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THE EXTENDED DOUBLE CONTROLLED CONE METRIC SPACES AND THEIR ASSOCIATED FIXED POINT RESULTS WITH APPLICATIONS

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Abstract. In this text, we investigate some fixed point results in double-controlled cone metric spaces using several contraction mappings such as the Ciric-Reich-Rus contraction, some rational type contraction, and their associated consequences. Also, a few examples are included to illustrate the results. Finally, we discuss some applications that support our main results in the domain of differential equations.

Keywords: fixed point; cone metric space; double controlled cone metric space; Ciric-Reich-Rus contraction; rational contraction mappings.

2020 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

By applying the concept of fixed point theory (*FPT*), approximate fixed point theory (*AFPT*) and best proximity point theory (*BPPT*), researchers from many fields have contributed to the progress of science and technology. Large scale problems requiring *FPT* are highly esteemed

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for their lightning-fast solutions. As a result, in recent years, many scholars have focused on developing *FPT* approaches and have provided various useful techniques for discovering *FPS*, *AFPS*, and *ABPPs* in complex issues (refer, [23], [26], [28]). These are currently crucial in many mathematics related areas and its applications, including economics, astronomy, dynamical systems, decision theory, and parameter estimation. The father of *FPT*, mathematician Brouwer [6], proposed *FP* theorems for continuous mappings on finite dimensional spaces. In 1922, Banach [4] established and confirmed the renowned Banach contraction principle (*BCP*). Following that, several authors used the *BCP* in numerous extended metric spaces and presented large amount of *FP* results [see, [5], [7], [8], [12], [13], [14], [19], [22], [24], [25], [27], [29]]. On the other hand, the concept of *b*-metric space was initiated by Bakhtin [3] in 1989, which was an interesting expansion of metric spaces. Recently, more and more extensions of *b*-metric spaces, such as $b_v(s)$ -metric spaces and *b*-rectangular metric spaces, were introduced, and some *FP* theorems are proved on these spaces [see, for examole, [16], [20], [30]]. Also, authors Mlaiki et al.[17] studied extension of the expanded *b*-metric spaces called controlled metric type spaces and proved some innovative *FP* theorems. Subsequently, Abdeljawad et al.[2] proposed some *FP* results using contraction mappings including α -contraction [see, Theorem 1], Kannan contraction [see, Theorem 3] and their consequences in double controlled cone metric spaces [*DCCMS*]. In [21], Shateri, proved many *FP* theorems on *DCCMS* including Reich type contraction mapping (see, Theorem [2.5]). Researchers in the field of *FPT* should consult manuscripts [10], [18], and the references therein for more relevent results on *DCCMS*.

The remaining parts of this manuscript are displayed as follows: In Section 2, some basic notions, such as notations, definitions and lemmas, are recalled from previous literature. In Section 3, we propose the main results of this work, where the existence and uniqueness of *FP's* are rigorously discussed in *DCCMS* using Ciric-Reich-Rus contraction mappings. In section 4, we extend the concept into rational type contraction mapping and derived new *FP* results. Finally, in Section 5, we present some conclusions.

2. PRELIMINARIES

In this section, some notations and basic notions, such as definitions and lemmas, from earlier research are recalled. These are then employed throughout the remainder of the main findings of this manuscript.

Definition 2.1. [1] *Let E be a real Banach space and P be a subset of E . P is called a cone if it satisfies the following conditions:*

(C1) P is closed, non-empty and $P \neq \{0\}$;

(C2) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b ; and

(C3) $P \cap (-P) = \{0\}$.

Remark 2.2. [1] *Consider a cone $P \subseteq E$. We can define a partial ordering $\preceq$ on E with respect to P by $x \preceq y$ if and only if $y - x \in P$. Here, $x \prec y$ indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ stand for $y - x \in \text{int}P$, in which $\text{int}P \neq \emptyset$ and $\preceq$ is a partial ordering with respect to P .*

Definition 2.3. [2] *The cone P is called normal if*

$$\inf\{\|x + y\| : x, y \in P, \|x\| = \|y\| = 1\} \succ 0$$

or equivalently, there is a constant number $M \succ 0$ such that for all $x, y \in E$ where $0 \preceq x \preceq y$ implies $\|x\| \preceq M\|y\|$. The least positive number satisfying above is called the normal constant of P .

Definition 2.4. [2] *Let T be a non-empty set. A mapping $d : T \times T \rightarrow E$ is said to be a cone metric (CM) on T if for all $p, q, z \in X$ the following hold:*

(CM1) $0 \prec d(p, q)$ and $d(p, q) = 0$ if and only if $p = q$;

(CM2) $d(p, q) = d(q, p)$;

(CM3) $d(p, q) \preceq d(p, z) + d(z, q)$.

