



Available online at <http://scik.org>
Adv. Fixed Point Theory, 2026, 16:27
<https://doi.org/10.28919/afpt/10095>
ISSN: 1927-6303

ON ENRICHED CONDENSED KANNAN-TYPE CONTRACTIONS IN NORMED SPACES WITH APPLICATIONS TO VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we introduce a new class of condensed Kannan-type contractions in normed spaces and establish several fixed point results for nonlinear operators satisfying the proposed contractive condition. By employing the Krasnoselskii iterative scheme, we establish an existence theorem for the proposed class of mappings that guarantees the approximation of solutions to both unique and non-unique fixed point problems. We further investigate the Ulam-Hyers stability of the associated fixed point problem. Several numerical examples are presented to illustrate the applicability and effectiveness of the theoretical findings. As an application, the results are utilized to study variational inequality problems. This demonstrates the usefulness of the proposed framework in nonlinear analysis and related optimization problems.

Keywords: fixed point; normed spaces; enriched condensed Kannan-type map; Krasnoselskii iteration.

2020 AMS Subject Classification: 47H10, 54H25.

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Received July 01, 2026

1. INTRODUCTION

Fixed point theory occupies a central position in nonlinear analysis and has found numerous applications in differential equations, integral equations, optimization, variational inequalities, and engineering sciences. The subject originated with the celebrated Banach contraction principle (See [1]) which guarantees the existence and uniqueness of a fixed point for a self-mapping $T : X \rightarrow X$ on a complete metric space (X, d) satisfying

$$(1) \quad d(Tx, Ty) \leq ad(x, y),$$

for all $x, y \in X$, where $a \in [0, 1)$. In [2], Kannan introduced a new class of contractions that provides an affirmative answer to the question of whether a discontinuous mapping could satisfy a contractive condition and still possess a fixed point in a complete metric space. A mapping $T : X \rightarrow X$ is called a Kannan contraction if

$$(2) \quad d(Tx, Ty) \leq c [d(x, Tx) + d(y, Ty)],$$

for all $x, y \in X$, where $c \in [0, \frac{1}{2})$. Since then, Kannan contraction has attracted considerable attention due to their theoretical significance and broad applicability. Over the years, several researchers have proposed generalized variants of Kannan contraction in order to describe more flexible contractive behaviours. These developments combine different contractive parameters and generalized distance functions to obtain broader classes of mappings while preserving fixed point properties; see, for example, Karapinar [3] proposed a Kannan-type called interpolative Kannan contraction in the complete metric space. This was later improved by Abbas *et al.* [4] by enriching the interpolative Kannan map and using the Krasnoselskii iteration [5] as a baseline sequence.

Definition 1.1. [4] *A mapping $T : X \rightarrow X$ is called an enriched interpolative Kannan-type map if there exist $b \in [0, \infty)$, $a \in [0, 1)$, and $\alpha \in (0, 1)$ such that for all $x, y \in X$,*

$$(3) \quad \|b(x - y) + Tx - Ty\| \leq a \|x - Tx\|^\alpha \|y - Ty\|^{1-\alpha}$$

Indeed, every enriched interpolative Kannan-type map is an interpolative Kannan-type map with $b = 0$ and not vice versa. For several results on the duo conditions, see [6, 7, 8, 9]. Despite

these significant advances, some limitations remain. In particular, both studies in [3, 4] only ensured the existence of non-unique fixed points. In this respect, Wahab *et al.* [10] introduced the concept of condensed Kannan-type contractions and established several fixed point results for operators that guarantee both unique and non-unique fixed points. Their study demonstrated the effectiveness of the condensed framework by providing a criterion for selecting an appropriate parameter $\alpha \in (0, 1)$ in the analysis of nonlinear operators. Subsequently, this concept has been extended and explored in different ways, see [11, 12, 13]. However, if the condensed Kannan-type map is employed under weaker assumptions, then the fixed points can no longer be guaranteed. Therefore, there is a need for a more relaxed technique to approximate the fixed point in this setting. The purpose of this paper is to introduce an enriched condensed Kannan-type contractions in normed spaces and to investigate their fixed point properties via the Krasnoselskii iterative process. The proposed approach extends and unifies several existing results in the literature.

2. MAIN RESULTS

The goal of this section is to establish and prove the existence properties of a new enriched condensed Kannan-type that can handle some operators that may not meet the hypotheses of previous Kannan-type conditions.

