ &\leq \frac{\delta\alpha M_2 T^r}{N(\alpha)\Gamma(r+1)} \left(\phi_1 \|h\|^q + \phi_2 \right). \end{aligned}$$

Thus, for all $h \in M$, $\mathcal{K}_2$ is bounded. For equicontinuity, let $t_1, t_2 \in [0, T]$ such that $t_2 < t_1$, then

$$\begin{aligned} |\mathcal{K}_2 h(t_1) - \mathcal{K}_2 h(t_2)| &= \frac{\delta \alpha}{N(\alpha) \Gamma(r)} \left| \int_0^{t_1} \frac{(t_1 - \tau)^{r-1}}{\omega(t_1)} \frac{\partial g(\varepsilon, \tau)}{\partial \tau} \omega(\tau) G(\tau, h(\tau)) d\tau \right. \\ &\quad \left. - \int_0^{t_2} \frac{(t_2 - \tau)^{r-1}}{\omega(t_2)} \frac{\partial g(\varepsilon, \tau)}{\partial \tau} \omega(\tau) G(\tau, h(\tau)) d\tau \right|. \end{aligned}$$

Therefore, $\lim_{t_2 \rightarrow t_1} |\mathcal{K}_2 h(t_1) - \mathcal{K}_2 h(t_2)| = 0$. Hence, $\mathcal{K}_2$ is equicontinuous. It follows from Arzela-Ascoli theorem that $\mathcal{K}_2$ is compact. Consequently, $\mathcal{K}_2$ is continuous since h is continuous.

Finally to verify the item (i) of Lemma 2.5, we notice that $\mathcal{K}_1$ is a contraction, then

$$\|h\| \leq \|(I - \mathcal{K}_1)h\|.$$

As $\mathcal{K}_2(M)$ is contained in a compact subset of $\mathcal{C}$, we deduce by appropriate construction of the subset M that for fixed h in M the contraction $h \mapsto \mathcal{K}_1 h + \mathcal{K}_2 h$ has a fixed point in M . According to Lemma 2.5, we conclude that model (7) has at least one solution. $\square$

Theorem 3.2. *If $\left(\frac{1-\alpha}{N(\alpha)} + \frac{\delta \alpha T^r}{N(\alpha) \Gamma(r+1)}\right) M_1 M_2 < 1$, then model (7) has a unique solution.*

Proof. Let $h, \tilde{h} \in C([0, T], R^3)$. We have

$$\begin{aligned} \|\mathcal{K}h - \mathcal{K}\tilde{h}\| &\leq \|\mathcal{K}_1 h - \mathcal{K}_1 \tilde{h}\| + \|\mathcal{K}_2 h - \mathcal{K}_2 \tilde{h}\| \\ &\leq \frac{(1-\alpha)M_2}{N(\alpha)} \max_{t \in [0, T]} |G(t, h(t)) - G(t, \tilde{h}(t))| \\ &\quad + \frac{\delta \alpha M_2}{N(\alpha) \Gamma(r)} \max_{t \in [0, T]} \left| \int_0^t (t - \tau)^{r-1} \frac{\omega(\tau)}{\omega(t)} G(\tau, h(\tau)) d\tau \right. \\ &\quad \left. - \int_0^t (t - \tau)^{r-1} \frac{\omega(\tau)}{\omega(t)} G(\tau, \tilde{h}(\tau)) d\tau \right| \\ &\leq \left(\frac{1-\alpha}{N(\alpha)} + \frac{\delta \alpha T^r}{N(\alpha) \Gamma(r+1)} \right) M_1 M_2 \|h - \tilde{h}\|. \end{aligned}$$

Since $\left(\frac{1-\alpha}{N(\alpha)} + \frac{\delta \alpha T^r}{N(\alpha) \Gamma(r+1)}\right) M_1 M_2 < 1$, we deduce that $\mathcal{K}$ is a contraction. Consequently, the Hattaf mixed fractal-fractional differential finance model (7) has a unique solution. $\square$

Next, we establish the equilibrium points of our model.

Theorem 3.3.

(i) If $1 - ab - \frac{b}{c} \leq 0$, then system (7) has only a trivial equilibrium of the form

$$E_1\left(0, \frac{1}{b}, 0\right).$$

(ii) If $1 - ab - \frac{b}{c} > 0$, then system (7) has besides to E_1 two other equilibrium points

$$E_2\left(\sqrt{1 - ab - \frac{b}{c}}, \frac{1 + ac}{c}, -\frac{1}{c}\sqrt{1 - ab - \frac{b}{c}}\right),$$

and

$$E_3\left(-\sqrt{1 - ab - \frac{b}{c}}, \frac{1 + ac}{c}, \frac{1}{c}\sqrt{1 - ab - \frac{b}{c}}\right).$$

Proof. Any equilibrium point of system (7) clearly satisfies the subsequent algebraic equations:

$$(12) \quad z + (y - a)x = 0,$$

$$(13) \quad 1 - by - x^2 = 0,$$

$$(14) \quad -x - cz = 0.$$

From (14), we get $z = -\frac{1}{c}x$ and replacing it in (12), we have $x = 0$ or $y = \frac{1 + ac}{c}$. So, we discuss two cases.