Then the pair (T, d) is called cone metric space (CMS).

Definition 2.5. [21] *Let T be a non-empty set and let $\mathbf{v} : T \times T \rightarrow [1, \infty)$ be a mapping. A mapping $d : T \times T \rightarrow E$ is said to be a controlled cone metric (CCM) with $\mathbf{v}$ if for all $p, q, z \in X$ the following hold:*

(CCM1) $0 \prec d(p, q)$ and $d(p, q) = 0$ if and only if $p = q$;

(CCM2) $d(p, q) = d(q, p)$;

(CCM3) $d(p, q) \preceq v(p, z)d(p, z) + v(z, q)d(z, q)$.

Then the pair (T, d) is called controlled cone metric space (CCMS). Note that each cone metric space is a controlled cone metric space with $v(p, q) = 1$.

Definition 2.6. [21] Let T be a non-empty set and let $v, \mu : T \times T \rightarrow [1, \infty)$ be non-comparable functions. A mapping $d : T \times T \rightarrow E$ is said to be a double controlled cone metric (DCCM) with respect to v and μ if for all $p, q, z \in X$ the following hold:

(DCCM1) $0 \prec d(p, q)$ and $d(p, q) = 0$ if and only if $p = q$;

(DCCM2) $d(p, q) = d(q, p)$;

(DCCM3) $d(p, q) \preceq v(p, z)d(p, z) + \mu(z, q)d(z, q)$.

Then the pair (T, d) is called double controlled cone metric space (DCCMS). It is clear that each CCMS is a DCCMS. But there exists DCCMS which are not CCMS.

Example 2.7. Let $E = \mathbb{R}^2$, $P = \{(p, q) \in E : p, q \succeq 0\}$, $T = \mathbb{R}^+$ and $d : T \times T \rightarrow E$ be defined by

$$d(p, q) = \begin{cases} (0, 0) & \text{iff } p = q \\ (\frac{1}{p}, \frac{1}{3}) & \text{if } p \succeq 1, q \in [0, 1) \\ (\frac{1}{3}, \frac{1}{q}) & \text{if } q \succeq 1, p \in [0, 1) \\ (1, 1) & \text{otherwise.} \end{cases}$$

Define $v, \mu : T \times T \rightarrow [1, \infty)$ by:

$$v(p, q) = \begin{cases} p & \text{if } p, q \succeq 1 \\ 1 & \text{otherwise.} \end{cases}$$

and

$$v(p, q) = \begin{cases} 1 & \text{if } p, q \preceq 1 \\ \max\{p, q\} & \text{otherwise.} \end{cases}$$

Then (T, d) is a double controlled cone metric space. Also, we have

$$d(0, \frac{1}{2}) = (1, 1) \succ (\frac{2}{3}, \frac{2}{3}) = v(0, 3)d(0, 3) + v(3, \frac{1}{2})d(3, \frac{1}{2}).$$

Which implies that d is not a controlled cone metric space when $v = \mu$.

Definition 2.8. [21] Let (T, d) be a DCCMS with respect to v and μ .

(i) A sequence $\{p_n\}$ is convergent to some p in T , if for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(p_n, p) \ll c$, then $\{p_n\}$ is said to be convergent and $\{p_n\}$ converges to p , and p is the limit of $\{p_n\}$. It is written as $\lim_{n \rightarrow \infty} p_n = p$.

(ii) A sequence $\{p_n\}$ is Cauchy, if for every $c \in E$ with $0 \ll c$ there is N such that for all $m, n > N$, $d(p_n, p) \ll c$.

(iii) (T, d) is said to be complete if every Cauchy sequence is convergent.

Lemma 2.9. [21] Let (T, d) be a DCCMS with respect to v and μ , P be a normal cone with normal constant M . Let $\{p_n\}$ be a sequence in T . Then $\{p_n\}$ converges to p if and only if $\lim_{n \rightarrow \infty} d(p_n, p) = 0$.

Lemma 2.10. [21] Let (T, d) be a DCCMS with respect to v and μ , P be a normal cone with normal constant M . Let $\{p_n\}$ be a sequence in T such that $\{p_n\}$ converges to p and q . If $\lim_{n \rightarrow \infty} v(p_n, p)$ and $\lim_{n \rightarrow \infty} \mu(p_n, q)$ exist and are finite, then $p = q$.