Definition 2.1. Let $(X, \|\cdot\|)$ be a normed space and $T : X \rightarrow X$ be a (a, b, c, α) -enriched condensed Kannan-type mapping if there exist $a \in [0, 1)$, $b \in [0, \infty)$, $c \in [0, \frac{1}{2})$, and $\alpha \in (0, 1)$ such that

$$(4) \quad aL[\alpha, x, y] \leq \|b(x - y) + Tx - Ty\| \leq cU[\alpha, x, y],$$

where $L[\alpha, x, y] = \|x - Tx\|^\alpha \|y - Ty\|^{1-\alpha}$, $U[\alpha, x, y] = \|x - Tx\|^{2\alpha} + \|y - Ty\|^{2(1-\alpha)}$, for all $x, y \in X \setminus \text{Fix}(T)$.

The following are consequence of inequality (4):

- I. If $b = 0$, then condition (4) reduces to the condition in [10].
- II. By replacing $X \setminus \text{Fix}(T)$ with X or by setting $a = 0$, inequality (4) becomes

$$(5) \quad \|b(x - y) + Tx - Ty\| \leq cU[\alpha, x, y]$$

III. By setting $\alpha = \frac{1}{2}$ and replacing $X \setminus \text{Fix}(T)$ by X , we recover the condition in [4].

Example 2.2. [4] Let $X = L^2(\chi, \mu)$ be a collection of all Borel measurable functions $f : \chi \rightarrow \mathbb{R}$ equipped with the norm $\|f\|_X = \left(\int_{\chi} |f|^2 d\mu \right)^{\frac{1}{2}}$. Define $T : L^2(\chi, \mu) \rightarrow L^2(\chi, \mu)$ by $Tf = 1 - 3f$, for $f \in L^2(\chi, \mu)$. Then,

- i. T forms a $(0.5, 2, 0.25, 0.45)$ -enriched condensed Kannan-type map.
- ii. T does not satisfy $(0.5, 2, 0.25, 0.45)$ -enriched interpolative Kannan-type map.

i. To see this, let $f, g \in L^2(\chi, \mu)$ such that $f(y) = 0$ and $g(y) = 1$ for all $y \in \chi$. Also, set $a = 2c = 0.5$, $b = 2$, and $\alpha = 0.45$, then using (4):

$$\|b(f - g) + Tf - Tg\|_X = \|-f + g\|_X = 1$$

$$aL[\alpha, f, g] = 0.5 \|4f - 1\|_X^{\alpha} \|4g - 1\|_X^{1-\alpha} \approx 0.9149.$$

$$cU[\alpha, f, g] = 0.25 \left(\|4f - 1\|_X^{2\alpha} + \|4g - 1\|_X^{2(1-\alpha)} \right) \approx 1.0871.$$

That is, $0.9149 < 1 < 1.0871$.

ii. Since $\|b(f - g) + Tf - Tg\|_X = 1 > 0.9149 \approx aL[\alpha, f, g]$ in part i., then T is not a $(0.5, 2, 0.25, 0.45)$ -enriched interpolative Kannan-type map.

We now present the results as follows:

Theorem 2.3. Let $(X, \|\cdot\|)$ be a normed spaces and $T : X \rightarrow X$ be a (a, c, b, α) -enriched condensed Kannan type mapping for all $x, y \in X \setminus \text{Fix}(T)$. Then,

- I. T admits a unique fixed point in X .
- II. for $x_0 \in X \setminus \text{Fix}(T)$, the sequence $\{x_n\}$ generated by Krasnoselskii iteration converges to the unique fixed point.

Proof. I. Let $x_0 \in X$ be a generic point and $\{x_n\}$ is the Krasnoselskii iteration:

$$(6) \quad x_{n+1} = (1 - \lambda)x_n + \lambda Tx_n, \quad n \geq 0,$$

Set $\lambda = \frac{1}{b+1} \in (0, 1)$ and define $T_{\lambda}x = (1 - \lambda)x + \lambda Tx$, then $x_{n+1} = T_{\lambda}x_n$.