- If $x = 0$, then $y = \frac{1}{b}$ and $z = 0$. Hence, the first financial equilibrium is $E_1(0, \frac{1}{b}, 0)$.
- If $y = \frac{1 + ac}{c}$, we replace y in (13) and if $1 - ab - \frac{b}{c} > 0$, we find two equilibrium points $E_2(\sqrt{1 - ab - \frac{b}{c}}, \frac{1 + ac}{c}, -z^*)$ and $E_3(-\sqrt{1 - ab - \frac{b}{c}}, \frac{1 + ac}{c}, z^*)$, where $z^* = \frac{1}{c}\sqrt{1 - ab - \frac{b}{c}}$. $\square$

4. NUMERICAL METHOD

In this section, we propose a numerical method for our fractal-fractional financial model with the new HMFF derivative, which is described by system (7).

The system (7) can be expressed as follows:

$$(15) \quad \begin{cases} \mathcal{D}_{0,\omega,\delta}^{\alpha,\beta,r,\varepsilon} x(t) = h_1(t, x(t), y(t), z(t)), \\ \mathcal{D}_{0,\omega,\delta}^{\alpha,\beta,r,\varepsilon} y(t) = h_2(t, x(t), y(t), z(t)), \\ \mathcal{D}_{0,\omega,\delta}^{\alpha,\beta,r,\varepsilon} z(t) = h_3(t, x(t), y(t), z(t)), \end{cases}$$

where $x(t)$, $y(t)$ and $z(t)$ are the state variables of the system, and

$$(16) \quad \begin{cases} h_1(t, x(t), y(t), z(t)) = z(t) + (y(t) - a)x(t), \\ h_2(t, x(t), y(t), z(t)) = 1 - by(t) - x^2(t), \\ h_3(t, x(t), y(t), z(t)) = -x(t) - cz(t). \end{cases}$$

Hence,

$$(17) \quad \begin{cases} {}^R D_{0,t,\omega,\delta}^{\alpha,\beta,r,r} x(t) = \frac{\partial g(\epsilon, t)}{\partial t} h_1(t, x(t), y(t), z(t)), \\ {}^R D_{0,t,\omega,\delta}^{\alpha,\beta,r,r} y(t) = \frac{\partial g(\epsilon, t)}{\partial t} h_2(t, x(t), y(t), z(t)), \\ {}^R D_{0,t,\omega,\delta}^{\alpha,\beta,r,r} z(t) = \frac{\partial g(\epsilon, t)}{\partial t} h_3(t, x(t), y(t), z(t)), \end{cases}$$

where ${}^R D_{0,t,\omega,\delta}^{\alpha,\beta,r,r}$ is the Hattaf mixed fractional derivative operator in the Riemann-Liouville sense defined in [1].

Substituting ${}^R D_{0,t,\omega,\delta}^{\alpha,\beta,r,r}$ by ${}^C D_{0,t,\omega,\delta}^{\alpha,\beta,r,r}$ in order to make use of the integer-order initial conditions. For simplicity, we study the case of $\beta = 1$ and operating the Hattaf mixed fractal-fractional integral on both sides of (17), we have

$$\begin{aligned} x(t) &= \frac{\omega(0)x(0)}{\omega(t)} + \frac{1-\alpha}{N(\alpha)} H_1(t, x(t), y(t), z(t)) \\ &\quad + \frac{\delta\alpha}{N(\alpha)\Gamma(r)} \frac{1}{\omega(t)} \int_0^t (t-\tau)^{r-1} \omega(\tau) H_1(\tau, x(\tau), y(\tau), z(\tau)) d\tau, \\ y(t) &= \frac{w(0)y(0)}{\omega(t)} + \frac{1-\alpha}{N(\alpha)} H_2(t, x(t), y(t), z(t)) \\ &\quad + \frac{\delta\alpha}{N(\alpha)\Gamma(r)} \frac{1}{\omega(t)} \int_0^t (t-\tau)^{r-1} \omega(\tau) H_2(\tau, x(\tau), y(\tau), z(\tau)) d\tau, \\ z(t) &= \frac{\omega(0)z(0)}{\omega(t)} + \frac{1-\alpha}{N(\alpha)} H_3(t, x(t), y(t), z(t)) \\ &\quad + \frac{\delta\alpha}{N(\alpha)\Gamma(r)} \frac{1}{\omega(t)} \int_0^t (t-\tau)^{r-1} \omega(\tau) H_3(\tau, x(\tau), y(\tau), z(\tau)) d\tau, \end{aligned}$$

where $H_j(t, x(t), y(t), z(t)) = \frac{\partial g(\epsilon, t)}{\partial t} h_j(t, x(t), y(t), z(t))$ for $j = 1, 2, 3$.