3. RESULTS FOR CONTRACTION MAPPINGS

In this section, we prove some *FP* results in *DCCMS* using various contraction mappings. For that, assume (T, d) be a complete *DCCMS* with respect to the functions $v, \mu : T \times T \rightarrow [1, \infty)$ and P be a normal cone with normal constant M . Following is the Ciric-Reich-Rus contraction mapping considered here to prove our first *FP* theorem in *DCCMS*.

Theorem 3.1. Let $K : T \rightarrow T$ be a map satisfy the contraction condition

$$(3.1) \quad d(Kp, Kq) \preceq \alpha d(p, q) + \beta [d(p, Kp) + d(q, Kq)]$$

for all $p, q \in T$, where $\alpha, \beta \in [0, 1)$ with $\alpha + 2\beta \prec 1$. For $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$(3.2) \quad \sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{v(p_{i+1}, p_{i+2})}{v(p_i, p_{i+1})} \mu(p_{i+1}, p_m) \prec \frac{1 - \beta}{\alpha + \beta}.$$

If for each $p \in T$,

$$(3.3) \quad \lim_{n \rightarrow \infty} v(p, p_n) \text{ exists, is finite and } \lim_{n \rightarrow \infty} \mu(p_n, p) \prec \frac{1}{\beta}.$$

Then K has an UFP in T .

Proof. Consider a sequence $\{p_n\}$ in T satisfies the hypothesis of the theorem. From (3.1), we obtain

$$\begin{aligned} d(p_n, p_{n+1}) &= d(Kp_{n-1}, Kp_n) \\ &\preceq \alpha d(p_{n-1}, p_n) + \beta [d(p_{n-1}, Kp_{n-1}) + d(p_n, Kp_n)] \\ &\preceq \alpha d(p_{n-1}, p_n) + \beta [d(p_{n-1}, p_n) + d(p_n, p_{n+1})] \\ &= \lambda d(p_{n-1}, p_n), \text{ where } \lambda = \frac{\alpha + \beta}{1 - \beta} \\ &\preceq \lambda^2 d(p_{n-2}, p_{n-1}) \\ &\dots \\ &\preceq \lambda^n d(p_0, p_1) \end{aligned}$$

That is, $d(p_n, p_{n+1}) = \lambda^n d(p_0, p_1)$, for all $n \geq 0$. Let m, n be integers such that $m \succ n$. Show that $\{p_n\}$ is a Cauchy sequence. Consider,

$$\begin{aligned} d(p_n, p_m) &\preceq v(p_n, p_{n+1})d(p_n, p_{n+1}) + \mu(p_{n+1}, p_m)d(p_{n+1}, p_m) \\ &\preceq v(p_n, p_{n+1})d(p_n, p_{n+1}) + \mu(p_{n+1}, p_m)v(p_{n+1}, p_{n+2})d(p_{n+1}, p_{n+2}) \\ &\quad + \mu(p_{n+1}, p_m)\mu(p_{n+2}, p_m)d(p_{n+2}, p_m) \\ &\preceq \dots \\ &\preceq v(p_n, p_{n+1})d(p_n, p_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1})d(p_i, p_{i+1}) \end{aligned}$$

$$\begin{aligned}
& + \prod_{k=n+1}^{m-1} \mu(p_k, p_m) d(p_{m-1}, p_m) \\
& \preceq v(p_n, p_{n+1}) \left(\frac{\alpha + \beta}{1 - \beta} \right)^n d(p_0, p_1) \\
& + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1}) \left(\frac{\alpha + \beta}{1 - \beta} \right)^i d(p_0, p_1) \\
& + \prod_{k=n+1}^{m-1} \mu(p_k, p_m) \left(\frac{\alpha + \beta}{1 - \beta} \right)^{m-1} d(p_0, p_1) \\
& \preceq v(p_n, p_{n+1}) \left(\frac{\alpha + \beta}{1 - \beta} \right)^n d(p_0, p_1) \\
& + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1}) \left(\frac{\alpha + \beta}{1 - \beta} \right)^i d(p_0, p_1) \\
& \preceq v(p_n, p_{n+1}) \left(\frac{\alpha + \beta}{1 - \beta} \right)^n d(p_0, p_1) \\
& + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1}) \left(\frac{\alpha + \beta}{1 - \beta} \right)^i d(p_0, p_1)
\end{aligned}$$

This implies,

$$\begin{aligned}
\|d(p_n, p_m)\| & \preceq M \left\| v(p_n, p_{n+1}) \left(\frac{\alpha + \beta}{1 - \beta} \right)^n d(p_0, p_1) \right. \\
& \left. + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1}) \left(\frac{\alpha + \beta}{1 - \beta} \right)^i d(p_0, p_1) \right\|.
\end{aligned} \tag{3.4}$$