By hypothesis,

$$\|b(x - y) + Tx - Ty\| = \frac{1}{\lambda} \|T_{\lambda}x - T_{\lambda}y\|.$$

Thus, using the $U[x, y, \alpha]$, we have

$$\|T_\lambda x - T_\lambda y\| \leq c_1 [\|x - Tx\|^{2\alpha} + \|y - Ty\|^{2(1-\alpha)}].$$

where $c_1 = \lambda c$. Set $x = x_n, y = x_{n-1}$. Then

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq c_1 [\|x_n - Tx_n\|^{2\alpha} + \|x_{n-1} - Tx_{n-1}\|^{2(1-\alpha)}] \\ (7) \qquad \qquad &= c_1 [\|x_n - x_{n+1}\|^{2\alpha} + \|x_{n-1} - x_n\|^{2(1-\alpha)}]. \end{aligned}$$

Similarly, we have from $L[x, y, \alpha]$ that

$$\begin{aligned} \|x_{n+1} - x_n\| &\geq a_1 \|x_n - T_\lambda x_n\|^\alpha \cdot \|x_{n-1} - T_\lambda x_{n-1}\|^{1-\alpha} \\ (8) \qquad \qquad &= a_1 \|x_n - x_{n+1}\|^\alpha \|x_n - x_{n-1}\|^{1-\alpha}. \end{aligned}$$

By combining inequalities (7) and (8), we have

$$\begin{aligned} a \|x_n - x_{n+1}\|^\alpha \|x_n - x_{n-1}\|^{1-\alpha} &\leq c_1 [\|x_n - x_{n+1}\|^{2\alpha} + \|x_n - x_{n-1}\|^{2(1-\alpha)}] \\ &\leq 2c_1 \max \left\{ \|x_n - x_{n+1}\|^{2\alpha}, \|x_n - x_{n-1}\|^{2(1-\alpha)} \right\} \end{aligned}$$

Since $a_1 \in [0, 1)$ and $c_1 \in [0, \frac{1}{2})$, then $a_1 = 2c_1$. Thus,

$$(9) \qquad \|x_n - x_{n+1}\|^\alpha \|x_n - x_{n-1}\|^{1-\alpha} \leq \max \left\{ \|x_n - x_{n+1}\|^{2\alpha}, \|x_n - x_{n-1}\|^{2(1-\alpha)} \right\}$$

This above inequality suggests two cases, namely,

Case A: If $\|x_n - x_{n+1}\|^\alpha \|x_n - x_{n-1}\|^{1-\alpha} \leq \|x_n - x_{n+1}\|^{2\alpha}$, then

$$(10) \qquad \|x_n - x_{n-1}\|^{1-\alpha} \leq \|x_n - x_{n+1}\|^\alpha$$

Case B: If $\|x_n - x_{n+1}\|^\alpha \cdot \|x_n - x_{n-1}\|^{1-\alpha} \leq \|x_n - x_{n-1}\|^{2(1-\alpha)}$, then

$$(11) \qquad \|x_n - x_{n+1}\|^\alpha \leq \|x_n - x_{n-1}\|^{1-\alpha}$$

From inequalities (10) and (11), the following holds:

$$(12) \qquad \|x_n - x_{n+1}\|^\alpha = \|x_n - x_{n-1}\|^{1-\alpha}$$

By using (12) in (7), we obtain

$$\|x_n - x_{n+1}\| \leq c_1 [\|x_n - x_{n+1}\|^{2\alpha} + \|x_n - x_{n-1}\|^{2(1-\alpha)}]$$

$$\begin{aligned}
&= 2c_1 \|x_n - x_{n-1}\|^{2(1-\alpha)} \\
&= a_1 \|x_n - x_{n-1}\|^{1-\alpha} \|x_{n+1} - x_n\|^\alpha
\end{aligned}$$

This further gives

$$\|x_n - x_{n+1}\| \leq a_1^{\frac{1}{1-\alpha}} \|x_n - x_{n-1}\|$$

Let $\rho = a_1^{\frac{1}{1-\alpha}} < 1$ for any $\alpha \in (0, 1)$, then

$$(13) \quad \|x_{n+1} - x_n\| \leq \rho \|x_n - x_{n-1}\| \leq \cdots \leq \rho^n \|x_1 - x_0\|$$

If $n \rightarrow \infty$, then $\|x_{n+1} - x_n\| \rightarrow 0$. Suppose, on contrary, that $x_m \neq x_n$ for $m > n$, $m \in \mathbb{N}$. Then,

$$(14) \quad \|x_n - x_m\| \leq \sum_{j=n}^{m-1} \|x_j - x_{j+1}\| \leq \|x_1 - x_0\| \sum_{j=n}^{m-1} \rho^j$$