Let $t_n = n\Delta t$, with $n = 0, 1, \dots, N$ and Δt is the time step duration. Based in the numerical scheme [1], we find

$$\begin{aligned} x(t_{n+1}) &= \frac{\omega(0)x(0)}{\omega(t_n)} + \frac{1-\alpha}{N(\alpha)} H_1(t_n) \\ &\quad + \frac{\delta\alpha(\Delta t)^r}{N(\alpha)\Gamma(r+2)\omega(t_n)} \sum_{i=0}^n \left[\omega(t_i)H_1(t_i)A_{n,i} + \omega(t_{i-1})H_1(t_{i-1})B_{n,i} \right], \\ y(t_{n+1}) &= \frac{\omega(0)y(0)}{\omega(t_n)} + \frac{1-\alpha}{N(\alpha)} H_2(t_{n+1}) \\ &\quad + \frac{\delta\alpha(\Delta t)^r}{N(\alpha)\Gamma(r+2)\omega(t_n)} \sum_{i=0}^n \left[\omega(t_i)H_2(t_i)A_{n,i} + \omega(t_{i-1})H_2(t_{i-1})B_{n,i} \right], \\ z(t_{n+1}) &= \frac{\omega(0)z(0)}{\omega(t_n)} + \frac{1-\alpha}{N(\alpha)} H_3(t_{n+1}) \\ &\quad + \frac{\delta\alpha(\Delta t)^r}{N(\alpha)\Gamma(r+2)\omega(t_n)} \sum_{i=0}^n \left[\omega(t_i)H_3(t_i)A_{n,i} + \omega(t_{i-1})H_3(t_{i-1})B_{n,i} \right], \end{aligned}$$

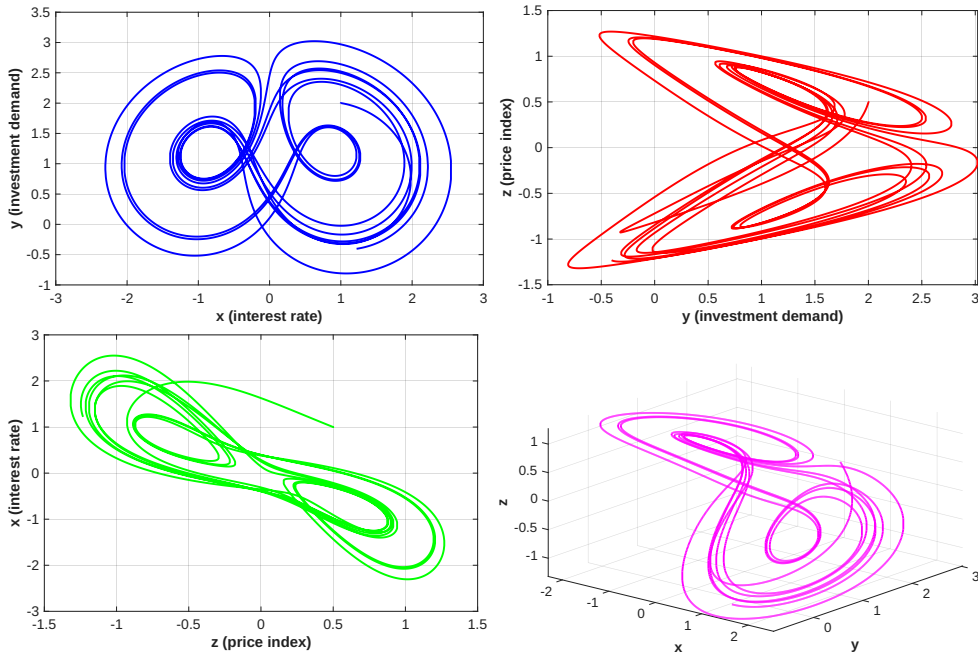
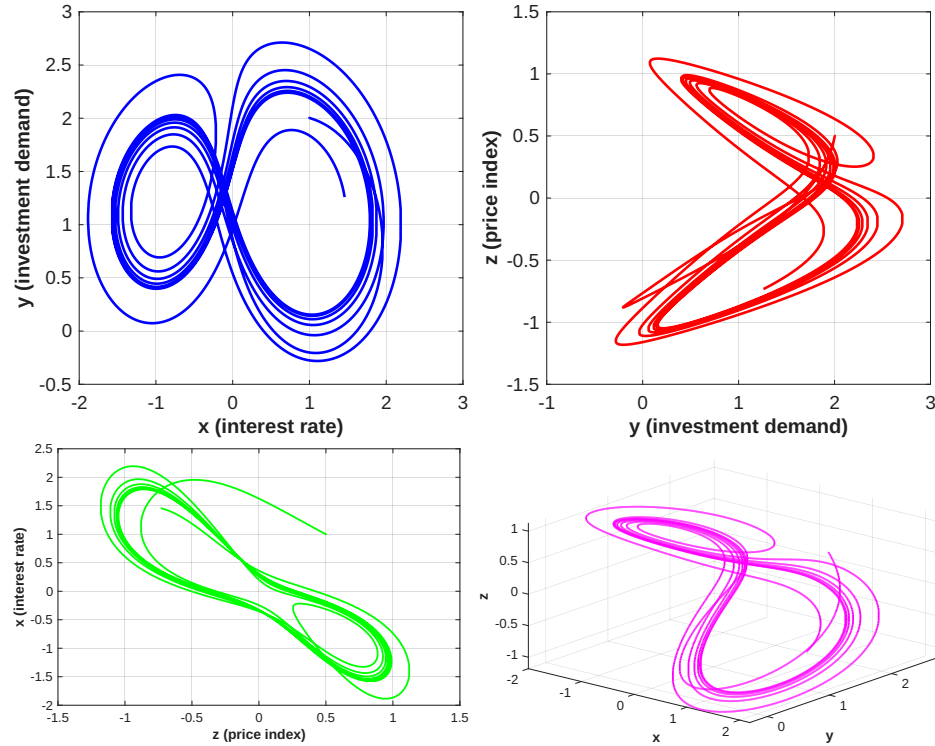
where