Choose $N_l = \sum_{i=0}^l \left(\prod_{j=0}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1}) \left(\frac{\alpha + \beta}{1 - \beta} \right)^i$, then we get

$$\|d(p_n, p_m)\| \preceq M \left\| d(p_0, p_1) \left[\left(\frac{\alpha + \beta}{1 - \beta} \right)^n v(p_n, p_{n+1}) + (N_{m-1} - N_n) \right] \right\|.$$

Then (3.2) implies that the limit of the sequence $\{N_n\}$ exists and so $\{N_n\}$ is Cauchy. Letting $m, n \rightarrow \infty$ in (3.3) gives $\lim_{m, n \rightarrow \infty} d(p_n, p_m) = 0$, and so $\{p_n\}$ is a Cauchy sequence. By using the completeness of K , there exists $p \in K$ such that $\lim_{n \rightarrow \infty} p_n = p$. We claim that $Kp = p$. It follows from (DCCM3) and (3.1) that

$$0 \prec d(p, Kp)$$

$$\begin{aligned}
&\preceq v(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)d(p_{n+1}, Kp) \\
&\preceq v(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)d(Kp_n, Kp) \\
&\preceq v(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)[\alpha d(p_n, p) + \beta[d(p_n, Kp_n) + d(p, Kp)]] \\
&\preceq v(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)[\alpha d(p_n, p) + \beta[d(p_n, p_{n+1}) + d(p, Kp)]]
\end{aligned}$$

and so

$$\begin{aligned}
&0 \prec \|d(p, Kp)\| \\
(3.5) \quad &\preceq M \|v(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)[\alpha d(p_n, p) + \beta[d(p_n, p_{n+1}) + d(p, Kp)]]\|.
\end{aligned}$$

Now, making use of the condition (3.3) and passing to the limit on (3.5) we get

$$0 \prec \|d(p, Kp)\| \prec \beta \|d(p, Kp)\|$$

which is a contradiction, therefore $Kp = p$. Suppose that K has another FP (say, q), then

$$\begin{aligned}
d(p, q) = d(Kp, Kq) &\preceq \alpha d(p, q) + \beta[d(p, Kp) + d(q, Kq)] \\
&\preceq \alpha d(p, q) + \beta[d(p, p) + d(q, q)] \\
&\preceq \alpha d(p, q)
\end{aligned}$$

But $\alpha \prec 1$. Therefore, our supposition is wrong. Hence, K has an UFP in T . □

Corollary 3.2. *Let $K : T \rightarrow T$ be a map satisfy the contraction condition*

$$(3.6) \quad d(Kp, Kq) \preceq \alpha d(p, q)$$

for all $p, q \in T$, where $\alpha \in [0, 1)$. For $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$(3.7) \quad \sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{v(p_{i+1}, p_{i+2})}{v(p_i, p_{i+1})} \mu(p_{i+1}, p_m) \prec \frac{1}{\alpha}.$$

If for each $p \in T$, $\lim_{n \rightarrow \infty} v(p, p_n)$ and $\lim_{n \rightarrow \infty} \mu(p_n, p)$ exist and are finite, then K has a UFP in T .

Proof. Substituting $\beta = 0$ in Theorem 3.1 completes this corollary. □

Corollary 3.3. *Let $K : T \rightarrow T$ be a map satisfy the contraction condition*

$$(3.8) \quad d(Kp, Kq) \preceq \beta[d(p, Kp) + d(q, Kq)]$$

for all $p, q \in T$, where $\beta \in [0, 1)$ with $2\beta \prec 1$. For arbitrary $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$(3.9) \quad \sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{v(p_{i+1}, p_{i+2})}{v(p_i, p_{i+1})} \mu(p_{i+1}, p_m) \prec \frac{1 - \beta}{\beta}.$$

If for each $p \in T$, $\lim_{n \rightarrow \infty} v(p, p_n)$ is exists, finite and $\lim_{n \rightarrow \infty} \mu(p_n, p) \prec \frac{1}{\beta}$ then K has a UFP in T .

Proof. Substituting $\alpha = 0$ in Theorem 3.1 completes this corollary. □

Corollary 3.4. *Let $K : T \rightarrow T$ be a map satisfy the contraction condition*

$$(3.10) \quad d(Kp, Kq) \preceq \beta[d(p, Kq) + d(q, Kp)]$$

for all $p, q \in T$, where $\beta \in (0, 1)$ with $2\beta \prec 1$. For arbitrary $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$(3.11) \quad \sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{v(p_{i+1}, p_{i+2})}{v(p_i, p_{i+1})} \mu(p_{i+1}, p_m) \prec \frac{1 - \beta}{\beta}.$$

If for each $p \in T$, $\lim_{n \rightarrow \infty} v(p, p_n)$ is exists, finite and $\lim_{n \rightarrow \infty} \mu(p_n, p) \prec \frac{1}{\beta}$ then K has a UFP in T .