Since $\sum_{j=n}^{m-1} \rho^j$ converges as $m, n \rightarrow \infty$, the sequence $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence. Thus, there exists $x^* \in X$ such that

$$(15) \quad \lim_{n \rightarrow \infty} x_n = x^*$$

Next, suppose that $x^* \neq T_\lambda x^*$, then

$$(16) \quad \|x^* - T_\lambda x^*\| \leq \|x^* - x_{n+1}\| + \|T_\lambda x_n - T_\lambda x^*\|$$

But by the hypothesis and (15), we have

$$0 \leq \|T_\lambda x_n - T_\lambda x^*\| \leq c_1 \|x^* - T_\lambda x^*\|^{2(1-\alpha)}$$

Thus, taking limit across (16) as $n \rightarrow \infty$, we have

$$\|x^* - T_\lambda x^*\|^{2\alpha-1} \leq c_1 < 1,$$

which is a contradiction. Hence, $x^* = T_\lambda x^*$.

Assume that x^* and y^* are fixed points such that $x^* \neq y^*$, then

$$(17) \quad 0 \leq \|x^* - y^*\| = \|T_\lambda x^* - T_\lambda y^*\| \leq a_1 [\|x^* - T_\lambda x^*\|^{2\alpha} + \|y^* - T_\lambda y^*\|^{2(1-\alpha)}] = 0.$$

Hence, $x^* = y^*$. This completes part I.

II. Since T is an enriched condensed Kannan-type map and by setting $x = x_n$ and $y = x^*$, then $L[\alpha, x_n, x^*] = 0$ and

$$(18) \quad \|x_n - x^*\| \leq \rho^n \|x_0 - x^*\|$$

Thus,

$$\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$$

This gives the desired result. $\square$

Remark 1. Inequality (18) is the error rate of the enriched condensed Kannan-type map via the sequence (6).

Theorem 2.4. Let $(X, \|\cdot\|)$ be a normed spaces and $T : X \longrightarrow X$ be a (b, c, α) -enriched condensed Kannan type mapping for all $x, y \in X$ such that

$$(19) \quad \|b(x - y) + Tx - Ty\| \leq c[\|x - Tx\|^{2\alpha} + \|y - Ty\|^{2(1-\alpha)}].$$

Then,

I. T admits a unique fixed point in X .

II. for $x_0 \in X$, the sequence $\{x_n\}$ generated by Krasnoselskii iteration converges to the unique fixed point.

Proof. Let $b = \frac{1}{\lambda} - 1 > 0$ for $\lambda \in (0, 1)$. Since T satisfies (19), then

$$\|T_\lambda x - T_\lambda y\| \leq c_1 [\|x - Tx\|^{2\alpha} + \|y - Ty\|^{2(1-\alpha)}].$$

where $c_1 = \lambda c \in [0, \frac{1}{2})$. Setting $x = x_{n-1}$, $y = x_n$, we have

$$\begin{aligned} \|x_n - x_{n+1}\| &\leq c_1 [\|x_{n-1} - Tx_{n-1}\|^{2\alpha} + \|x_n - Tx_n\|^{2(1-\alpha)}] \\ &= c_1 [\|x_{n-1} - x_n\|^{2\alpha} + \|x_n - x_{n+1}\|^{2(1-\alpha)}] \\ (20) \quad &\leq 2c_1 \max_{\alpha} \{\|x_{n-1} - x_n\|^{2\alpha}, \|x_n - x_{n+1}\|^{2(1-\alpha)}\}. \end{aligned}$$

If $\|x_n - x_{n+1}\| \leq a_1 \|x_n - x_{n+1}\|^{2(1-\alpha)}$, then $\|x_n - x_{n+1}\|^{2\alpha-1} \leq a_1 < 1$ is a contradiction for $\alpha \in (0, 1)$.