$$\begin{aligned} A_{n,i} &= (n+1-i)^r(n-i+2+r) - (n-i)^r(n-i+2+2r), \\ B_{n,i} &= (n-i)^r(n-i+1+r) - (n+1-i)^{r+1}. \end{aligned}$$

5. NUMERICAL SIMULATIONS

In this section, we simulate the new discretization proposed in the previous section, by fixing $a = 0.5$, $b = 0.2$, $c = 1.2$, $g(\varepsilon, t) = t^\varepsilon$ and $N(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

Figures 1-9 demonstrate the dynamics of our financial system (7) for different values of the fractional order α and the fractal dimension ε order, their effects on the system dynamics were examined through two-dimensional and three-dimensional phase portraits. These parameters play a fundamental role in characterizing the memory properties and scaling behavior of the financial system, in which the numerical results illustrate a significant difference between the classical and fractal-fractional approaches. In the fractal-fractional framework ($\alpha, \varepsilon \leq 1$), the system develops complex dynamics characterized by the appearance of self-similar attractors, irregular oscillations and multi-turn structures. Furthermore, a progressive decrease in the parameters α and ε leads to a transition from the chaotic regime to periodic and then stable behavior. This evolution underscores the crucial role of these parameters in controlling the system's dynamics and their ability to modulate the intensity of the chaotic behavior.

FIGURE 1. Phase portraits of our fractal-fractional financial model for $\alpha = 1$, $r = 1$ and $\varepsilon = 1$.FIGURE 2. Phase portraits of our fractal-fractional financial model for $\alpha = 1$, $r = 1$ and $\varepsilon = 0.95$.

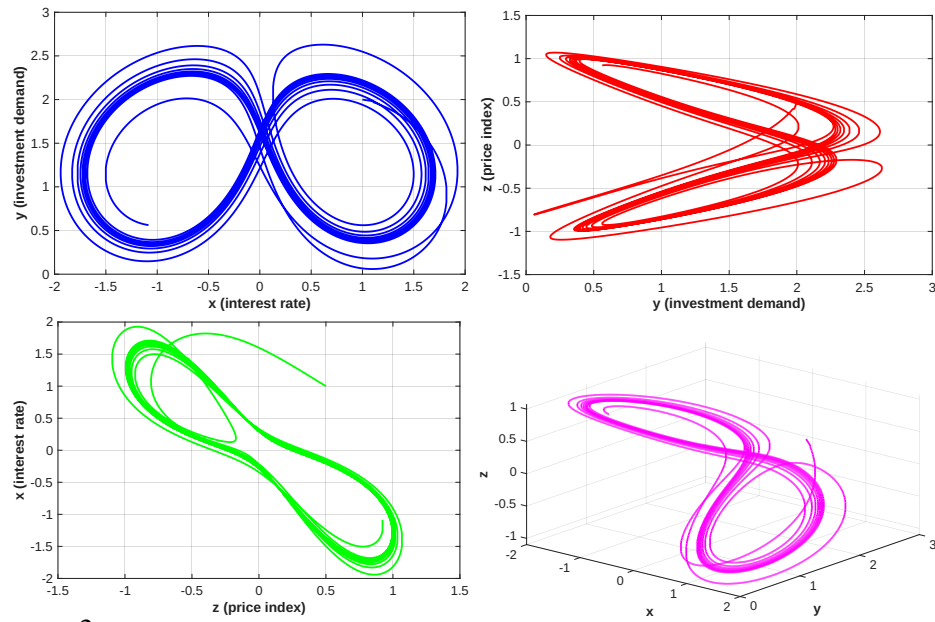


FIGURE 3. Phase portraits of our fractal-fractional financial model for $\alpha = 0.95$, $r = 0.98$ and $\varepsilon = 1$.