Proof. Substituting $\alpha = 0$ and interchanging $kp = kq$ vice-versa in Theorem 3.1 completes this corollary. □

Corollary 3.5. *Let $K : T \rightarrow T$ be a map satisfy the contraction condition*

$$(3.12) \quad d(Kp, Kq) \preceq \alpha[d(p, Kp) + d(q, Kq) + d(p, q)]$$

for all $p, q \in T$, where $\alpha \in [0, 1)$ with $3\alpha \prec 1$. For arbitrary $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$(3.13) \quad \sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{v(p_{i+1}, p_{i+2})}{v(p_i, p_{i+1})} \mu(p_{i+1}, p_m) \prec \frac{1 - \alpha}{2\alpha}.$$

If for each $p \in T$, $\lim_{n \rightarrow \infty} v(p, p_n)$ exists, finite and $\lim_{n \rightarrow \infty} \mu(p_n, p) \prec \frac{1}{\alpha}$ then K has a UFP in T .

Proof. Substituting $\alpha = \beta$ and interchanging $kp = kq$ in Theorem 3.1 completes this corollary. $\square$

Example 3.6. Define a partial order relation $\preceq$ on E as follows:

$$p \preceq q \text{ if and only if } \|p\|_2 \preceq \|q\|_2.$$

Let $E = l^2$, $X = l^2$ and $P = \{p \in E : p(i) \succeq 0\}$, then P is a cone. Consider a map $d : X \times X \rightarrow E$ is defined by

$$d(p, q) = \begin{cases} z & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

where $z(i) = |p(i)| + |q(i)|$. Let $v, \mu : T \times T \rightarrow [1, \infty)$ be defined by $v(p, q) = 1$ and $\mu(p, q) = 1$, for all $(p, q) \in K \times K$. It can be easily verified that d is a DCCM. Define $K : l^2 \rightarrow l^2$ by $Kp = p/3$. Thus, $d(Kp, Kq) = d(p/3, q/3) = z/3$, where $z = |p(i)| + |q(i)|$. Choose $\alpha = 1/2$; $\beta = 3/16$ then $2\beta \preceq 1$. Also,

$$\begin{aligned} \alpha d(p, q) + \beta [d(p, Kp) + d(q, Kq)] &= (\alpha + \frac{2}{3}(\alpha + \beta))|p(i)| + |q(i)| \\ &= |p(i)| + |q(i)| \\ &= z \end{aligned}$$

Therefore, K is a Ciric-Reich-Rus contraction operator. Hence, K has a FP. More concretely, 0 is the only FP.

4. RESULTS FOR RATIONAL CONTRACTION MAPPINGS

We prove some FP results in DCCMS using rational contraction mappings which were discussed mainly in [9] and [11]. For that, assume (T, d) be a complete DCCMS with respect to the functions $v, \mu : T \times T \rightarrow [1, \infty)$ and P be a normal cone with normal constant M .

Theorem 4.1. *Let $K : T \rightarrow T$ be a map satisfy the rational contraction condition*

$$(4.1) \quad d(Kp, Kq) \preceq \frac{\alpha[1 + d(p, Kp)]d(q, Kq)}{1 + d(p, q)} + \beta d(p, q) + \gamma d(p, Kp) + \delta d(q, Kq)$$

for all $p, q \in T$, with $1 + d(p, q) \neq 0$ where $\alpha, \beta, \gamma, \delta \in [0, 1]$ and $\alpha + \beta + \gamma + \delta < 1$. For $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$(4.2) \quad \sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{v(p_{i+1}, p_{i+2})}{v(p_i, p_{i+1})} \mu(p_{i+1}, p_m) \prec \frac{\beta + \gamma}{1 - \alpha - \delta}.$$

If for each $p \in T$,

$$(4.3) \quad \lim_{n \rightarrow \infty} v(p, p_n) \text{ exists, is finite and } \lim_{n \rightarrow \infty} \mu(p_n, p) \prec \frac{1}{\beta}$$

then K has a UFP in T .