But if $\|x_n - x_{n+1}\| \leq a_1 \|x_{n-1} - x_n\|^{2\alpha}$ and since $\|x_{n-1} - x_n\|$ is bounded, say $\|x_{n-1} - x_n\| < 1$, then for $\alpha \geq \frac{1}{2}$, we have

$$(21) \quad \|x_n - x_{n+1}\| \leq a_1 \|x_{n-1} - x_n\|^{2\alpha} \leq a_1 \|x_{n-1} - x_n\|$$

This completes the proof. On other hand, if $\|x_{n-1} - x_n\|$ is bounded and $\alpha < 1$, then

$$(22) \quad \begin{aligned} \|x_n - x_{n+1}\| &\leq a_1 \|x_{n-1} - x_n\|^{2\alpha} \\ &\leq \dots \\ &\leq a_1^{\sum_{j=0}^{n-1} (2\alpha)^j} \|x_0 - x_1\|^{(2\alpha)^n} \equiv \kappa_n \|x_0 - x_1\|^{(2\alpha)^n}. \end{aligned}$$

where $\kappa_n \equiv a_1^{\sum_{j=0}^{n-1} (2\alpha)^j}$. By taking limit as $n \rightarrow \infty$, then $\sum_{j=0}^{n-1} (2\alpha)^j \rightarrow \infty$ and $\kappa_n \rightarrow 0$. Therefore,

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0.$$

Thus, rest of the proof is a routine. □

Remark 2. For $x^* \in \text{Fix}(T)$ and $\{x_n\} \subset X$, the error rate for the inequality (19) is semilinear derived as:

$$\|x_n - x^*\| \leq \kappa_n \|x_0 - x^*\|^{(2\alpha)^n}.$$

Corollary 1. Let $T : X \longrightarrow X$ be a $(b, a, c, \frac{1}{2})$ -Kannan-type map satisfying

$$a \|x - Tx\|^{\frac{1}{2}} \|y - Ty\|^{\frac{1}{2}} \leq \|b(x - y) + Tx - Ty\| \leq c[\|x - Tx\| + \|y - Ty\|]$$

for all $x, y \in X \setminus \text{Fix}(T)$, $a \in [0, 1)$, $c \in [0, \frac{1}{2})$. Then, T possesses a unique fixed point.

Proof. The proof follows from Theorem 1 with $\alpha = \frac{1}{2}$. □

Corollary 2. Let $T : X \longrightarrow X$ be a $(b, c, \frac{1}{2})$ -Kannan-type map satisfying

$$\|b(x - y) + Tx - Ty\| \leq c[\|x - Tx\| + \|y - Ty\|]$$

for all $x, y \in X$, $c \in [0, \frac{1}{2})$. Then, T possesses a unique fixed point.

Proof. By exchanging $X \setminus \text{Fix}(T)$ for X , the proof follows from Corollary 1. □

Corollary 3. *Let $T : X \longrightarrow X$ be a $(c, \frac{1}{2})$ -Kannan-type map satisfying*

$$\|Tx - Ty\| \leq c[\|x - Tx\| + \|y - Ty\|]$$

for all $x, y \in X$, $c \in [0, \frac{1}{2})$. Then, T possesses a unique fixed point.

Proof. The proof is immediate by setting $b = 0$ in Corollary 2. □

Example 2.5. *Let $X = [0, 1]$ be equipped with the usual normed and define $T : X \rightarrow X$ by $Tx = 1 - x$, for all $x \in [0, 1]$. Then,*

- i. *T fulfills all hypotheses of Theorem 2.3 for $x, y \in X \setminus \text{Fix}(T)$.*
- ii. *T fulfills all hypotheses of Theorem 2.4 for $x, y \in X$.*

It suffices to show that T satisfies inequality (4) and (5). Indeed, for $a = 2c = 0.6$, $b = 2$, and $\alpha = 0.7$, then by setting $x = \frac{2}{3}$ and $y = 1$,

Inequality (4): $0.2781 < 0.3333 < 0.3644$.