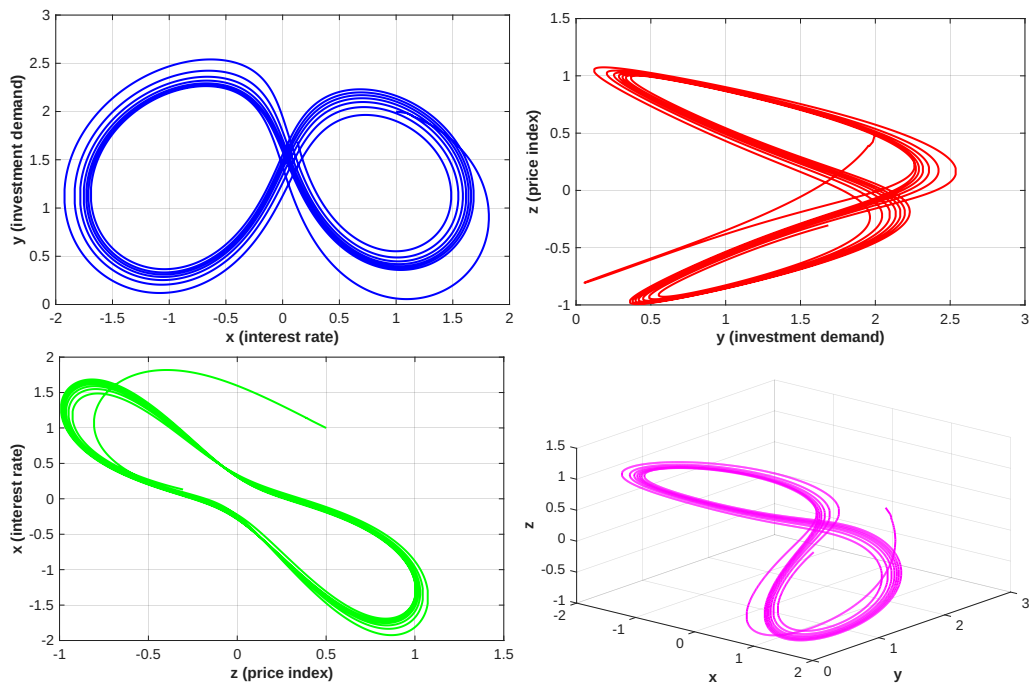


FIGURE 4. Phase portraits of our fractal-fractional financial model for $\alpha = 0.95$, $r = 0.98$ and $\varepsilon = 0.95$.

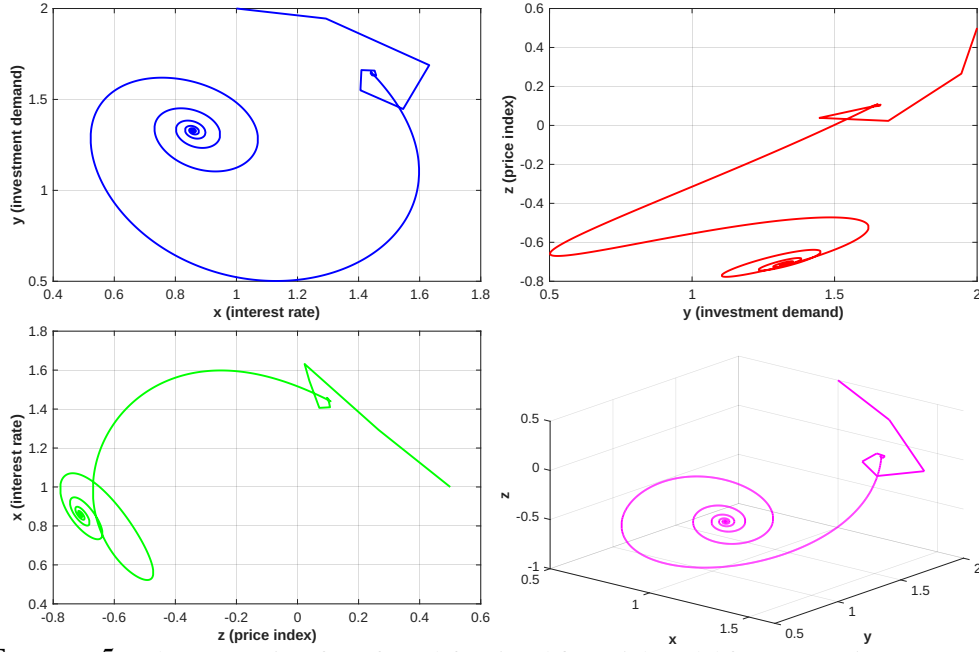


FIGURE 5. Phase portraits of our fractal-fractional financial model for $\alpha = 0.85$, $r = 0.94$ and $\varepsilon = 0.9$.