Proof. Consider a sequence $\{p_n\}$ in T satisfies the hypothesis of the theorem. From (3.1), we obtain

$$\begin{aligned} d(p_n, p_{n+1}) &= d(Kp_{n-1}, Kp_n) \\ &\preceq \frac{\alpha[1 + d(p_{n-1}, Kp_{n-1})]d(p_n, Kp_n)}{1 + d(p_{n-1}, p_n)} + \beta d(p_{n-1}, p_n) \\ &\quad + \gamma d(p_{n-1}, Kp_{n-1}) + \delta d(p_n, Kp_n) \\ &= \frac{\alpha[1 + d(p_{n-1}, p_n)]d(p_n, p_{n+1})}{1 + d(p_{n-1}, p_n)} + \beta d(p_{n-1}, p_n) \\ &\quad + \gamma d(p_{n-1}, p_n) + \delta d(p_n, p_{n+1}) \\ &= \alpha d(p_n, p_{n+1}) + \beta d(p_{n-1}, p_n) + \gamma d(p_{n-1}, p_n) + \delta d(p_n, p_{n+1}) \end{aligned}$$

That is, $d(p_n, p_{n+1}) \preceq \frac{\beta + \gamma}{1 - \alpha - \delta} d(p_{n-1}, p_n)$. By using induction on n , we get

$$(4.4) \quad d(p_n, p_{n+1}) \preceq \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^n d(p_0, p_1), \text{ for all } n \geq 0.$$

Now, to prove $\{p_n\}$ is a Cauchy sequence. Using (DCCM3) and (4.1), for all $m, n \in \mathbb{N}$ implies that

$$\begin{aligned} d(p_n, p_m) \\ \preceq v(p_n, p_{n+1})d(p_n, p_{n+1}) + \mu(p_{n+1}, p_m)d(p_{n+1}, p_m) \end{aligned}$$

$$\begin{aligned}
& \preceq v(p_n, p_{n+1})d(p_n, p_{n+1}) + \mu(p_{n+1}, p_m)v(p_{n+1}, p_{n+2})d(p_{n+1}, p_{n+2}) \\
& \quad + \mu(p_{n+1}, p_m)\mu(p_{n+2}, p_m)d(p_{n+2}, p_m) \\
& \preceq \dots \\
& \preceq v(p_n, p_{n+1})d(p_n, p_{n+1}) \\
& \quad + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1})d(p_i, p_{i+1}) + \prod_{k=n+1}^{m-1} \mu(p_k, p_m)d(p_{m-1}, p_m) \\
& \preceq v(p_n, p_{n+1}) \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^n d(p_0, p_1) \\
& \quad + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1}) \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^i d(p_0, p_1) \\
& \quad + \prod_{k=n+1}^{m-1} \mu(p_k, p_m) \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^{m-1} d(p_0, p_1) \\
& \preceq v(p_n, p_{n+1}) \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^n d(p_0, p_1) \\
& \quad + \sum_{i=n+1}^{m-1} \left(\prod_{j=n+1}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1}) \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^i d(p_0, p_1) \\
& \preceq v(p_n, p_{n+1}) \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^n d(p_0, p_1) \\
& \quad + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1}) \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^i d(p_0, p_1)
\end{aligned}$$

This implies,

$$\begin{aligned}
\|d(p_n, p_m)\| & \preceq M \left\| v(p_n, p_{n+1}) \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^n d(p_0, p_1) \right. \\
(4.5) \quad & \quad \left. + \sum_{i=n+1}^{m-1} \left(\prod_{j=0}^i \mu(p_j, p_m) \right) v(p_i, p_{i+1}) \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^i d(p_0, p_1) \right\|.
\end{aligned}$$

Choose $N_l = \sum_{i=0}^l (\prod_{j=0}^l \mu(p_j, p_m)) v(p_i, p_{i+1}) \left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^i$, then we get

$$\|d(p_n, p_m)\| \preceq M \left\| d(p_0, p_1) \left[\left(\frac{\beta + \gamma}{1 - \alpha - \delta} \right)^n v(p_n, p_{n+1}) + (N_{m-1} - N_n) \right] \right\|.$$