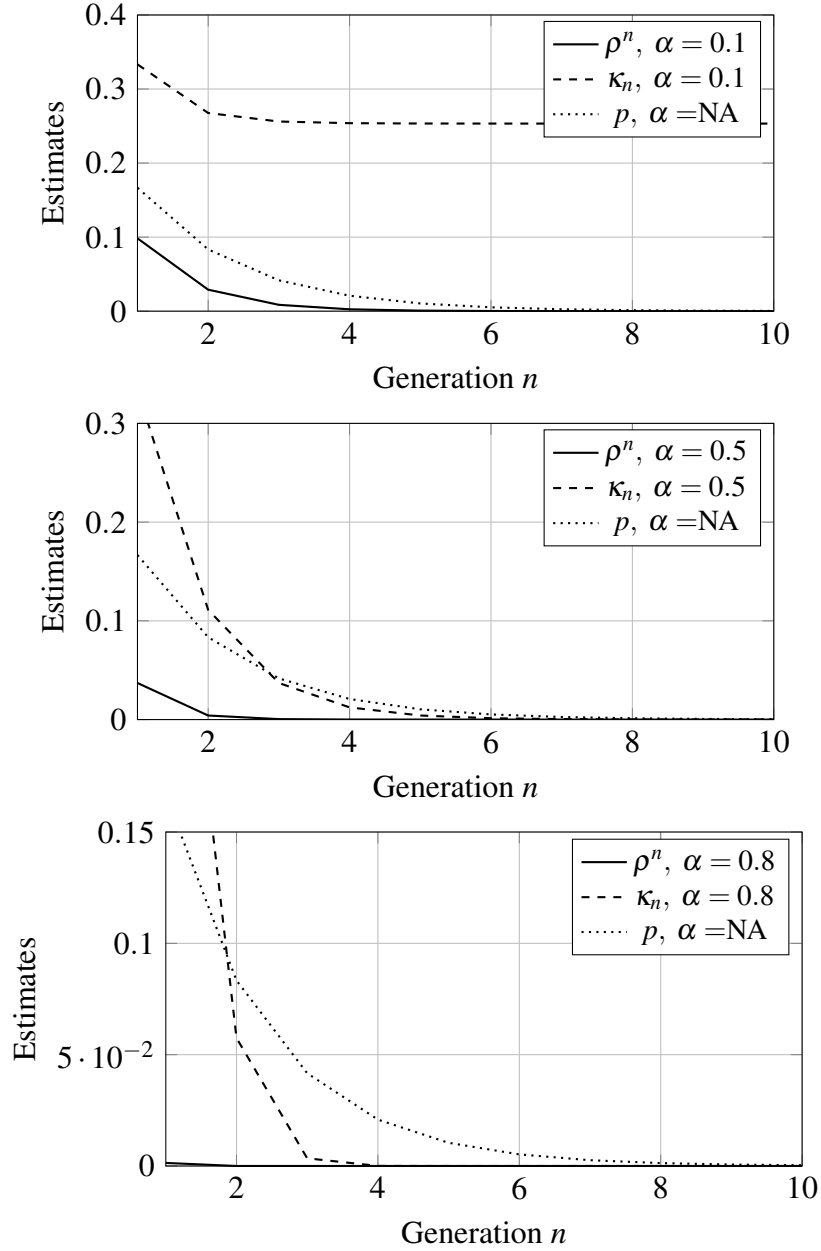
Inequality (5): $0.3333 < 0.3644$.

Obviously, T is not an enriched interpolative Kannan-type map.

Now, for any initial guess $x_0 \in [0, 1]$, $\lambda = \frac{1}{3}$, and $n = 10$, we demonstrate the error estimates for the conditions (2), (4), and (5), using the Krasnoselskii procedure for each value of $\alpha = 0.2, 0.5, 0.8$. The three cases (of α) behave quite differently as summarised in Figure 2. The term p represents the estimate for condition (2). The estimate ρ^n tends to zero exponentially as α increases and fewer generations. Also, κ_n tends to zero exponentially as α and n increase, but slower when compared to ρ^n . The estimate p maintains a steady decay to zero as n increases. Overall, estimate ρ^n decays much faster than estimates κ_n and p , followed by κ_n which performs better than p as α approaches 1.

2.1. Ulam-Hyers Stability. In this subsection, we study the Ulam-Hyers-type stability of T satisfying (5). See [14, 15, 16] for some related works.

Definition 2.6. *Let $(X, \|\cdot\|)$ be a normed spaces, A mapping $T : X \rightarrow X$ be a (a, b, c, α) -enriched condensed Kannan-type map and $\varepsilon > 0$, the fixed point problem is called generalized Ulam-Hyers stable if and only if there exists an increasing and continuous function $\phi : [0, \infty) \rightarrow [0, \infty)$*

FIGURE 1. Graph of error estimates vs generations n

with $\phi(0) = 0$ such that for each ε solution $w^* \in X$ of the fixed point equation $Tx = x$, there exists a solution $x^* \in X$ such that

$$(23) \quad \|x^* - w^*\| \leq \phi(\varepsilon)$$

In the Definition 2.6, if $\phi(\varepsilon) = \varepsilon t$ for $t \geq 0$ and $\varepsilon > 0$, then the fixed point equation $Tx = x$ is said to be Ulam-Hyers stable.