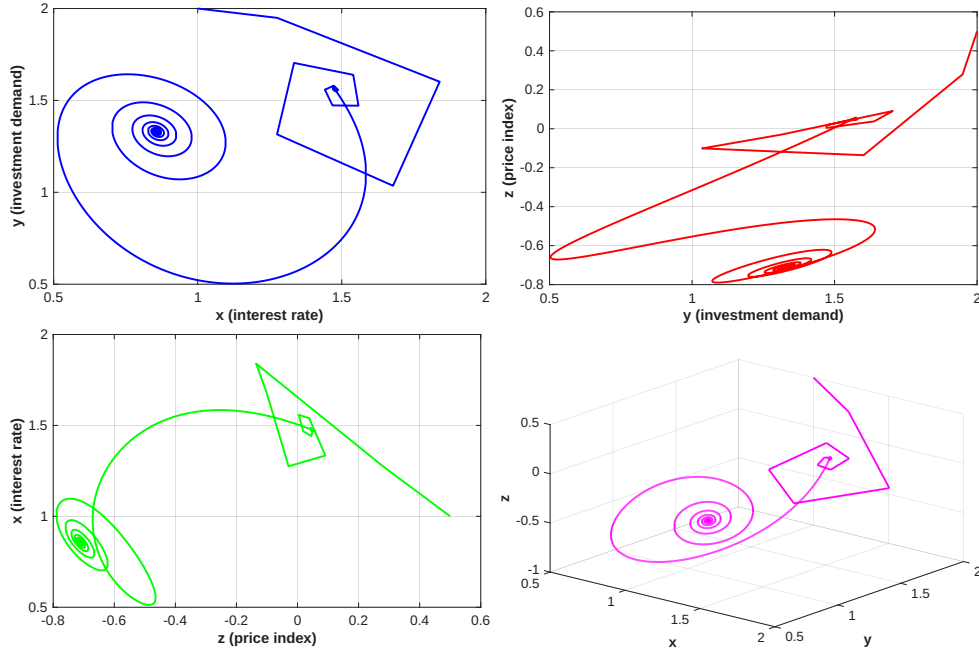


FIGURE 6. Phase portraits of our fractal-fractional financial model for $\alpha = 0.85$, $r = 0.94$ and $\varepsilon = 0.85$.

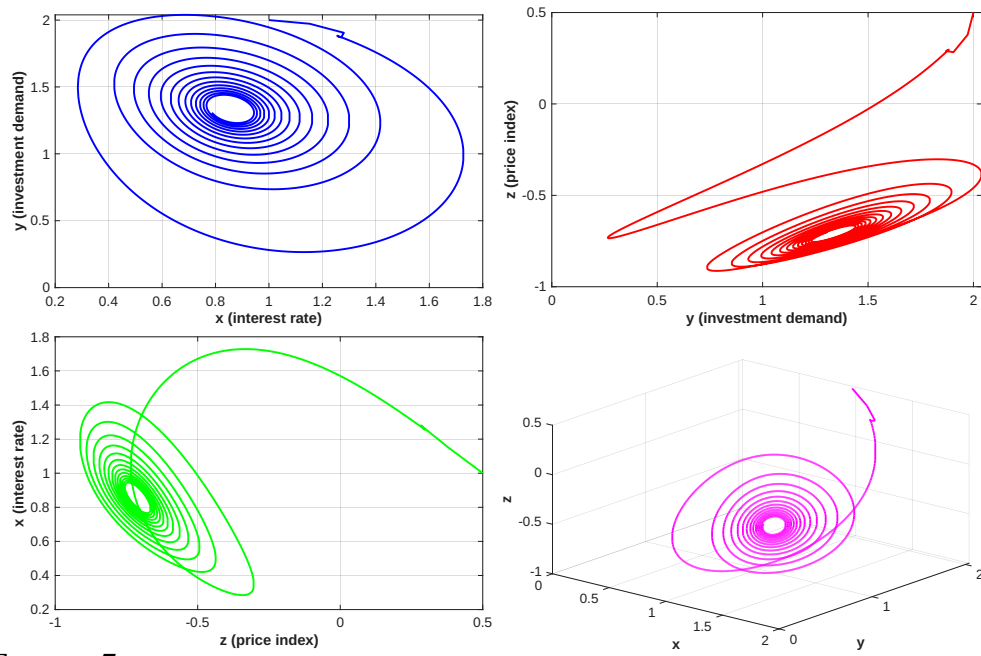


FIGURE 7. Phase portraits of our fractal-fractional financial model for $\alpha = 0.92$, $r = 0.95$ and $\varepsilon = 0.92$.