Since $\left(\frac{\beta + \gamma}{1 - \alpha - \delta}\right) \prec 1$. Which implies that the limit of the sequence $\{N_n\}$ exists and so $\{N_n\}$ is Cauchy. Letting $m, n \rightarrow \infty$ in (4.5) gives $\lim_{m, n \rightarrow \infty} d(p_n, p_m) = 0$, and so $\{p_n\}$ is a Cauchy sequence. Using the completeness of K , there exists $p \in K$ such that $\lim_{n \rightarrow \infty} p_n = p$. We claim that $Kp = p$. It follows from (DCCM3) and (4.1) that

$$\begin{aligned}
0 &\prec d(p, Kp) \\
&\preceq v(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)d(p_{n+1}, Kp) \\
&\preceq v(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp)d(Kp_n, Kp) \\
&\preceq v(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp) \\
&\quad \left[\frac{\alpha d(p, Kp)[1 + d(p_n, p_{n+1})]}{1 + d(p_n, p)} + \beta d(p_n, p) + \gamma(p_n, Kp_n) + \delta d(p, Kp) \right]
\end{aligned}$$

and so

$$\begin{aligned}
0 &\prec \|d(p, Kp)\| \\
&\preceq M \left\| v(p, p_{n+1})d(p, p_{n+1}) + \mu(p_{n+1}, Kp) \right. \\
(4.6) \quad &\quad \left. \left[\frac{\alpha d(p, Kp)[1 + d(p_n, p_{n+1})]}{1 + d(p_n, p)} + \beta d(p_n, p) + \gamma(p_n, p_{n+1}) + \delta d(p, Kp) \right] \right\|.
\end{aligned}$$

Now, making use of the condition (4.3) and passing to the limit on (4.6) we get

$$0 \prec \|d(p, Kp)\| \prec (\beta + \delta)\|d(p, Kp)\|$$

which is a contradiction, therefore $Kp = p$. Suppose that K has another fixed point (say, q), then

$$\begin{aligned}
d(p, q) &= d(Kp, Kq) \\
&\preceq \frac{\alpha[1 + d(p, Kp)]d(q, Kq)}{1 + d(p, q)} + \beta d(p, q) + \gamma d(p, Kp) + \delta d(q, Kq) \\
&\preceq \beta d(p, q)
\end{aligned}$$

But $\beta \prec 1$. Therefore, our supposition is wrong. Hence, K has a *UFP* in T . $\square$

Corollary 4.2. *Let $K : T \rightarrow T$ be a map satisfy the rational contraction condition*

$$(4.7) \quad d(Kp, Kq) \preceq \frac{\alpha[1 + d(p, Kp)]d(q, Kq)}{1 + d(p, q)} + \beta d(p, q)$$

for all $p, q \in T$, with $1 + d(p, q) \neq 0$ where $\alpha, \beta \in [0, 1)$ and $\alpha + \beta < 1$. For $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$(4.8) \quad \sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{v(p_{i+1}, p_{i+2})}{v(p_i, p_{i+1})} \mu(p_{i+1}, p_m) \prec \frac{1 - \alpha}{\beta}.$$

If for each $p \in T$,

$$(4.9) \quad \lim_{n \rightarrow \infty} v(p, p_n) \text{ exists, is finite and } \lim_{n \rightarrow \infty} \mu(p_n, p) \prec \frac{1}{\alpha}$$

then K has a UFP in T .

Proof. Substituting $\gamma = \delta = 0$ in the above Theorem 4.1 completes this corollary. $\square$

Corollary 4.3. *Let $K : T \rightarrow T$ be a map satisfy the rational contraction condition*

$$(4.10) \quad d(Kp, Kq) \preceq \beta d(p, q) + \gamma d(p, Kp) + \delta d(q, Kq)$$

for all $p, q \in T$, where $\alpha, \beta \in [0, 1)$ with $\beta + \gamma + \delta < 1$. For $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$(4.11) \quad \sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{v(p_{i+1}, p_{i+2})}{v(p_i, p_{i+1})} \mu(p_{i+1}, p_m) \prec \frac{1 - \delta}{\alpha + \beta}.$$

If for each $p \in T$,

$$(4.12) \quad \lim_{n \rightarrow \infty} v(p, p_n) \text{ exists, is finite and } \lim_{n \rightarrow \infty} \mu(p_n, p) \prec \frac{1}{\delta}$$

then K has a UFP in T .

Proof. Substituting $\alpha = 0$ in the above Theorem 4.1 completes this corollary. $\square$

Corollary 4.4. *Let $K : T \rightarrow T$ be a map satisfy the rational contraction condition*

$$(4.13) \quad d(Kp, Kq) \preceq \beta d(p, q) + \gamma d(p, Kp)$$

for all $p, q \in T$, where $\gamma, \beta \in [0, 1)$ with $\beta + \gamma < 1$. For $p_0 \in T$, choose $p_n = K^n p_0$. Suppose that

$$(4.14) \quad \sup_{m \geq 1} \lim_{i \rightarrow \infty} \frac{v(p_{i+1}, p_{i+2})}{v(p_i, p_{i+1})} \mu(p_{i+1}, p_m) \prec \frac{1 - \gamma}{\beta}.$$

If for each $p \in T$,

$$(4.15) \quad \lim_{n \rightarrow \infty} v(p, p_n) \text{ exists, is finite and } \lim_{n \rightarrow \infty} \mu(p_n, p) \prec \frac{1}{\gamma}$$

then K has a UFP in T .