Theorem 2.7. *Let $(X, \|\cdot\|)$ be a normed spaces and T fulfills all hypotheses of Theorem 2.3, then the fixed point problem is Ulam-Hyers stable.*

Proof. Since $\text{Fix}(T) = \text{Fix}(T_\lambda)$, it follows that the fixed point problem T is equivalent to the fixed point problem $x = T_\lambda x$. Let ω^* be ε -solution of T_λ , that is,

$$(24) \quad \|\omega^* - T_\lambda \omega^*\| \leq \varepsilon.$$

Then, we have from (5), (23), and (24) that

$$\begin{aligned} \|x^* - \omega^*\| &= \|T_\lambda x^* - \omega^*\| \\ &\leq \|T_\lambda x^* - T_\lambda \omega^*\| + \|T_\lambda \omega^* - \omega^*\| \\ &\leq c[\|x^* - T_\lambda x^*\|^{2\alpha} + \|\omega^* - T_\lambda \omega^*\|^{2(1-\alpha)}] + \varepsilon \\ &\leq c\varepsilon^{2(1-\alpha)} + \varepsilon \equiv \phi(\varepsilon). \end{aligned}$$

where $\phi(\varepsilon) = c\varepsilon^{2(1-\alpha)} + \varepsilon$ is semilinear, for $c \in [0, \frac{1}{2})$, $\alpha \in (0, 1)$, and $\varepsilon > 0$, then the fixed point equation is Ulam-Hyers stable. $\square$

2.2. Application. Many real life problems arising in social science, applied science, and engineering can be modelled using variational inequality problems (VIPs). Let $\mathbb{H}$ be a Hilbert space and $F : C \rightarrow C$ be an operator for $C \subset \mathbb{H}$. The VIP with respect to F and C is to seek a solution $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0 \text{ for all } x \in \mathbb{H}.$$

One of most used method for solving the VIPs is the projection method $y = P_C(I - vF)x$ for $v > 0$. But this method turns out to be repulsive if some of the underlying conditions are weak. Many robust techniques have been introduced to approximate the solutions of VIPs. In this subsection, we aim to study the existence property of VIPs using Theorem 2.3, that is, adopting a discontinuous operators that ensures both unique and non-unique fixed solution with better convergent rate.

Theorem 2.8. *Let $F : C \rightarrow C$ be an operator associated with the VIP and assume that $P_C(I - vF)$ is a self-operator fulfilling all hypotheses of Theorem 2.3. Then, P_C has a unique solution*

in C . In addition, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = (I - \lambda)x_n + \lambda P_C(I - vF)x_n$, $n \geq 0$ converges strongly to the unique solution x^* for any $x_0 \in X$.

Proof. Let X be induced by an inner product which obeys the parallelogram equality in Theorem 2.3, then the proof follows if we set $X := C$ and $T := P_C(I - vF)$. $\square$

Example 2.9. Let $X = [1, 2] \times [-1, 1]$ be given by the Euclidean norm for any $x, y \in X$, and define $F : X \rightarrow X$ by $F(x) = \frac{1}{v}(1 - x_1, x_2)$, for $v > 0$ and $x = (x_1, x_2) \in X$. Now, define a projection-type operator $P_C : X \rightarrow X$ as

$$P_C(x) = \begin{cases} (2x_1 - 1, 0) & x_2 \geq 0 \\ (2x_1 - \frac{1}{2}, 0) & x_2 < 0 \end{cases}$$

Obviously, P_C has a jump discontinuity along x_2 tends to zero. But then, P_C is a (b, c, α) -enriched condensed Kannan-type map on X . By Theorem 2.4, P_C has a unique solution in X .

3. CONCLUSION

The study introduced the class of (a, b, c, α) -enriched condensed Kannan-type maps and established fixed point theorems for the existence of unique and non-unique fixed points of operators in normed spaces. We showed that the estimate deduced in Theorem 2.3 (or Theorem 2.4) via the Krasnoselskii-type procedure has better convergent rate. We also presented some examples to demonstrate that the class of enriched condensed Kannan-type maps strictly subsumes the class of enriched interpolative Kannan-type maps, in the sense that there exist operators which are not enriched interpolative Kannan-type but belong to the class of enriched condensed Kannan-type maps. As an application, we introduced a Krasnoselskii projection-type operator for solving VIPs within the framework of enriched condensed Kannan-type maps.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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