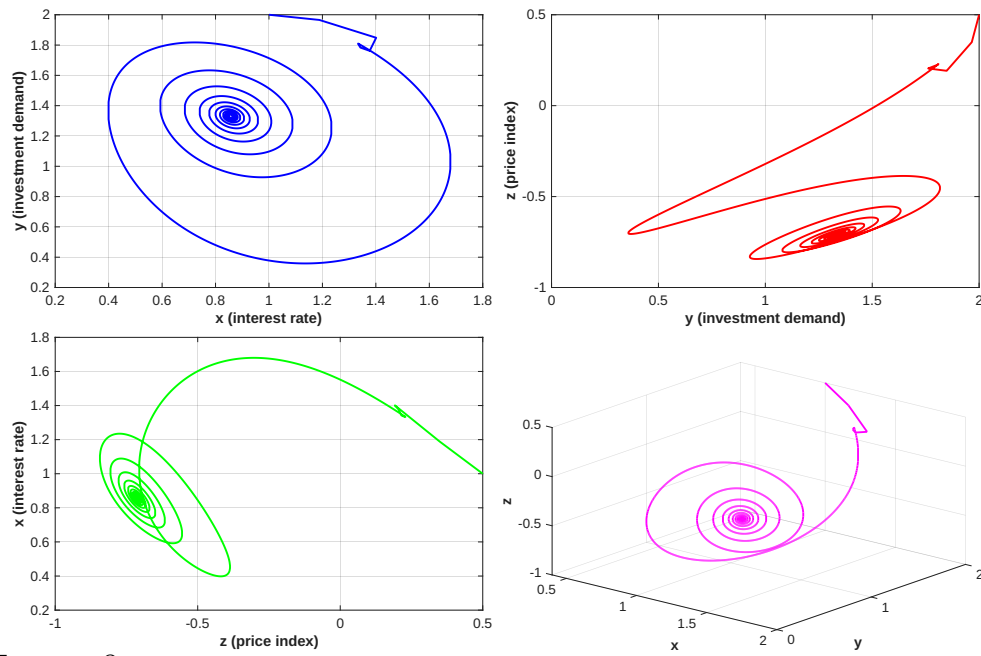


FIGURE 8. Phase portraits of our fractal-fractional financial model for $\alpha = 0.9$, $r = 0.95$ and $\varepsilon = 0.9$.

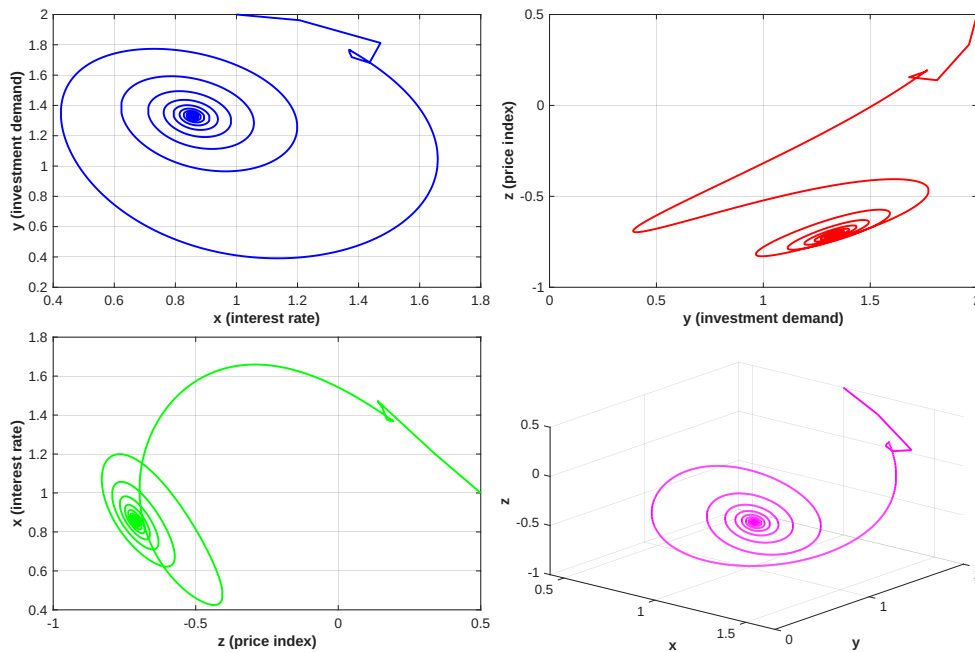


FIGURE 9. Phase portraits of our fractal-fractional financial model for $\alpha = 0.89$, $r = 0.94$ and $\varepsilon = 0.89$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