Proof. Substituting $\alpha = \delta = 0$ in the above Theorem 4.1 completes this corollary. $\square$

Remark 4.5. All the results proved in the Section 3 and Section 4 hold in complete double controlled cone metric spaces, too.

5. APPLICATIONS

The *FPT* covers a wide range of applications in the field of mathematics, particularly operator theory, optimization problems, and so on. By reading [15] and references therein, one can find a variety of applications involving *FP* results in several domains. The examples below demonstrate how to apply *FP* findings in differential equations.

Example 5.1. Consider $p^{(4)}b = p(b) - 8e^b(1+b)$ with the boundary conditions $p(0) = p(1) = 0$ and $p''(0) = 0; p''(1) = -4e$. Here the exact solution is $p_E b = b(1-b)e^b$. Define $K : C^{(3)}[0, 1] \rightarrow C^{(3)}[0, 1]$ by:

$$(5.1) \quad K(p) = p + \int_0^1 G(b, \iota) [p^{(4)}(\iota) - p(\iota) + 8e^\iota(1+\iota)] d\iota$$

Here p is a solution of $p^{(4)}(b) = 0, u(b) = \frac{2e}{3}(b - b^3)$.

$$\begin{aligned} K(p) &= \frac{2e(b - b^3)}{3} + \int_0^1 G(b, \iota) p^{(4)}(\iota) d\iota - \int_0^1 G(b, \iota) [p(\iota) - 8e^\iota(1+\iota)] d\iota \\ &= \frac{2e(b - b^3)}{3} - \int_0^1 G(b, \iota) [p(\iota) - 8e^\iota(1+\iota)] d\iota \end{aligned}$$

Consider,

$$|K(p) - K(q)| = \left| \int_0^1 G(b, \iota) [p(\iota) - 8e^\iota(1+\iota)] d\iota - \int_0^1 G(b, \iota) [q(\iota) - 8e^\iota(1+\iota)] d\iota \right|$$

$$\begin{aligned}
&= \left| \int_0^1 G(b, t) [p(t) - q(t)] dt \right| \\
(5.2) \quad |K(p) - K(q)| &\leq \left[\int_0^1 |G(b, t)|^2 dt \right]^{\frac{1}{2}} \left[\int_0^1 |p(t) - q(t)|^2 dt \right]^{\frac{1}{2}}
\end{aligned}$$

where,

$$G(b, t) = (0.1667) \begin{cases} t(b-1)(t^2 - 2b + b^2) & 0 \leq t \leq b \\ b(t-1)(b^2 - 2t + t^2) & b \leq t \leq 1 \end{cases}$$

Now,

$$\begin{aligned}
\int_0^1 |G(b, t)|^2 dt &\leq 0.0278 \left(\int_0^b t^2 (b-1)^2 (t^2 - 2b + b^2)^2 dt + \int_b^1 b^2 (t-1)^2 (b^2 - 2t + t^2)^2 dt \right) \\
&= 0.0278 \left[0.0762b^2 - 0.2667b^4 + 0.5333b^6 - 0.4571b^7 + 0.1143b^8 \right]
\end{aligned}$$

Therefore,

$$\int_0^1 |G(b, t)|^2 dt \leq (0.0278)(0.0076)$$

Therefore (4.2) becomes,

$$\begin{aligned}
|K(p) - K(q)| &\leq 0.0208 \sqrt{0.4857} \left[\int_0^1 |p(t) - q(t)|^2 dt \right]^{\frac{1}{2}} \\
&\leq \left[\int_0^1 |p(t) - q(t)|^2 dt \right]^{\frac{1}{2}} \\
&= \|p - q\|
\end{aligned}$$

Hence, K is a contraction operator. Then, by Theorem 3.1, K has a fixed point.

6. CONCLUSION

This paper has introduced some new FP results that are applicable to both contraction and rational contraction operators on $DCCMS$. In particular, going in the same direction as [17] and [21], we provide the results in the setting of contraction mappings, namely the Ciric-Reich-Rus contraction. Additionally, we provide the FP results by using the rational contraction mappings. In order to confirm the presence of the UFP results, alternative discoveries presented in the later can be demonstrated in a lower environment.

AUTHOR CONTRIBUTIONS

All authors contributed equally, read and approved the final manuscript.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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