- [1] K. Hattaf, A New Mixed Fractional Derivative with Applications in Computational Biology, *Computation* 12 (2024), 7. <https://doi.org/10.3390/computation12010007>.
- [2] K. Toufiq, S. Lasfar, K. Hattaf, K. Adnaoui, Mathematical Modeling and Analysis of the Dynamics of a Nonlinear Finance System via Hattaf Fractional Derivative and Fixed Point Theory, *Adv. Fixed Point Theory* 16 (2026), 5. <https://doi.org/10.28919/afpt/9690>.
- [3] K. Hattaf, A New Generalized Definition of Fractional Derivative with Non-Singular Kernel, *Computation* 8 (2020), 49. <https://doi.org/10.3390/computation8020049>.
- [4] Z. Ali, F. Rabiei, K. Hosseini, A Fractal-Fractional-Order Modified Predator-Prey Mathematical Model with Immigrations, *Math. Comput. Simul.* 207 (2023), 466–481. <https://doi.org/10.1016/j.matcom.2023.01.006>.
- [5] M. Aslam, M. Farman, H. Ahmad, T.N. Gia, A. Ahmad, S. Askar, Fractal Fractional Derivative on Chemistry Kinetics Hires Problem, *AIMS Math.* 7 (2022), 1155–1184. <https://doi.org/10.3934/math.2022068>.
- [6] M.M. El-Dessoky, M.A. Khan, Modeling and Analysis of an Epidemic Model with Fractal-Fractional Atangana-Baleanu Derivative, *Alex. Eng. J.* 61 (2022), 729–746. <https://doi.org/10.1016/j.aej.2021.04.103>.

- [7] E.A. Algehyne, M. Ibrahim, Fractal-Fractional Order Mathematical Vaccine Model of COVID-19 under Non-Singular Kernel, *Chaos Solitons Fractals* 150 (2021), 111150. <https://doi.org/10.1016/j.chaos.2021.111150>.
- [8] H. El Mamouni, Z. Hajhouji, K. Hattaf, Existence of Solutions for a New Class of Hattaf Mixed Fractal-Fractional Equations Modeling the HIV/AIDS Disease, *Adv. Fixed Point Theory* 16 (2026), 1. <https://doi.org/10.28919/afpt/9654>.
- [9] A. Atangana, Fractal-Fractional Differentiation and Integration: Connecting Fractal Calculus and Fractional Calculus to Predict Complex System, *Chaos Solitons Fractals* 102 (2017), 396–406. <https://doi.org/10.1016/j.chaos.2017.04.027>.
- [10] A. Atangana, Modelling the Spread of COVID-19 with New Fractal-Fractional Operators: Can the Lockdown Save Mankind before Vaccination?, *Chaos Solitons Fractals* 136 (2020), 109860. <https://doi.org/10.1016/j.chaos.2020.109860>.
- [11] K. Hattaf, A New Class of Generalized Fractal and Fractal-Fractional Derivatives with Non-Singular Kernels, *Fractal Fract.* 7 (2023), 395. <https://doi.org/10.3390/fractalfract7050395>.
- [12] W. Chen, Time-Space Fabric Underlying Anomalous Diffusion, *Chaos Solitons Fractals* 28 (2006), 923–929. <https://doi.org/10.1016/j.chaos.2005.08.199>.
- [13] X.J. Yang, New General Calculi with Respect to Another Functions Applied to Describe the Newton-Like Dashpot Models in Anomalous Viscoelasticity, *Therm. Sci.* 23 (2019), 3751–3757. <https://doi.org/10.2298/TSCI180921260Y>.
- [14] T.A. Burton, A Fixed-Point Theorem of Krasnoselskii, *Appl. Math. Lett.* 11 (1998), 85–88. [https://doi.org/10.1016/S0893-9659\(97\)00138-9](https://doi.org/10.1016/S0893-9659(97)00138-9).
- [15] M.A. Krasnosel'skiĭ, Some Problems of Nonlinear Analysis, *Amer. Math. Soc. Transl. Ser. 2* 10 (1958), 345–409. <https://doi.org/10.1090/trans2/010/13>.
- [16] A. Agathiyan, Vinothkumar B.A. Akgul, F.S. Alshammari, Fractal-Fractional Modeling and Chaos Analysis of a Financial System with Generalized Memory Kernels, *Results Control Optim.* 21 (2025), 100632. <https://doi.org/10.1016/j.rico.2025.100632>.